Canad. J. Math. Vol. **55** (5), 2003 pp. 1019–1079

More Eventual Positivity for Analytic Functions

David Handelman

Abstract. Eventual positivity problems for real convergent Maclaurin series lead to density questions for sets of harmonic functions. These are solved for large classes of series, and in so doing, asymptotic estimates are obtained for the values of the series near the radius of convergence and for the coefficients of convolution powers.

This paper began as a continuation of [H1], dealing with eventual positivity problems, *e.g.*, if f and P are convergent Maclaurin series in one variable with real coefficients, and if the coefficients of P are nonnegative, under what conditions does there exist an integer m such that all coefficients of $P^m f$ are nonnegative? However, it evolved somewhat into a discussion of problems concerning the trace space of an algebra associated to P, the asymptotic behaviour of $P(r)$ for r a positive real number "near" the radius of convergence of P (which could be infinite), and the growth of the coefficients in powers of P.

Associated to a convergent Maclaurin series P with no negative coefficients and with radius of convergence $\rho \in (0, \infty]$ is a compact set, $X(P)$, together with a natural embedding of $[0, \rho)$ into $X(P)$. If P happens to be lacunary, the image need not be dense; on the other hand, if P is continuous at ρ and some additional conditions (log convexity of the coefficients) hold, then $X(P)$ is just the one-point compactification of $[0, \rho)$, *i.e.*, a closed interval. However, for P as simple as $(1 - x)^{-1}$, the situation is far more complicated.

We prove that for series whose coefficients grow slowly (including $(1 - x)^{-1}$), the image is dense. We also prove it for many fast-growing series (such as $P = \exp\big(1/(1 - x)\big)$). The latter arguments require asymptotic estimates on the coefficients in powers of P, and also on the values of $P(r)$ for r a positive real number near ρ. The general idea is to relate the behaviour of $\liminf_{r\uparrow\rho} g(r)/P(r)$ to the ratios of the coefficients of g to those of P, for a suitable family of power series g whose coefficients are nonnegative and grow no faster than those of P. Problems arise because the ratio, g/P, although bounded, can oscillate arbitrarily badly.

If $f = \sum a_j x^j$ is a convergent Maclaurin series with real coefficients, we use the notation $(f, x^k) = a_k$. Associated to a Maclaurin series P with no negative coefficients is an ordered **R**-algebra,

$$R_P = \left\{ f/P^k \;\middle|\; |(f, x^m)| \leq K(P^k, x^m) \text{ for some } K > 0, \text{ for all } m \right\}$$

Received by the editors February 6, 2002; revised September 10, 2002.
Supported in part by an operating grant from NSERC (Canada).
AMS subject classification: Primary 30B10; Secondary 30D15, 30C50, 13A99, 41A58, 42A16.

(this is called A_P in [H1]). This admits an ordering in a natural way,

$$R_P^+ = \left\{ f/P^k \in R_P \;\middle|\; \text{there exists } n \text{ such that } (P^n f, x^m) \geq 0 \text{ for all } m \right\}.$$

Of particular importance for the eventual positivity problem are the (pure) traces; these are the positive (ring) homomorphisms $\tau\colon R_P \to \mathbf{R}$. We denote by $X(P)$ the set of all such τ, with the point-open (weak) topology. If ρ in $(0, \infty]$ denotes the radius of convergence of P, we obtain a pure trace $\tau_s\colon R_P \to \mathbf{R}$ for each s in $[0, \rho)$ via evaluation at s: $\tau_s(a) = a(s)$ for a in R_P; such pure traces are called *point evaluation traces*. If P happens to be left continuous at ρ, then τ_ρ is also defined. In some cases, these are all of the pure traces, *e.g.*, [H1, Theorems 7 and 9] yield such results under conditions related to log convexity of the coefficients of P.

However, if P is not left continuous at ρ (which here means that $P(\rho)$ is infinite), then there must be other pure traces, as the pure trace space is weakly compact, and the weak topology on $[0, \rho)$ agrees with the usual topology as a subset of the reals. The question then arises, under what circumstances is the set of the point evaluations dense in the set of all pure traces? Aside from idle curiosity about maximal ideal spaces, there is a positivity result motivating it. If $a = f/P^k$ is an element of R_P such that $a(s)$ is bounded below away from zero for $0 \leq s < \rho$ and the set of point evaluations is dense in the pure trace of R_P, then there exists an integer n such that $P^n f$ has no negative coefficients (actually more is true: there exists $\delta > 0$ such that $P^n(f - \delta P^k)$ has no negative coefficients, and this is true, with sufficiently large n depending on δ, for any $\delta < \inf a\big|[0, \rho)$).

The first section shows that a slightly stronger property than density holds for many choices of P with relatively slow growth—*e.g.*, for $P = (1-x)^{-1}, \ln\bigl(1/(1-x)\bigr)$, and variations on these. The next section concerns the equivalence relation $P \sim Q$, which means that $\{(P, x^k)/(Q, x^k)\}$ is bounded above and below (away from zero). While it need not be true that R_P and R_Q even admit a homomorphism between them, there is a canonical homeomorphism between their trace spaces when both have density of the point evaluations. We also present a weird example wherein $P \sim Q$, but R_Q^+ is properly contained in R_P^+. This is related to the phenomenon that if Q has no zeroes within the open disk of convergence and there exists an integer such that $P^n Q^{-1}$ has no negative coefficients in its Maclaurin series, then whatever f is made positive by Q or some power, is also made positive by some power of P (more succinctly, if Qf has no negative coefficients, then there exists n such that $P^n f$ has no negative coefficients).

This last problem is considered in the special case that $P = (1-x)^{-1}$. Results along these lines can be deduced from versions of Tauberian theorems, but the most useful ones come from summability techniques, such as Cesàro limits. The next section again deals with $P = (1-x)^{-1}$, but this time, how the positivity results are affected by perturbations (not small) of the multiplying power series P.

Up to this point, we are essentially working with slowly growing power series—the coefficients of powers of $(1-x)^{-1}$ grow polynomially. When we permit faster (or much much slower) growth, we have to restrict somewhat the behaviours of the coefficients. A class that is fairly extensive and for which theorems can be obtained includes P whose coefficients form a log concave distribution with a mild long-term smoothness condition (this includes many reasonable fast-growing distributions). In this case, we also have to obtain asymptotics for the coefficients of powers of P, and for the behaviour of $P(r)$ as $r \uparrow \rho$, before we can obtain the density theorems. Almost by accident, we also obtain asymptotic estimates for some nasty-looking combinatorial expressions, by reverse engineering the asymptotics of the coefficients.

Notation If f is a Maclaurin series, we denote the coefficient of x^k by (f, x^k). All Maclaurin series discussed here are *real* series, *i.e.*, the coefficients are real. If P has no negative coefficients and there exists $K > 0$ such that for all k, $|(f, x^k)| \leq K(P, x^k)$, then we say that f *is subequivalent to* P, denoted $f \prec P$. If Q also has no negative coefficients and both $P \prec Q$ and $Q \prec P$, then we say P and Q are *equivalent*, denoted $P \sim Q$.

Statement of Results Section 1 contains the density result for $P = (1-x)^{-1}$, $P = -\ln(1-x)/x$, and their relatives; explicitly for these relatively slow-growing Maclaurin series, the set of point evaluations from $[0, 1)$ is dense in the pure trace space of R_P. Section 2 contains results relating to the pure trace spaces of R_P and R_Q, when for example, $P \sim Q$—in this case, if the density results apply to R_P and R_Q, then there is a canonical homeomorphism between the pure trace spaces, even though there need not be any relevant algebra homomorphisms between R_P and R_Q.

Section 3 analyzes some interesting aspects of R_P when $P = (1-x)^{-1}$; *e.g.*, an example is presented wherein $Q \sim P = (1-x)^{-1}$ but R_Q^+ is strictly contained in R_P^+. This has an interpretation in terms of the eventual positivity problem. A class of results, related to Tauberian theorems, is obtained for the problem, if Qf has no negative coefficients, then $P^k f$ does for some k (but all f with radius of convergence at least one). This boils down to eventual positivity of $P^k Q^{-1}$ (and a necessary condition is that Q have no zeroes in the open unit disk, hence the radius of convergence of Q^{-1} is at least one). This applies if, *e.g.*, the coefficients of Q are monotonic non-increasing, or if they are absolutely summable and Q has no zeroes on the closed unit disk; there are other results of this type, emanating from Tauberian or other summability theorems (Cæsarian?). Section 4 discusses perturbation-type results, of the form, if P' is "close" to P, and $P'f$ has no negative coefficients, does there exists an n such that $P^n f$ has no negative coefficients?

Section 5 deals with relatively fast-growing Maclaurin coefficients, typically (P, x^n) growing in n faster than any polynomial (but slower than exponential, to ensure the radius of convergence is at least one). When $n \mapsto (P, x^n)$ is a log concave function and has a long distance approximation property (that holds for reasonable choices of Q), called FLRA, then we can obtain sharp asymptotic estimates for (P^k, x^N) ($N \to \infty$, k fixed), and for $P(r)$ ($r \uparrow \rho$ where ρ is the radius of convergence of P). An unexpected consequence is an asymptotic estimate for a horrible sum coming out of attempting to determine the Maclaurin coefficients of $\exp\big(1/(1-x)\big)$. Section 6 uses the asymptotic formulæ of Section 5 in order to obtain density results on the corresponding R_P. In this case, the methods are far more complicated than their counterparts in Section 1 (although based on the same set of ideas). Section 7 contains a set of relatively easy density results when the growth of the coefficients is particularly "spiky" (this applies to a class of entire functions). Some examples are computed in Section 8.

There are brief comments on how to extend (some) of these results to Laurent power series and several variables in Section 9.

Appendix A solves two specific problems: (a) determines for which complex numbers z does $(1-x)^{-1}\big((x-z)(x-\overline{z})\big)^{-1}$ have no negative coefficients, and (b) shows that if $p(x)$ is a polynomial of degree d with no roots in the open unit disk, then $(1-x)^{-d}p^{-1}$ has no negative Maclaurin coefficients. Appendix B discusses some consequences of the presence of anomalous homomorphisms $R_P \to \mathbf{C}$; in particular, their existence often prohibits order unit cancellation from holding, as well as other properties. An invariant, $\Psi(P)$, formalizes this.

1 Slow-Growing/Decaying Coefficients

Some basic relevant results for ordered rings and pure traces can be found in the initial sections of [H3] (where *traces* are called *states*). For more specific results and definitions related to rings of the form R_P where P has a Maclaurin series (with nonzero radius of convergence), see the initial sections of [H1].

Recall from *e.g.*, [H3], that a *(normalized) trace* (older notation: *state*) on a unital ordered ring R is an additive homomorphism $\tau\colon R \to \mathbf{R}$ such that $\tau(R^+) \subseteq \mathbf{R}^+$ and $\tau(R^+) \neq 0$ (*i.e.*, τ is a positive unital additive homomorphism; if R is also a real ordered vector space, then τ is automatically linear). A trace is *pure* (or *extremal*) if it cannot be represented as a non-trivial convex-linear combination of other traces—a standard result [H4, Theorem 1.1] asserts that a trace is pure if and only if it is multiplicative (when 1 is an order unit of R). The set of pure traces, $\partial_e \mathrm{T}(R, 1)$, known as the *pure trace space,* is thus a compact set (with the topology of weak convergence, *i.e.*, the point-open topology)—(compactness emanates from the ordered ring structure on $R, 1$).

When $R = R_P$, the pure traces are multiplicative. A *point evaluation trace* on R_P is a function of the form τ_r for r with $0 \leq r \leq \rho$, where $\tau_r(a) = a(r)$. If $0 \leq r < \rho$, then τ_r is defined and a pure trace (τ_ρ is defined only if $P(\rho)$ is finite, something that is discussed in detail in [H1], but is largely irrelevant here). With the weak topology, the set of point evaluation traces (excluding the problematic τ_ρ) is naturally homeomorphic to the interval $[0, \rho)$ with its usual topology. Density of the set of point evaluation traces in the pure trace space is a natural question. In this section, we show that for some choices of P, density holds. This includes the most important case, that $P = (1-x)^{-1}$.

In our context, an *order unit* of R_P is an element of R_P of the form f/P^k for which there exist a positive integer m and a positive real number δ such that for all l, $(fP^m, x^l) \geq \delta(P^{k+m}, x^l)$. A more general definition (consistent with this one) will be given later. The set of order units in R_P is denoted R_P^{++}.

The first lemma requires a notion that will not be an issue in the rest of the paper. For P a Maclaurin series with no negative coefficients, let $\operatorname{supp} P = \{k \mid (P, x^k) \neq 0\}$. If $P(0) \neq 0$, then $\operatorname{supp} P \subseteq \operatorname{supp} P^2 \subseteq \operatorname{supp} P^3 \subseteq \cdots$, but this sequence may be strictly increasing. For example, if $P = 1 + \sum_{n=0}^{\infty} x^{2^n}$, then $2^m - 1$ belongs to $\operatorname{supp} P^m$, but not to $\operatorname{supp} P^{m-1}$. In most of what follows however, the P to which the lemma will be applied already satisfy $\operatorname{supp} P = \mathbf{N}$. Even some (admittedly, modestly) lacunary series can satisfy the $\operatorname{supp} P^s = \operatorname{supp} P^{s+1}$ condition—*e.g.*, if $P = \sum x^{n^2}$, then $\operatorname{supp} P^4 = \mathbf{N}$, as every positive integer is a sum of four squares.

Lemma 1.1 *Suppose that f and P are Maclaurin series with no negative coefficients and nonzero radius of convergence, and there exists s such that* $\operatorname{supp} P^s = \operatorname{supp} P^{s+1}$. *If $f(0) > 0$, $f \prec P^k$ for some k, and* $\liminf_{n\to\infty}(f,x^n)/(P^k,x^n) > 0$, *then f/P^k is an order unit of R_P.*

Proof The first two properties of f yield that f/P^k belongs to R_P^+, and the third says that there exist $\delta > 0$ and an integer N such that whenever $n \geq N$, $(f,x^n) \geq \delta(P^k,x^n)$. If $m \geq 2N$, we calculate

$$\begin{aligned}(fP^k,x^m) &= \sum_{i=0}^{m}(f,x^i)(P^k,x^{m-i})\\ &\geq \sum_{i=N}^{m}(f,x^i)(P^k,x^{m-i})\\ &\geq \delta\sum_{i=N}^{m}(P^k,x^i)(P^k,x^{m-i})\\ &\geq \delta\sum_{m\geq i\geq m/2}(P^k,x^i)(P^k,x^{m-i})\\ &\geq \delta(P^{2k},x^m)/2.\end{aligned}$$

Repeating this process yields that for all integers t, if $m \geq 2^tN$, then

$$(fP^{(2^t-1)k},x^m) \geq \frac{\delta}{2^t}(P^{2^tk},x^m).$$

Select t so that $(2^t - 1)k > s$. Then $f(0) > 0$ and the nonnegativity of the coefficients of f entail that $\operatorname{supp} fP^{(2^t-1)k}$ contains $\operatorname{supp} P^{(2^t-1)k}$, and this equals $\operatorname{supp} P^s$ and thus equals $\operatorname{supp} P^{2^tk}$. Hence if $m < 2^tN$, we have that $(P^{2^tk},x^m) > 0$ implies $(fP^{(2^t-1)k},x^m) > 0$. There thus exists $\eta > 0$ such that for all $m < 2^tN$, $(fP^{(2^t-1)k},x^m) \geq \eta(P^{2^tk},x^m)$. Thus with $\kappa = \min\{\eta,\delta/2^t\}$, we have that $(fP^{(2^t-1)k},x^m) \geq \kappa(P^{2^tk},x^m)$ for all m. Hence $f/P^k = fP^{(2^t-1)k}/P^{2^tk}$ is an order unit of R_P. ■

In the following, the difficult aspects for verification of $(*)$ concern the $\liminf$s (the second one is implicit—the condition on the r_i is simply $\liminf_{r\uparrow\rho}(f/P^k)(r) = 0$), and the fact that it must be verified for all k (sufficient is that it hold for an infinite set of k's).

Lemma 1.2 *Suppose that P is a Maclaurin series with no negative coefficients, the radius of convergence of P, ρ, is nonzero, and* $\operatorname{supp} P^s = \operatorname{supp} P^{s+1}$ *for some s. Sufficient in order that the set of point evaluation traces be dense in the pure trace space of R_P is*

$(*)$ *for all k, if $f \prec P^k$, $(f, x^n) \geq 0$, and* $\liminf_{n\to\infty}(Pf, x^n)/(P^{k+1}, x^n) = 0$, *then there exist $r_i \uparrow \rho$ such that* $\lim_i f(r_i)/P^k(r_i) = 0$.

Proof Suppose density fails; by [GH, Theorem 4.1], there exists a in R_P^+ such that for some $\delta > 0$, $a(t) > \delta$ for all t in $[0, \rho)$, yet a is not an order unit. Without loss of generality, we may write $a = f/P^k$ where $f \prec P^k$, and f has no negative coefficients. We may write $a = fP/P^{k+1}$, and apply Lemma 1.1; hence $\liminf_{n\to\infty}(Pf, x^n)/(P^{k+1}, x^n) = 0$. By $(*)$, there exist r_i in $(0, \rho)$ with $a(r_i) \to 0$. This contradicts the initial assumption on a. ■

Density of the set of point evaluation traces in the pure trace space is characterized by the following condition, weaker than $(*)$. (Section 7 contains examples that fail to satisfy $(*)$, but for which density occurs.)

$(*/\infty)$ For all k, if $f \prec P^k$, $(f, x^n) \geq 0$ for all n, and for all s,

$$\liminf_N (fP^s, x^N)/(P^{k+s}, x^N) = 0,$$

then $\liminf_{r\uparrow\rho} f(r)/P^k(r) = 0$.

So $(*)$ is this condition restricted to $s = 1$. The spiky distributions of Section 7 provide examples wherein the condition with $s = 3k$ applies, but for which $(*)$ fails.

Now we proceed to verify $(*)$ for selected power series. The most important case is that of $P = (1 - x)^{-1} = \sum x^j$. The general pattern for the proof is as follows. For $\epsilon > 0$, there are infinitely many integers N such that $(Pf, x^N)/(P^{k+1}, x^N) < \epsilon$; a "sufficiently large" choice is made (depending on the choice of P and usually a consequence of a minor point in a future part of the proof). Now decompose $f = f_0 + f_N$ into its truncation into the polynomial of degree N (f_0), and the remainder (f_N). We then obtain separate estimates for $f_0(r)/P^k(r)$ and $f_N(r)/P^k(r)$ for suitable choices of real r; these estimates are functions of ϵ (which should go to zero as ϵ does), and the values of r for which both estimates hold should lie in a non-empty intersection of two open subsets. In the case of $P = (1 - x)^{-1}$, we can rely on very explicit and simple formulæ for $(P^k, x^N) = \binom{k+N-1}{k-1}$, but for other Maclaurin series, the situation is more complicated. Difficulties arise from generic non-convergence (*i.e.*, oscillation) of $f/P^k(r)$ as $r \to 1$.

We begin with a very general function $J(r) = \sum h(t)r^t$ for r in the open unit interval and h the restriction to the positive integers of a reasonably smooth function, and constrain h as needed. We wish to determine a relatively explicit form for t_0 (depending on r and h) such that the sequence of terms $\{h(t)r^t\}$ is essentially decreasing beyond t_0—*i.e.*, there exists a positive real δ such that for all integers $t_2 > t_1 > t_0$, $h(t_1)r^{t_1} > \delta h(t_2)r^{t_2}$ ($\delta = 1$ just means that the sequence is decreasing beyond t_0). For example, if h is twice differentiable and $h''h - (h')^2 < 0$ on the positive reals (*i.e.*, h is log concave), then t_0 is determined as the (unique) solution to $h'(t)/h(t) = -\log r$.

For reasonable choices of P, this is easy to estimate. For example, if $P = (1 - x)^{-(k+1)}$, then $(P, x^i) = \binom{i+k}{k} \sim i^k/k!$; setting $h(t) = t^k/k!$, we arrive at

$t_0 = (-\log r/k)^{1/(k-1)}$, which is slightly less than $(-\log r)^{1/(k-1)}$. If

$$P = \left(-\log(1-x)/x\right)^{k+1} = \left(\sum x^i/(i+1)\right)^{k+1},$$

then $(P, x^i) \sim (\log i)^k/(i+1)$, and it is easy to see that $t_0 < e^k$ (regardless of the choice of r). For very rapidly growing choices for h, *e.g.*, $h(t) = e^{t^\beta}$, other techniques are developed in later sections.

Now suppose $0 \leq f \leq P^k$ coefficientwise. Set $\epsilon = 1/n$ for an integer n, and pick $N \equiv N(n) > N(n-1)$ such that $(Pf, x^N)/(P^{k+1}, x^N) < \epsilon$. Since for each ϵ, there exist infinitely many N such that $(Pf, x^N)/(P^{k+1}, x^N) < \epsilon$, we can also insist on some other inequalities.

Let f_0 be the truncation of f up to its x^N term, and $f_N = \sum_{i=1}^{\infty}(f, x^{N+i})x^{N+i}$. Obviously $(f_0 P, x^N) = (fP, x^N)$. Let $M(N) = \inf\{(P, x^i) \mid i \leq N\}$. We deduce

$$\begin{aligned}\epsilon(P^{k+1}, x^N) \geq (f_0 P, x^N) &= \sum_{i \leq N}(P, x^{N-i})(f_0, x^i)\\ &\geq M(N)\sum_{i \leq N}(f_0, x^i) = M(N) f_0(1).\end{aligned}$$

Hence $f_0(1) \leq \epsilon(P^{k+1}, x^N)/M(N)$, so that for any r in the interval $(0, 1)$,

$$\frac{f_0(r)}{P^k(r)} \leq \frac{f_0(1)}{P^k(r)} \leq \frac{\epsilon(P^{k+1}, x^N)}{M(N)P^k(r)}.$$

Now the remainder term, $f_N(r)/P^k(r)$, is bounded. Set $H(r) = \sum_{j=N+1}^{\infty}(P^k, x^j)r^j$. Define

$$r_0 \equiv r_0(N, k) = \inf\left\{\frac{(P^k, x^{N+j})}{(P^k, x^{N+j+1})} \;\middle|\; j \geq 1\right\}.$$

If $P = x^{-1}\ln\big(1/(1-x)\big)$ (or many similar functions), then for all sufficiently large N (depending on k), the sequence $\{(P^k, x^{N+j})\}_{j \geq 1}$ is decreasing, hence for N large enough, $r_0(N, k) = 1$. On the other hand, if $P = (1-x)^{-1}$, then

$$(P^k, x^{N+j})/(P^k, x^{N+j+1}) = (N+j+1)/(N+j+k),$$

and it follows that $r_0(N, k) = 1 - (k-1)/(N+1) \geq 1 - k/N$.

If r is a positive real number with $r < r_0$, then

$$(P^k, x^{N+j+1})r^{N+j+1} \leq (P^k, x^{N+1})r^{N+1} \cdot (r/r_0)^j.$$

Thus

$$\begin{aligned}H(r) &= \sum_{j=0}^{\infty}(P^k, x^{N+j+1})r^{N+j+1}\\ &\leq (P^k, x^{N+1})r^{N+1}\sum_{j=0}^{\infty}\left(\frac{r}{r_0}\right)^j = \frac{(P^k, x^{N+1})r^{N+1}r_0}{r_0 - r}.\end{aligned}$$

Now we impose the following condition on P:

(%) For all positive integers k, there exists $K \equiv K(k) > 0$ such that for all N,

$$(P^k, x^N) \leq \frac{KP^{k-1}(1-1/N)P'(1-1/N)}{N^2}.$$

For functions such as $P = (1-x)^{-1}, \ln\bigl(1/(1-x)\bigr)/x$ and many others (including products of them), not only is (%) satisfied, but the inequality can be replaced by asymptotic equality, for suitable values of $K(k)$.

Assuming that P satisfies (%) and $r < r_0$, we obtain

$$\frac{f_N(r)}{P^k(r)} \leq \frac{H(r)}{P^k(r)} \leq \frac{KP^{k-1}(1-1/N)P'(1-1/N)}{N^2 P^k(r)} \cdot \frac{r^{N+1} r_0}{r_0 - r}.$$

(Some of the $(N+1)$ terms have been replaced by N on the right side; the resulting factor can be incorporated into the constant K.) Now set $\gamma = 1/\bigl(2\ln(1/\epsilon)\bigr)$ and $r = 1 - 1/N\gamma$. Assuming $r < r_0$, we obtain

$$\begin{aligned} \frac{f_N(r)}{P^k(r)} &\leq \frac{KP^{k-1}(1-1/N)P'(1-1/N)}{N^2 P^k(1-1/N\gamma)} \cdot (1-1/N\gamma)^{N+1} \cdot \frac{r_0}{r_0 - r} \\ &\leq K' \left(\frac{P(1-1/N)}{P(1-1/N\gamma)}\right)^k e^{-\gamma} \frac{P'(1-1/N)}{N^2 P(1-1/N)} \cdot \frac{r_0}{r_0 - r}. \end{aligned}$$

If $P = (1-x)^{-1}$, then $r_0 \geq 1 - k/N$, so $r = 1 - 1/N\gamma < r_0$ (if N is chosen large enough), and the right side of the last displayed expression is bounded by a multiple of $\epsilon^2 \gamma^{-k+1}$, which of course goes to zero as ϵ does. If

$$P = x^{-1} \ln\bigl(1/(1-x)\bigr),$$

provided N is large enough, $r_0(N,k) = 1$. Maintaining $r = 1 - 1/N\gamma$ and selecting N sufficiently large that $|\ln\gamma| < (\ln N)/2$, the upper bound for $f_N(r)/P^k(r)$ simplifies to a multiple of $\epsilon^2\gamma/N$, which again goes to zero. A similar analysis holds for variations on these, *e.g.*, $\ln\bigl(x^{-1}\ln\bigl(1/(1-x)\bigr)\bigr)$, *etc.*

The upper bound estimate for $f_0(r)/P^k(r)$ simplifies (via (%)) to a multiple of $\epsilon P^k(1-1/N)P'(1-1/N)/N^2 M(N) P^k(r)$. When $P = (1-x)^{-1}$, the upper bound becomes $\epsilon\gamma^{-k}$, which again goes to zero in ϵ. When $P = x^{-1}\ln\bigl(1/(1-x)\bigr)$ (or its logarithm), we obtain a multiple of ϵ/N (if N is so large that $|\ln\gamma| < (\ln N)/2$).

Hence we have shown the following.

Theorem 1.3 *Each of the following satisfy* $(*)$*:* $(1-x)^{-1}$, $-\ln(1-x)/x$, *and* $\ln\bigl(-\ln(1-x)/x\bigr)/x$.

The following is an application of Theorem 1.3; it will be used in the proofs of Lemma 3.5(C) and Proposition 3.5(D).

Lemma 1.4 *Suppose that* $q = \sum_{n=0}^{\infty} a_n x^n$ *is a power series with radius of convergence at least one. Suppose that*

(a) $q\big|(0,1) > 0$
(b) $\liminf_{t\uparrow 1} q(t) > 0$
(c) *the set of partial sums,* $\{\sum_{n=0}^{N} a_n\}_{N\in\mathbf{N}}$, *is bounded.*

Then there exists a positive integer m such that $(1-x)^{-m}q$ has no negative coefficients in its Maclaurin expansion.

Proof If $q(0) = 0$, then q has a zero of some order, say k; then we may replace q by q/x^k, which satisfies (a)–(c), and in addition its value at zero exceeds zero, by continuity. Hence we may assume at the outset that q is strictly positive on $[0,1)$. It follows from (b) that there exists $\delta > 0$ such that $q\big|[0,1) > \delta$.

Let $P = (1-x)^{-1}$. Condition (c) is a restatement of $Pq \prec P$. Hence q belongs to R_P. The condition $q\big|[0,1) \geq \delta$ means that the value of any point evaluation trace on q is at least δ, so by density of the set of point evaluation traces, $\tau(q) \geq \delta$ for all pure traces τ. By [GH, Theorem 4.1], q belongs to R_P^{++}, which is slightly stronger than the desired conclusion. (In fact, for any $\epsilon < \delta$, $q - (\delta - \epsilon)$ is an order unit, which entails that there exist an integer $M \equiv M(\epsilon)$ such that $P^M q - (\delta-\epsilon)P^M$ has no negative coefficients in its Maclaurin expansion.) ■

2 Trace Spaces

The hypotheses in Sections 1, 5, and 6 on the power series, $P = \sum h(i)x^i$, specifically on h, in order to obtain $(*)$, are rather restrictive, in that they require some smoothness conditions on the coefficients given by h. The next elementary lemma resolves this—if $Q = \sum j(i)x^i$, and P satisfies $(*)$, then sufficient for Q to satisfy $(*)$ is simply that $\{j(i)/h(i)\}$ be bounded above and below (away from zero), with no smoothness conditions whatever on j.

Lemma 2.1 *If $P \sim Q$ and P satisfies $(*)$, then so does Q.*

Proof Let f have no negative coefficients in its Maclaurin expansion and suppose that $f \prec Q^k$; then $f \prec P^k$. As $(Qf, x^n) \geq K(Pf, x^n)$ for some $K > 0$ uniformly in n, and $(Q^{k+1}, x^n) \geq K'(P^{k+1}, x^n)$ for some $K' > 0$, it follows from $(*)$ applying to P that $f(r_i)/P^k(r_i) \to 0$ for some r_i tending to 1 from below. As $P^k(t) \leq K''Q^k(t)$ for some K'' uniform in t from $(0,1)$, we obtain that $f(r_i)/Q^k(r_i) \to 0$, verifying $(*)$. ■

We have much more—the pure trace spaces of R_P and of R_Q are naturally homeomorphic when $P \sim Q$ and they satisfy $(*)$. We recall some definitions.

An *order unit* u in a partially ordered abelian group A is an element of the positive cone of A such that for all a in A, there exists a positive integer N with $-Nu \leq a \leq Nu$. Let A, u be a partially ordered (unperforated) abelian group with order unit u. We denote by A^{++} its collection of order units, and by $T(A, u)$ (or simply $T(A)$ when u is understood), the set of normalized traces on A, u, equipped with the usual weak topology. Its extremal boundary (consisting of the pure traces) is denoted $\partial_e \mathrm{T}(A, u)$ or $\partial_e \mathrm{T}(A)$.

Let us say that for P and Q Maclaurin series with radius of convergence at least one, P *positively divides* Q if there exists a Maclaurin series R with no negative coefficients and radius of convergence at least as large as the minimum of those of P and Q, such that $PR = Q$. We say that P *eventually positively divides* Q if there exists a positive integer m such that P positively divides Q^m.

If P eventually positively divides Q, then there exists an embedding $R_P \subset R_Q$ with the additional property that $R_P^+ \subset R_Q^+$: If f/P^k in R_P, then there exists a positive integer l such that $fP^l \prec P^{l+k}$ (from the definition of R_P); by replacing f by fP^l and P^k by P^{l+k}, we may assume that $f \prec P^k$. Thus $fR^k \prec Q^{km}$ if $PR = Q^m$ and R has no negative coefficients, so that $f/P^k = fR^k/Q^{km}$ expresses the original element as a member of R_Q. If f/P^k lay in R_P^+, then without loss of generality, we could also have assumed that f has no negative coefficients (since there exists $l' \geq l$ such that $fP^{l'}$ has no negative coefficients and $fP^{l'} \prec P^{l'+k}$), so that neither does the numerator, fR^k, in the alternative expression—so f/P^k also lies in R_Q^+.

It is tempting to ask, if P merely divides a power of Q (*i.e.*, drop the positivity of the coefficients of the quotient), whether there is an embedding of algebras $R_P \subset R_Q$, not necessarily order-preserving. In fact, this already fails when P and Q are polynomials in one variable, even if their sets of exponents generate the positive integers as a semigroup. For example, set $Q = (1 - x + x^2 + x^3)(1 + x) = 1 + 2x^3 + x^4$, and $P = (1 + x)Q$. Then P divides Q^2, but there is no natural identification of R_P with a subring of R_Q. This can be made into an example with Maclaurin series having radii of convergence one.

The following is a trivial modification of [H2, Lemma 1.4, p. 78].

Lemma 2.2 *Let $\phi\colon A \to B$ be a group homomorphism of partially ordered abelian groups (not assuming $\phi(A^+) \subseteq B^+$), and suppose that $u \in A$ is an order unit for A and $\phi(u)$ is an order unit for B.*

(a) *If $A^{++} = \phi^{-1}(B^{++})$, then the restriction map $T\big(B, \phi(u)\big) \to T(A, u)$ given by $\tau \mapsto \tau \circ \phi$ is onto; in particular, every pure trace of A can be "extended" to a pure trace of B.*

(b) *If $\phi(A^+) \subset B^+$ and every (pure) trace of A extends to one of B, then $A^{++} = \phi^{-1}(B^{++})$.*

Proof (a) Define two new partially ordered abelian groups A_1 and B_1 by setting $A_1 = A$ and $A_1^+ = A^{++} \cup \{0\}$, and $B_1 = B$ and $B_1^+ = B^{++} \cup \{0\}$. It is immediate that A_1 and B_1 are both unperforated partially ordered abelian groups with u as order unit (and everything that is in the positive cone but not zero is now an order unit), and moreover $A_1^{++} = A_1^+ \setminus \{0\} \to B_1^+ \setminus \{0\} = B_1^{++}$. The result for traces on $A_1 \subseteq B_1$ now follows from [H2, Lemma 1.4, p. 78].

Now we show that the traces on (A_1, u) are exactly the traces on (A, u) (and similarly for the traces of the B's). If τ is a trace of (A, u), then $\tau(A^{++}) \subseteq \mathbf{R}^{++}$, so $\tau(A_1^+) \subseteq \mathbf{R}^+$, and thus is a trace of (A_1, u). Conversely, suppose that τ is a trace of (A_1, u) and a belongs to $A_l^+ \setminus \{0\}$. We may assume that u/n belongs to A (and thus to A^{++}) for all positive integers n (tensor with the rationals if necessary). Then for all positive integers n, $a + u/n$ lies in A^{++}, so $\tau(a + u/n) \geq 0$ for all positive n. Hence $\tau(a) \geq -1/n$ for all positive n, and thus $\tau(a) \geq 0$. Hence τ is a trace of (A, u).

(b) is already in [*op. cit.*]. ■

If we increase the structure imposed on our current A by permitting it to be an ordered commutative algebra with 1 as order unit, then $\partial_e \mathrm{T}(A)$ is now compact, and consists of the multiplicative traces on $A, 1$.

Lemma 2.3 *Let P and Q be Maclaurin series with no negative coefficients and the same radius of convergence ρ (which could be infinite, but must be nonzero). Suppose that $R_P \subset R_Q$ (as algebras, but not assuming $R_P^+ \subseteq R_Q^+$).*

(a) *If the set of point evaluations (from the interval $[0, \rho)$) is dense in the pure trace space of R_P, then every pure trace of R_P lifts to a pure trace of R_Q.*

(b) *If the set of point evaluations (from the interval $[0, \rho)$) is dense in the pure trace space of R_Q, then every trace of R_Q restricts to a trace on R_P, and pure traces restrict to pure traces.*

[Remark: Since we are not assuming that $R_P^+ \subset R_Q^+$, there is no reason that a trace of R_Q should restrict to a trace on R_P, so that (b) is not entirely tautological, although it is trivial. This remark also explains the rather awkward use of $M(B)$ in the proof of (a)—if $R_P^+ \subset R_Q^+$, the proof is completely elementary.]

Proof Let $A = R_P$ and $B = R_Q$. Let $M(B)$ be the closure of the collection of point evaluation traces on B within the family of all normalized traces of B. Then $M(B)$ is compact. There is a dense subset of $M(B)$ (namely the collection of point evaluation traces) that restrict to point evaluation traces of A, and it is clear that a limit point of point evaluation traces of B will restrict to a limit point of point evaluation traces (now with respect to weak convergence on A) of A. In particular, $M(B) \to T(A)$ is well-defined and continuous; moreover, its image is a compact subset containing the point evaluations as a dense subset. Since the pure trace space of A is compact, the image of $M(B)$ is exactly $\partial_e \mathrm{T}(A)$. This proves (a).

(b) Point evaluation traces restrict to point evaluation traces (since the radii of convergence are the same—all we really need is that $\rho(Q) \leq \rho(P)$), and the rest follows from the usual, and here easy, boundary theory. ■

At this point, we have derived sufficient conditions for $R_P \subseteq R_Q$ to induce an onto map $\partial_e \mathrm{T}(R_Q) \to \partial_e \mathrm{T}(R_P)$—for example, if P eventually positively divides a power of Q (so that $R_P^+ \subset R_Q^+$) and P satisfies $(*)$ (so that the set of point evaluations of R_P is dense in the latter's pure trace space).

However, life isn't that simple. Even if $P \sim Q$, there need not be an inclusion of rings or even a useful ring homomorphism $R_P \to R_Q$ or vice versa. However, we can form the new ring R_{PQ}, and now we have the maps induced by restriction, $\partial_e \mathrm{T}(R_{PQ}) \to \partial_e \mathrm{T}(R_P)$ and $\partial_e \mathrm{T}(R_{PQ}) \to \partial_e \mathrm{T}(R_Q)$ (and both of these are continuous and well-defined, as $R_P^+, R_Q^+ \subseteq R_{PQ}^+$). We obtain a homeomorphism $\partial_e \mathrm{T}(R_P) \to \partial_e \mathrm{T}(R_Q)$ whenever we can show both $\partial_e \mathrm{T}(R_{PQ}) \to \partial_e \mathrm{T}(R_P)$ and

$\partial_e \mathrm{T}(R_{PQ}) \to \partial_e \mathrm{T}(R_Q)$ are homeomorphisms. They are at least onto if both R_P and R_Q have the property that the point evaluations are dense in the multiplicative trace spaces (*e.g.*, if P satisfies $(*)$ and $P \sim Q$; or if both P and Q otherwise satisfy $(*)$). Criteria for the maps to be one to one are somewhat more algebraic.

Suppose that $A \subset B$ are partially ordered rings with common 1 as order unit, and there exists u in A such that $B = A[u^{-1}]$, *i.e.*, u is invertible in B, and everything in B can be expressed in the form $b = au^{-k}$ for some integer k. Then any ring homomorphism $B \to \mathbf{R}$ (or to any other ring) is determined by its effect on A—in other words, if the map $\partial_e \mathrm{T}(B) \to \partial_e \mathrm{T}(A)$ is well-defined, then it is one to one.

The simplest situation under which this occurs is when $P \sim Q$, $A = R_P$ (or R_Q) and $B = R_{PQ}$. Set $u = Q/P$ (if $A = R_P$); then u belongs to A, $u^{-1} = P/Q$ belongs to R_Q and thus to $B = R_{PQ}$, and we note that if b is an element of R_{PQ}, we may write $b = f/(PQ)^m$ for some integer $m > 0$ and $f \prec (PQ)^m$. Since $(PQ)^m \sim P^{2m}$, we see that $a = f/P^{2m}$ belongs to R_P and $b = au^{-m}$. This can be extended somewhat as follows.

Suppose that P and Q are, as usual, Maclaurin series with no negative coefficients. Now suppose that P and Q are related as follows: there exist $P_0 \sim P$ and $Q_0 \sim Q$ such that P_0 eventually positively divides Q and Q_0 eventually positively divides P. This is obviously a generalization of $P \sim Q$, it is easy to see that the preceding analysis applies to $R_P, R_Q \subset R_{PP_0QQ_0}$, and the latter is just $R_{PQ}[u^{-1}]$ where u is an order unit in R_{PQ}.

Under these fairly weak conditions, we obtain that the inclusions $R_P \subseteq R_{PQ}$ and $R_Q \subset R_{PQ}$ induce homeomorphisms $\partial_e \mathrm{T}(R_{PQ}) \to \partial_e \mathrm{T}(R_P), \partial_e \mathrm{T}(R_Q)$. Composing one with the inverse of the other, we obtain a homeomorphism $\partial_e \mathrm{T}(R_P) \to \partial_e \mathrm{T}(R_Q)$. Moreover, this is not just any homeomorphism, but one which sends point evaluations to point evaluations (fixing the same point)—the density hypotheses yields that this is the *only* homeomorphism with this property. We have the following.

Theorem 2.4 *Let P and Q be convergent Maclaurin series with no negative coefficients, and suppose that there exist $P_0 \sim P$ and $Q_0 \sim Q$ such that P_0 positively divides a power of Q, and Q_0 positively divides a power of P. Suppose in addition that both R_P and R_Q have the property that the set of point evaluation traces is dense in the pure trace space. Then there exists a unique homeomorphism $\partial_e \mathrm{T}(R_P) \to \partial_e \mathrm{T}(R_Q)$ which sends point evaluation traces to point evaluation traces (at the same point).*

In particular, if $P \sim Q$ and either one satisfies $(*)$, then the pure trace spaces of R_P and of R_Q are canonically homeomorphic in the sense described in the statement of the theorem.

3 $(1-x)^{-1}$

If P and Q are convergent Maclaurin series and P has no negative coefficients, we can ask (without expecting much of an answer except under further hypotheses) if whenever f is a convergent Maclaurin series such that $Q^n f$ has no negative coefficients, there exists m such that $P^m f$ has no negative coefficients. Typically, but not necessarily, Q will also have no negative coefficients. For example, if $P \sim Q$ (both having no negative coefficients), does this relation hold? Of course, we can rephrase it merely with $n = 1$.

This is motivated by some results dealing with polynomials, even in several variables. If $P = \sum \lambda(w)x^w$ is a polynomial in d variables ($x^w = x_1^{w(1)}x_2^{w(2)}\cdots$, the usual monomial notation) and the coefficients $\lambda(w)$ are all nonnegative, denote by $\operatorname{Log} P$ the set $\{w \in \mathbf{Z}^d \mid \lambda(w) \neq 0\}$. It was shown in [H3] that if $\operatorname{Log} P = \operatorname{Log} Q$ (and all the coefficients of Q are nonnegative), then for polynomials f, $Q^n f$ having no negative coefficients implies there exists m such that $P^m f$ has no negative coefficients. In fact, somewhat stronger asymmetric results are available, *e.g.*, [H3, Theorem II.1, p. 15]. The condition on polynomials, $\operatorname{Log} P = \operatorname{Log} Q$, is a restatement of $P \sim Q$ in this context. This motivates the original question. However, a more powerful motivation is simply the fact that it may be easier to determine the fs that become eventually positive under one power series than under another.

Alas, the conjecture fails, although with appropriate additional hypotheses, it can be somewhat resuscitated. First we give some results in the positive direction.

For the following, Q will denote a real Maclaurin series with nonzero radius of convergence and $Q(0) \neq 0$. Then Q^{-1} will admit a Maclaurin series of some positive radius of convergence, and the obstructions to it being at least as large as that of Q are precisely the zeroes of Q. For some problems, the zeroes do not present a real problem. In the following, we do not require Q itself to have no negative coefficients.

Lemma 3.1 *Suppose that P is a real Maclaurin series with no negative coefficients and positive radius of convergence. Suppose that there exists N such that all coefficients of $P^N Q^{-1}$ are nonnegative. If f is a real Maclaurin series with positive radius of convergence and $Q^n f$ has no negative coefficients for some n, then there exists m such that $P^m f$ has no negative coefficients.*

Proof Consider $P^{Nn} f = (P^N Q^{-1})^n (Q^n f)$ as a product of Maclaurin series without negative coefficients. ∎

This begs the study of these quotients $P^N Q^{-1}$, which we will consider after the following example. We observe that if $P = (1 - x)^{-1}$, then $fP^{-1} = (1 - x)f$ has no negative coefficients if and only if the coefficients of f are nonnegative and nondecreasing.

We say that the ordered ring R satisfies *order unit cancellation* (*e.g.*, [H1, p. 337] and the references cited there) if whenever u is an order unit of R and a is an element of R such that ua lies in R^+, then a belongs to R^+. We do not know whether R_P satisfies order unit cancellation if $P = (1 - x)^{-1}$, but the following example shows that order unit cancellation fails for general R_P (see also the discussion in Appendix B).

Example 3.2 A Q with no negative coefficients such that $Q \sim P = (1-x)^{-1}$ but no power of Q has increasing coefficients, *i.e.*, for all n, not all coefficients of $(1 - x)Q^n$ are nonnegative:

Set $Q = (2 + x)(1 - x^2)^{-1}$; the coefficients are alternately 2 and 1; *i.e.*, $Q = 2 + x + 2x^2 + x^3 + 2x^4 + \cdots$. It can be proved by direct computation that for any n and all sufficiently large t, $(Q^n, x^{2t}) > (Q^n, x^{2t+1})$. It follows that $(1-x)Q^n$ has alternating signs after discarding an initial segment. The computation-free arguments in Proposition B.2(d) and Corollary B.6 yield the weaker but sufficient result that $(1-x)Q^n$ has negative coefficients for every n.

In this example, $PQ^{-1} = (1+x)/(2+x)$, and it is easy to check that for some n, P^nQ^{-1} has no negative coefficients (because $(2+x)^{-1}$ belongs to R_P^{++} by [H1, Lemma 1(e), p. 318]). A particular consequence is that Q eventually positively divides P, so that $R_Q^+ \subset R_P^+$. This yields $R_Q \subset R_P$.

On the other hand, for every n, $(1-x)Q^n$ has at least one negative coefficient. Now we can show that R_Q is a proper subring of R_P. Observe that $QP^{-1} = (1-x)Q$ belongs to R_P and is an order unit thereof. Since $Q \sim P$ and both satisfy $(*)$, the point evaluation traces are dense in the pure traces spaces of both R_P and R_Q. As an order unit of R_P, $f := QP^{-1}$ satisfies $\inf_{t\in[0,1)} f(t) = \delta > 0$. If f were also in R_Q, from the point evaluation traces being dense we would deduce that f is an order unit of R_Q—however, we know that f is not even in the positive cone of R_Q, since $(1-x)Q^n$ has at least one negative coefficient for each n. This contradiction means that f is in $R_P^{++} \setminus R_Q$.

A consequence of this and a later result (without any further computation) is the following. For $s > 0$, for the power series $Q_s := (1-x)^{-1} + s(1-x^2)^{-1}$ (for the example above, set $s = 1$), no power has increasing coefficients. Order unit cancellation fails in this example; set $u = PQ^{-1} \in R_Q^{++}$, and $a = 1-x \in R_Q \setminus R_Q^+$.

The following is a more general statement. The symmetric version is that $R_P = R_Q$ implies $R_P^+ = R_Q^+$ in the presence of $(*)$.

Lemma 3.3 *If $P \sim Q$ and both satisfy $(*)$, then $R_P \subseteq R_Q$ implies $R_P^+ \subseteq R_Q^+$.*

Proof From $P \sim Q$, it follows that QP^{-1} is in R_P and is an order unit thereof. In particular, QP^{-1} belongs to R_Q, and moreover is bounded below away from zero on the point evaluation traces of R_P, which are precisely the same as those of R_Q. As the point evaluation traces of R_Q are dense in the latter's pure trace space, then QP^{-1} is an order unit of R_Q. Hence there exists an integer n such that Q^nP^{-1} has no negative coefficients. Lemma 3.1 now yields $R_P^+ \subseteq R_Q^+$. ■

Without growth conditions on the target functions, simple examples are available. For example, the linear functions $1+x$ and $2+x$ render different power series eventually positive although when the targets are restricted to polynomials, the sets rendered positive are the same—*i.e.*, even though

$$\{f \in \mathbf{R}[x] \mid \text{there exists } n \text{ such that } (x+1)^n f \text{ has no negative coefficients}\}$$
$$= \{f \in \mathbf{R}[x] \mid \text{there exists } m \text{ such that } (x+2)^m f \text{ has no negative coefficients}\},$$

there exist convergent power series f and g such that $(x+1)f$ and $(x+2)g$ have no negative coefficients, but for no n do either of $(x+1)^n g$ or $(x+2)^n f$ have no negative coefficients.

Lemma 3.4 *Suppose that p is a real power series with radius of convergence ρ in $(0, \infty]$, and let $0 < \alpha$ be such that $1/\alpha < \rho$ and $p(-1/\alpha) \neq 0$. Then the coefficients of $(1+\alpha x)^{-1} p$ have alternating signs, after deleting an initial set.*

Proof Let p_k be the truncation to x^k of the Maclaurin series for p. Expand $(1+\alpha x)^{-1} p$ in its Maclaurin series (radius of convergence is $1/\alpha$); we obtain the kth coefficient is given by

$$\sum_{j \leq k} (p, x^j)(-\alpha)^{k-j} = (-\alpha)^k p_k(-\alpha^{-1}).$$

Since $\{p_k(-1/\alpha)\}$ converges to $p(-1/\alpha)$ and the latter is nonzero, for all sufficiently large k, $\operatorname{sgn} p_k(-\alpha^{-1}) = \operatorname{sgn} p(-\alpha^{-1})$. Hence for k sufficiently large, the sign of the kth coefficient is $(-1)^k \operatorname{sgn} p(-\alpha^{-1})$. ■

Applying this to $p = (1+x)^n$, we see that provided $\alpha \neq 1$, $(1+x)^n(1+\alpha x)^{-1}$ never has only positive coefficients. Hence we can set $f = (1+x)^{-1}$ and $g = (2+x)^{-1}$. (Obviously $(1+x)f$ and $(2+x)g$ have no negative coefficients!)

We now present four results (Lemmas 3.5 (A)–(C) and Proposition 3.5 (D)) on the nonnegativity of the coefficients of $P^n Q^{-1}$, where $P = (1-x)^{-1}$. The first is little more than a tautology. The second is elementary; the third requires a little complex analysis and the density theorem, and the fourth makes use of a lemma of Wiener.

Lemma 3.5 (A) *Let $P = (1-x)^{-1}$. Suppose that the (real) Maclaurin series Q has no zeroes in the open unit disk, $Q(0) > 0$, and there exists a positive integer K such that $(\ln Q, x^n) \leq K/(n+1)$ for all n. Then $P^K Q^{-1}$ has no negative coefficients.*

Proof Since the open disk is simply connected and Q does not vanish on it, $\ln Q$ is holomorphic thereon. Now note that $\ln P = \sum_{j=1}^{\infty} x^j/j$, so that $K \ln P - \ln Q$ has no negative coefficients. Hence its exponential, $P^K Q^{-1}$, has no negative coefficients. ■

The expansion of $-\ln(1-x) = \sum_{k=1}^{\infty} x^k/k$ has no negative coefficients, so that for positive real α, the function $(1-x)^{-\alpha} = \exp\bigl(-\alpha \ln(1-x)\bigr)$ similarly has no negative coefficients. Generally, if Q is holomorphic in the open unit disk and has no zeroes therein, then $\ln Q$ is defined and holomorphic there; thus so is $Q^\alpha = \exp \alpha \ln Q$ for any real α. If Q is real-valued and positive on $(-1, 1)$, then $\ln Q$ can be chosen to be real-valued thereon, and therefore will have real Maclaurin coefficients, and thus so will Q^α.

Lemma 3.5 (B) *Suppose that $Q = \sum_j a_j x^j$ where $a_j \geq a_{j+1} \geq 0$. For any positive real α, $(1-x)^{-\alpha} Q^{-\alpha}$ has only nonnegative Maclaurin series coefficients. If k is the ceiling function of α, then $(1-x)^{-k} Q^{-\alpha}$ has only nonnegative coefficients. Moreover, $(1-x)^{-1} Q^{-1} \prec (1-x)^{-1}$.*

Proof Without loss of generality, we may assume that $a_0 = 1$. Then $(1-x)Q = 1-f$ where f has only nonnegative coefficients in its Maclaurin expansion, $f(0) = 0$ and $f(1) = 1$. It is immediate that $|f(z)| < 1$ for all z in the open unit disk (in fact, Schwarz's lemma gives $|f(z)| \leq |z|$, but we don't require this). In particular, $(1-x)Q$ has no zeroes in the open unit disk, and thus neither does Q—so $\ln Q$ and Q^α are defined, and have real coefficients. Hence $\sum_1^k f^k/k$ converges uniformly on compact subsets of the open unit disk, and of course this is just $-\ln(1-f)$. Obviously the Maclaurin series expansion has no negative coefficients. For any positive α, the same is true of the expansion of $-\alpha \ln\big((1-x)Q\big) = -\alpha\ln(1-x) - \alpha \ln Q$. Exponentiating this, we obtain $(1-x)^{-\alpha}Q^{-\alpha}$ has no negative coefficients.

As $(1-x)^{-\beta}$ has no negative coefficients for any $\beta > 0$, in particular, if $\beta = k - \alpha$, so does $(1-x)^{-k}Q^{-\alpha} = (1-x)^{-\beta}(1-x)^{-\alpha}Q^{-\alpha}$.

That $(1-x)^{-1}Q^{-1} \prec (1-x)^{-1}$, *i.e.*, that the coefficients of $(1-f)^{-1}$ are bounded above, is a well-known result in renewal theory—for each n, the expression $\sum_{i=0}^{\infty}(f^i, x^n)$ is the probability that a particle, initially at zero will eventually land at n under the IID with distribution $k \mapsto (f, x^k)$. Hence for each n, $\sum_i (f^i, x^n) \leq 1$. ■

This has some interesting consequences. For example, with $P = (1-x)^{-1}$, $Q := -\big(\ln(1-x)\big)/x$ has decreasing but positive Maclaurin coefficients, so that all coefficients of $P_1 := PQ^{-1}$ are positive, and the same is true of $P_\alpha := PQ^{-\alpha}$ for $\alpha \leq 2$ (the coefficients of Q^2 are nonincreasing). It is a routine consequence of the preceding lemma that P_α positively divides P for these α, but what is more interesting is that P positively divides a power of P_α only for $0 < \alpha < 2$ (*i.e.*, not for $\alpha = 2$). In particular, $R^+_{P_\alpha} = R^+_P$ if $0 < \alpha < 2$, but R_{P_2} is strictly contained in R_P. To see the latter, note that $Q^2 = 1 + x + \cdots$, so that $(1-x)Q^2$ has no x coefficient, and it easily follows that neither does its inverse, P_2. It is now easy to see that neither x nor Q^{-2} belongs to R_{P_2}, although both belong to R_P.

Lemma 3.5 **(*C*)** *Suppose that Q is a function analytic on the open disk $|z| < 1 + \epsilon$ for some $\epsilon > 0$, except possibly with poles. Additionally, assume that zero is neither a pole nor a zero of Q, the Maclaurin series coefficients of Q are real, and Q has no zeroes or poles on the open interval $(0, 1)$. Then there exists an integer m such that $(1-x)^{-m}Q^{-1}$ has no negative coefficients if and only if both of the following hold.*

(a) $Q(0) > 0$;
(b) *Q has no zeroes in the open unit disk.*

Proof Set $P := (1-x)^{-m}Q^{-1}$, which we assume has no negative coefficients. If Q has a zero in $|z| < 1$, let r denote the infimum of the moduli of zeroes in the unit disk. Then $r > 0$ (since $Q(0) \neq 0$), and Q^{-1} is analytic with real Maclaurin coefficients on the disk $|z| < r$. Thus P is analytic on this disk, and the radius of convergence of P is r. Since the coefficients of P are nonnegative, either $P(r) < \infty$, so that P is continuous on $|z| \leq r$, or r is a pole of P. The former contradicts the existence of a zero of Q on $|z| = r$, and the latter says $Q(r) = 0$, contradicting one of the assumptions. Hence (b) holds, and (a) is trivial (since $Q(0) \neq 0$).

Now suppose that (a) and (b) hold. Then Q^{-1} has no singularities in $|z| < 1$ and $Q(0) > 0$. Since Q has no poles on $[0, 1)$, Q is continuous thereon, and since it is never zero there, $Q\big|[0, 1) > 0$. Thus $Q^{-1}\big|[0, 1) > 0$. There exists $\epsilon' < \epsilon$ such that no zeroes or poles of Q lie in the annulus $1 < |z| < 1 + \epsilon'$ (since there are only finitely many zeroes and poles in the closed disk $|z| \leq 1 + \epsilon/2$). We may thus write $Q = pq^{-1}Q_0$, where Q_0 has no poles or zeroes in $|z| < 1 + \epsilon'$, and p and q are monic polynomials whose zeroes with multiplicity are respectively the zeroes of Q and the poles of Q (each with multiplicity).

In particular, Q_0^{-1} has radius of convergence exceeding 1, its Maclaurin series coefficients are absolutely summable, and it follows from [H2, Proposition 10, p. 332] or Lemma 1.4 that there exists $m(1)$ such that $(1 - x)^{-m(1)}Q_0^{-1}$ has no negative coefficients. The factors with constant coefficient one of q are either of the form $1 + sx$ (if $-1/s$ is a pole of Q; necessarily $1 \leq s$) or a quadratic polynomial (allowing repetitions). The first type already have no negative coefficients, and the second type do not change sign on $[0, 1)$, hence are positive on that interval, and since we can absorb poles or zeroes at 1 into the power of $1 - x$, the same result [*op. cit.*] applies to both types of factor. Thus there exists $m(2) \geq 0$ such that $(1 - x)^{-m(2)}q$ has no negative Maclaurin series coefficients.

The monic irreducible (real) factors of p are of the form $x \pm 1$ or $1 + x^2 - 2x\cos\theta$ (where $0 < \theta \leq \pi$). Obviously $(1 - x)^{-1}(1 + x)^{-1} = (1 - x^2)^{-1}$ has no negative coefficients. Set $f = 1 + x^2 - 2x\cos\theta$. Then f^{-1} admits the Maclaurin expansion $\sum_{n\geq 0} x^n \sin(n + 1)\theta/\sin\theta$ (this is an easy exercise for second year students; partial fractions with complex coefficients seems easiest, but it can also be done by diagonalizing a size two matrix, or by verifying the obvious difference equation). Then

$$\begin{aligned}\big((1 - x)^{-1}f, x^N\big) &= \frac{1}{\sin\theta}\sum_{j=0}^{N}\sin(j + 1)\theta\\ &= \frac{1}{\sin\theta}\,\mathrm{Im}\Big(\sum_{j=0}^{N}\exp i(j + 1)\theta\Big),\end{aligned}$$

and as is well known, the absolute value of this is bounded as N increases. Hence, $(1 - x)^{-1}f \prec (1 - x)^{-1}$, and thus f is an element of R_P for $P = (1 - x)^{-1}$. We observe that at the point evaluation traces (evaluating at s in $[0, 1)$), $f(s) > 0$, and $\lim_{s\uparrow 1} f(s) = 2(1 - \cos\theta) > 0$. Hence at the point evaluation traces on R_P, γ, $\gamma(f)$ is bounded below away from zero. By Lemma 1.4, there exists $n \equiv n(f)$ such that $(1-x)^{-n}f$ has no negative coefficients. Thus there exists an integer $m(3)$ (obtained by summing all the $n(f)$ arising from the monic quadratic and linear factors, including multiplicities) such that $(1 - x)^{-m(3)}p^{-1}$ has no negative coefficients.

Finally, set $m = m(1) + m(2) + m(3)$. ∎

For more discussion (to put it mildly) concerning $P^n f^{-1}$, see the first appendix.

Proposition 3.5 (D) *Suppose that $Q = \sum_{n\geq 0} a_n x^n$ with $\sum |a_n| < \infty$, and moreover that Q has no zeroes on the closed unit disk. Then $PQ^{-1} \prec P$ where $P = (1 - x)^{-1}$. If $Q\big|[0, 1] > 0$, then there exists n such that P^nQ^{-1} has no negative coefficients.*

https://doi.org/10.4153/CJM-2003-041-7 Published online by Cambridge University Press

Proof Since the coefficients of Q are absolutely summable, Q is continuous on the closed unit disk, and as a function on the circle, its Fourier expansion is $\sum_{n=0}^{\infty} a_n e^{in\theta}$. We may apply Wiener's lemma [R, Theorem 18.21, p. 363], which asserts that the reciprocal function on the unit circle has a Fourier expansion (possibly involving negative n's) with summable coefficients. As $1/Q$ is holomorphic on the open disk and continuous on the circle (no zeroes in the closed unit disk), its Fourier expansion cannot contain any exponentials for negative n's, and as the coefficients are summable, define a holomorphic function on the disk; by uniqueness, the Fourier coefficients must be the Maclaurin coefficients. Hence the Maclaurin coefficients of Q^{-1} are summable, which is stronger than $PQ^{-1} \prec P$. The rest follows from Lemma 1.4, as (a)–(c) are satisfied. ■

Condition (c) of Lemma 1.4 suggests that some Tauberian theorems or their variants might be applied usefully to obtain results along these lines. For example, suppose that $q = \sum a_n x^n$ is the expansion for Q^{-1}. If $a_n = \mathbf{O}(1/n)$ and $q(t)$ is bounded as $t \uparrow 1$, then condition (c) holds [B, Exercise 18.4, p. 151]; this is a variation on Littlewood's Tauberian theorem with a weakened hypothesis and a weakened conclusion. Fatou's theorem [B, p. 155] asserts that if $a_n \to 0$ and q can be continued beyond 1, then $\sum a_n$ converges (more than we need); this type of result would not be useful when the coefficients of $Q = q^{-1}$ are nonnegative, but would be in other cases, such as $Q = \ln(1+x)/x$ (note that PQ has no negative coefficients). The problem in trying to apply this type of theorem to our situation is that we must prove results about the behaviour of the coefficients of Q^{-1} ($\mathbf{O}(1/n)$ or $\mathbf{o}(1)$ respectively).

A fairly nasty example to which Proposition 3.5(D) applies is the lacunary $1 + \sum_{n=0} x^{2^n}/2^{n+1}$ (this function is continuous on the closed unit disk, and has no zeroes there). A related but nastier example is $Q_1 = 1 + x/2 + x^3/4 + \cdots = \sum_{n=0} x^{2^n - 1}/2^n$. The coefficients are summable, but there is a zero at -1 (and nowhere else in the closed unit disk). It would be sufficient to prove Lemma 1.4(c). From the lacunarity, Q_1 and therefore Q_1^{-1} cannot be continued beyond 1, so even if the coefficients of Q_1^{-1} went to zero—which I wasn't able to prove—Fatou's theorem could not be applied, and application of Littlewood's weakened theorem requires $|(Q_1^{-1}, x^n)| = \mathbf{O}(1/n)$, which appears unlikely and even if true, appears to be very difficult. Fortunately, a few other tricks are available, as in the following amusing result. To apply it to this example, set $Q = PQ_1/2$.

Lemma 3.6 *Suppose $P = (1-x)^{-1}$, and Q has Maclaurin series $\sum a_n x^n$ with $0 < a_0 \le a_1 \le a_2 \le \cdots$ and $a_n \uparrow 1$. If $a_0 \ge 1/2$, then there exists m such that $P^m Q^{-1}$ has no negative coefficients.*

Proof Set $R = (1-x)Q$; as the coefficients of Q are increasing, obviously those of R are nonnegative, and obviously $R = a_0 + \sum_{n=1}^{\infty}(a_n - a_{n-1})x^n$, R has absolutely summable coefficients, and $R(1) = 1$. If R has no zeroes on the closed unit disk, we can obviously apply Proposition 3.5(D). Since $a_0 \ge 1/2$, the only way R can have a zero on the closed unit disk is if $a_0 = 1/2$ and all the even order terms are zero, and in that case the lone zero is at $x = -1$. Hence we may write $S := 2R = 1 + xf$ where $f = \sum c_n x^n$ (of course $c_{2n} = 2(a_{2n+1} - a_{2n})$, but the notation becomes an obstruction), with $c_n \geq 0$ and $\sum c_n = 1$. For a complex number z with $|z| < 1$, $|zf(z)| \leq |z|$, so that the series for S^{-1}, $1 + \sum_{n=1}^{\infty}(-xf)^n$, converges absolutely and uniformly on compact subsets of the open unit disk. We show that the set of partial sums, $\{\sum_{k=0}^{N}(S^{-1}, x^k)\}_N$, is bounded.

For an integer $k > 0$, (S^{-1}, x^k) is $\sum_{j=1}^{k}(-1)^j(f^j, x^{k-j})$. Hence (discarding the $(S^{-1}, x^0) = 1$ term),

$$\begin{aligned}
\sum_{k=1}^{N}(S^{-1}, x^k) &= \sum_{k=1}^{N}\sum_{j=1}^{k}(-1)^j(f^j, x^{k-j}) \\
&= \sum_{j=1}^{N}(-1)^j\sum_{k=j}^{N}(f^j, x^{k-j}) \\
&= \sum_{j=1}^{N}(-1)^j\sum_{k=0}^{N-j}(f^j, x^k).
\end{aligned}$$

Set $A_j = \sum_{k=0}^{N-j}(f^j, x^k)$; we claim that $A_j \geq A_{j+1}$, so that the series is alternating, and thus the partial sums are bounded. That $A_j \geq A_{j+1}$ is obvious, once it is put in the right context. Let $F\colon \mathbf{N} \to \mathbf{R}^+$ be a summable sequence with $\sum F(n) = 1$. Form the convolution powers $F^{(j)} = F * F^{(j-1)}$. Then

$$\begin{aligned}
\sum_{k=0}^{N-j}F^{(j)}(k) &= \sum_{k=0}^{N-j}\sum_{i=0}^{k}F^{(j-1)}(i)F(k-i) \\
&= \sum_{i=0}^{N-j}F^{(j-1)}(i)\sum_{k=i}^{N-j}F(k-i) \\
&\leq \sum_{i=0}^{N-j}F^{(j-1)}(i) \\
&\leq \sum_{i=0}^{N-j+1}F^{(j-1)}(i).
\end{aligned}$$

This applies with $F(n) = (f, x^n)$. Hence $\{\sum_{k=0}^{N}(S^{-1}, x^k)\}_N$ is bounded, so that Lemma 1.4 applies to S^{-1} and thus to R^{-1}. So there exists an integer M such that $P^M R^{-1} = P^{M+1}Q^{-1}$ has no negative coefficients. ∎

In this result, the condition $a_0 \geq 1/2$ cannot be dropped without adding further assumptions, as Q might acquire a zero in the open unit disk. This occurs if $Q = (1/2 - \epsilon) + \sum_{j=1}^{\infty} x^j = (1-x)^{-1} - (1/2 + \epsilon)$.

The condition, $(1-x)^{-1}q \prec (1-x)^{-1}$, is related to Tauberian results, which typically are difficult and restrictive. If instead, we consider the condition $(1-x)^{-2}q \prec$

$(1-x)^{-2}$ (which also yields $q \in R_P$ if $P = (1-x)^{-1}$), then Cesàro summability results are applicable, and these are easier and more general. Throughout D will denote the open unit disk in $\mathbf{C}$, and $\overline{D}$ its closure.

Proposition 3.7 *Let $q \colon D \to \mathbf{C}$ be bounded and holomorphic with real Maclaurin coefficients, and* $\sup\{ |q(z)| \mid z \in D \} := M < \infty$. *Then for all integers m,*

$$\left| \left((1-x)^{-2} q, x^m \right) \right| \leq M \left((1-x)^{-2}, x^m \right).$$

In particular, if $P = (1-x)^{-1}$, then $P^2 q \prec P^2$ and q belongs to R_P.

Proof Write $q = \sum a_j x^j$. Following [T, p. 236], set $s_n(z) = \sum_{j=0}^{n} a_j z^j$, and $\sigma_n(z) = \left(\sum_{k=0}^{n-1} s_k(z) \right)/n$. By *op. cit.*, $|\sigma_n(z)| \leq M$ for all z in D for all n. Fix n; then σ_n is a polynomial, so that $|\sigma_n(1)| \leq M$. However, $\sigma_n(1) = \sum_{k=0}^{n-1} s_k(1)/n$, which of course is just $(P^2 q, x^{n-1})/n$. We conclude with the observation that $(P^2, x^{n-1}) = n$. ∎

Of course, boundedness of q on D is not necessary in order that q belong to R_P— an obvious example is $(1+x)^{-1}$, as has been noted previously. A necessary condition is that q be bounded on the interval $[0,1)$, but this is far from sufficient—*e.g.*, $\exp\left(-(1+x)^{-1}\right)$ provides an immediate if awkward example. A minor modification of the boundedness argument yields a somewhat larger class of Q's for which Q^{-1} is an element of R_P.

Proposition 3.8 *Suppose that Q is holomorphic on D, and there exist real polynomials p and p_1 such that p has no zeroes in D and Qp_1/p is bounded below away from zero on D (i.e.,* $\inf_{z \in D} |Qp_1/p(z)| > 0$*). Then Q^{-1} belongs to R_P and one of $\pm Q^{-1}$ is in R_P^+.*

Proof Form $q = p_1^{-1} p Q^{-1}$; by the preceding, q belongs to R_P. We note that p_1, being a polynomial, belongs to R_P, so that $p_1 q = pQ^{-1}$ does. Now the zeroes of p lie either on the boundary of D or outside it; its irreducible factors (over the reals) are thus of the form $1 \pm x$, $x^2 - 2x\cos\theta + 1$, or $x^2 - 2xr\cos\theta + r^2$ (with $r > 1$). We have already seen, in the proof of Lemma 3.5(C), that the inverses of the first two classes are in R_P, and that of the third class has coefficients with exponential decay, so belongs to R_P; all are in fact positive. As p^{-1} is a product of these inverses, we deduce that p^{-1}, and thus Q^{-1} belongs to R_P. It follows from the hypotheses that Q has no zeroes in D; assume $Q(0) > 0$; then Q^{-1} is positive on $[0,1)$. If $\liminf_{t \uparrow 1} Q^{-1}(t) = 0$, then $\limsup Q(t) = \infty$; this entails that $p_1(1) = 0$. Factor $p_1 = (1-x)^k p_2$ where $p_2(1) \neq 0$, and replace Q by $Q_0 = QP^k$, so that $Q_0 p_2/p$ is bounded below; thus Q_0^{-1} lies in R_P, but this time $\liminf_{t \uparrow 1} Q_0^{-1}(t) \neq 0$, and positivity of Q_0 (thus of Q) follows from Lemma 1.4. ∎

For example, if $Q = \left(\ln(1+x)\right)/x$, then PQ has no negative coefficients (by a trivial calculation), but also, Q^{-1} is in R_P^+ (that is, $P^m Q^{-1}$ has no negative coefficients for some m), since Q is bounded below on D away from zero. None of the earlier results could be made to apply to this case.

It is reasonable to conjecture that if Q has polynomial growth on its coefficients (that is, $(Q, x^n) = \mathbf{O}(n^k)$ for some integer k) and has no zeroes in D, then $P^2 Q^{-1} \prec P^2$, or at least that $Q^{-1} \in R_P$, where $P = (1-x)^{-1}$. This fails (Example 4.8). It might hold if we require the coefficients of Q^{-1} to grow at most polynomially.

4 Perturbations

If Q has a convergent Maclaurin series with no negative coefficients, it is of interest to determine whether $P^n f$ has no negative coefficients (with $P = (1-x)^{-1}$) implies there exists m such that $Q^m f$ has no negative coefficients. By Lemma 3.1, sufficient is that $(1-x)Q^k$ have no negative coefficients, and necessity is obvious. Now $(1-x)Q^k$ has no negative coefficients if and only if the coefficients of Q^k are increasing. We also saw in Example 3.2 that reasonable bounded perturbations of P may fail to have this property. Now we examine which perturbations maintain the property.

Proposition 4.1 *Suppose A and B are Maclaurin series with no negative coefficients, and radius of convergence at least one, and $B \leq A$ (coefficientwise), and in addition $xA \prec A$ and $A \prec A - B$. Finally assume that for some m, both $(1-x)A^m$ and $(1-x)(A+B)^m$ have no negative coefficients. Then there exists n such that $(1-x)(A-B)^n$ has no negative coefficients.*

For the proof, we need several results. A partially ordered abelian group A is *unperforated* if whenever a belongs to A and na belongs to A^+ for some positive integer n, then a belongs to A^+. Ordered vector spaces automatically have this property.

Lemma 4.2 **([H2, Corollary 1.3, p. 77])** *Suppose that R is an unperforated partially ordered commutative ring with 1 as order unit. Suppose that a and $\{u_i\}$ are elements of R such that u_i, $u_i a$ are all in R^+ and the ideal $\sum R u_i$ is improper. Then a belongs to R^+.*

Proof By [*op. cit.*], there exist order units v_i and a positive integer m such that $\sum u_i v_i = m$. Thus $ma = \sum v_i u_i a$ is in the positive cone, so that a is also. ■

Proof of Proposition 4.1 Set $Q = A - B$; then $A \prec A - B$ means that AQ^{-1} belongs to R_Q, and is obviously positive (in fact, an order unit). Similarly, so does $(A+B)Q^{-1}$, as well as BQ^{-1}. Now $x(A-B) \prec xA \prec A \prec A - B$, and thus x itself belongs to R_Q, whence $a := 1 - x$ does as well. Set $u = A/Q$ and $v = (A+B)/Q$. We note that au^m and av^m are both nonnegative. Obviously $2u - v = 1$, whence $uR_Q + vR_Q = R_Q$, and so $u^m R_Q + v^m R_Q = R_Q$. Everything is now in place for Lemma 4.2 to apply, so that $1 - x$ is in R_Q^+. ■

Nonnegativity of both $(1-x)^m(A+B)$ and $(1-x)^m A$ for some m are obviously implied by nonnegativity of $(1-x)^k B$ and $(1-x)^l A$ for some k and l.

Corollary 4.3 *If $\{(P, x^i)\}$ is monotone decreasing and $\lim(P, x^i) > 0$, then there exists n such that the coefficients of P^n are monotone increasing.*

Proof Set $c = \lim(P, x^i)$, and select $d > (P, x^0) - c$; then $P = \sum(c+d)x^j - \left(d - \left((P, x^j) - c\right)\right)x^j$ expresses P as a difference of two monotone increasing series, each of which is subequivalent to P (because $c > 0$), and Proposition 4.1 applies. ■

The ultimate result in this direction obtainable from this is the following.

Let P be a Maclaurin series with radius of convergence one and with no negative coefficients. Define for $i = 0, 1, 2, \ldots$, the sequence $\{c(i)\}$ via $c(0) = 0$, $c(i) = -\min\{(P, x^i) - (P, x^{i-1}), 0\}$ (*i.e.*, $c(i) = 0$ if $(P, x^i) \geq (P, x^{i-1})$, and the negative of the difference otherwise). Now define $C := \sum_i c(i)x^i$; since the coefficients of C are nonnegative, those of $B := (1-x)^{-1}C$ are monotone increasing. Set $A = B + P$, so $P = A - B$. Now $(1-x)P = (1-x)A - C$; on the other hand, if we write $(1-x)P = F - G$ where F and G have no support in common and both have nonnegative coefficients (*i.e.*, breaking it into its positive and negative parts), we see that $G = C$. Hence $(1-x)A = F$, so that $(1-x)A$ has no negative coefficients; obviously the same is true of $(1-x)B$.

In order for Proposition 4.1 to apply, we require that $A \prec P = A - B$; of course this does not always hold. We obviously require that P (or some power) has a lower bound away from zero on its coefficients, and $xP \prec P$ (*i.e.*, $\liminf (P, x^i)/(P, x^{i+1}) > 0$), but more is needed. A sufficient additional condition is $\sum c(i) := c < \infty$. Then we simply note that $B \leq c(1-x)^{-1}$ and $B \prec P$, and the rest is easy to check.

Corollary 4.4 *If some power of Q has increasing coefficients, and $R = Q + D$ where D is absolutely summable and R has a lower bound away from zero on its coefficients, then some power of R has monotone increasing coefficients.*

In contrast, Example 3.2 provides a limitation on results of this type. In fact, with no additional effort, we obtain a continuum of such examples. For $s > 0$, define the power series $Q_s := (1-x)^{-1} + s(1-x^2)^{-1}$ (for the example above, set $s = 1$). For no s does a power of Q have increasing coefficients. First, assume $s < 1$. We observe that $Q_s \equiv (1-x)^{-1}$, and we may write $Q_s = sQ_1 + (1-s)(1-x)^{-1}$; this realizes Q_1 as the difference $Q_s - (1-s)(1-x)^{-1}$, each constituent of which has some power strictly increasing and Q_1, Q_s and $(1-x)^{-1}$ are mutually subequivalent. Hence by Proposition 4.1, this would force some power of Q_0 to have its coefficients increasing, a contradiction. If $s > 1$, write $Q_s = Q_1 + (s-1)(1-x^2)^{-1}$. The argument of the example, Q_1 showed for that all even powers, all sufficiently large coefficients of even terms exceed the coefficient of their immediate successor. Expanding powers of Q_s, we see that these differences are exacerbated (made bigger).

Now we consider perturbation results for more general choices for P. In what follows, if R is a commutative ring and $\{r_i\}$ is a subset of R, then $\langle r_i \rangle$ denotes the ideal generated by R (in case R is an ordered ring, there is another, different notion, namely the order ideal generated by R; the latter is not going to be considered here).

Corollary 4.2 (A) *If R is an unperforated ordered commutative ring with* 1 *as order unit, and s is an element of R such that the ideal of R generated by $\{r \in R^+ \mid rs \in R^+\}$ is improper, then $s \in R^+$.*

Proof Immediate from Lemma 4.2. ■

In fact, Corollary 4.2(A) can be improved; let X denote the pure trace space of R; this is a compact space (with the weak topology). Suppose that X is connected. Let s be an element of R such that $\{x \in X \mid x(s) = 0\}$ is nowhere dense in X. If the ideal generated by $\{r \in R \mid rs \in R^+\}$ is improper, then one of $\pm s$ belongs to R^+. The difference is that the hypotheses here do not require the "r" that multiply s to a positive element to be positive themselves.

Lemma 4.2 (B) *Let $R \subseteq S$ be an inclusion of ordered unital unperforated rings (i.e., $R^+ \subseteq S^+$), with 1 an order unit of R, and R commutative. Denote the multiplicative trace space of R by X, and that of S by Y. Assume X is connected. Suppose that s is an element of S such that*

$$\{x \in X \mid \textit{there exists } y \in Y \textit{ such that } y|R = x \textit{ and } y(s) \neq 0\}$$

is dense in X. If the ideal of R generated by $J := \{r \in R \mid rs \in S^+\}$ is improper, then one of $\pm s$ belongs to S^+.

Proof Set

$$A = \{x \in X \mid \text{there exists } y \in Y \text{ such that } y|R = x \text{ and } y(s) > 0\}$$
$$B = \{x \in X \mid \text{there exists } y \in Y \text{ such that } y|R = x \text{ and } y(s) < 0\}.$$

For r in J and any x in X, y in Y such that $y|R = x$, we have $0 \leq y(rs) = x(r)y(s)$. If $x = x_0$ belongs to A and y_0 is one of its extensions with $y_0(s) > 0$ (we do not hypothesize that *all* extensions of x are positive at s), we infer that $x_0(r) \geq 0$; in particular, if x_0 belongs to the closure of A, then $x_0(r) \geq 0$ for all r in J. Similarly, if x_0 lies in the closure of B, then $x_0(r) \leq 0$ for all r in J.

Therefore, if x_0 belongs to the closure of A and to the closure of B, then $x_0(r) = 0$ for all r in J. This is impossible, as J generates the improper ideal—there exist finite subsets $\{r_i\}$ of J and $\{a_i\}$ of R such that $\sum r_i a_i = 1$. Hence the closure of A has empty intersection with the closure of B. Since $A \cup B$ is dense in X and $\mathrm{cl}(A) \cap \mathrm{cl}(B)$ is empty, we obtain a disconnection of X, which forces one of A or B to be empty.

Without loss of generality, we may assume B is empty, so that A is dense. Thus, for any r in J, $x(r) \geq 0$ for all x in X. As $1 = \sum r_i a_i$, for each x, there exists $i \equiv i(x)$ such that $x(r_i) > 0$. Set $t_0 = \sum r_i$, and for each i, set $t_i = t_0 + r_i$. Each of these is strictly positive (when evaluated at any element of X), hence is an order unit; moreover, each belongs to J, and obviously the ideal generated by $\{t_0\} \cup \{t_i\}$ contains all the r_i, hence is improper. By [H2, Corollary 1.3, p. 77], there exists a positive integer m together with order units $\{v_0\} \cup \{v_i\}$ such that $m = \sum v_j t_j$. Hence $ms = \sum v_j \cdot (t_j s) \in S^+$. As S is unperforated, s belongs to S^+. ■

In our case, $R = S = R_P$; if we assume that point evaluations are dense in the pure trace space, then the connnectedness hypothesis holds, and since all the elements of R_P are at least meromorphic (and analytic in a neighbourhood of the relevant interval of nonnegative reals), nowhere density of the zero set also applies. It can also be applied where R is a proper subset of S. If $R_P \subset S = R_Q$ for some Q with the same radius of convergence as P and R_P satisfies $(*)$ (or more generally, point evaluations are dense), then as point evaluation traces obviously extend to S, the displayed condition on s is satisfied whenever s is not zero.

Lemma 4.5 *Suppose that P and T are convergent Maclaurin series, the former with no negative coefficients, $T(0) > 0$, and $T \prec P^t$ for some integer t. Let f be a convergent Maclaurin series such that $f(0) \neq 0$, $f \prec P^k$ for some integer k, and Tf has no negative coefficients. Let $c = 1/P$ and $q = T/P^t$. Both c and q belong to R_P, and if $\langle c, q\rangle = R_P$, then there exists n such that $P^n f$ has no negative coefficients.*

Proof Set $a = f/P^k$; then obviously each of a, c, and q belong to R_P, and c belongs to R_P^+. Since the constant terms of T and f are both nonzero, so is the constant term of Tf; hence there exists $\epsilon > 0$ such that $Tf \geq \epsilon 1$ (coefficientwise). Thus $P^k T f \geq \epsilon P^k$ (coefficientwise), and as $f \prec P^k$, $P^k \geq -\delta f$ for some positive real δ. Therefore, $TP^k f \geq -\epsilon\delta f$, or in other words, $(TP^k + \delta\epsilon 1)f \geq 0$ (coefficientwise). Dividing by P^{2k+t}, we obtain that $(q + \delta\epsilon c^{t+k})a \in R_P^+$. We also have that $qa \in R_P^+$ by hypothesis (divide Tf by P^{k+t}), so that a will belong to R_P^+ if $\langle q, q + \delta\epsilon c^{t+k}\rangle = R_P$. This will obviously occur if and only if $\langle q, c\rangle = R_P$. Hence $a \in R_P^+$, and the final statement is a translation of this. ■

So the problem revolves around determining when $\langle q, 1/P\rangle = R_P$. Here is a simple situation. If for some m, $(P, x^n)/(n+1)^m$ is bounded above and below away from zero, and $Q = \sum a_i x^i$ with $\sum |a_i| < \infty$, then $QP \prec P$, as is easy to check (in fact, sufficient for the following to apply in the presence of such a P is that Q be bounded on the open unit disk). In case Q is merely a polynomial, sufficient for $QP \prec P$ is that $xP \prec P$, *i.e.*, $\sup(P, x^{k-1})/(P, x^k) < \infty$.

Proposition 4.6 *Suppose that P and Q are convergent Maclaurin series with nonzero radii of convergence, the former series with no negative coefficients, $P(0) \neq 0$, and $Q \prec P$ and $Q^n P^t \prec P^{n-1} P^t$ for some integers $t \geq 0$ and $n \geq 1$. Let f be a Maclaurin series with nonzero radius of convergence such that $f(0) \neq 0$, $f \prec P^k$ for some integer k, and there exists s such that $(P+Q)^s f$ has no negative coefficients. Then there exists m such that $P^m f$ has no negative coefficients.*

Proof Set $a = f/P^k \in R_P$. Set $T = (P + Q)^s$. Let $d = 1 + Q/P$ and $q = d^s$. Then q is also an element of R_P^+. We note that $(Q/P)^n = (Q^n/P^{n-1})(1/P)$. Hence $(Q/P)^n \in \langle d, c\rangle$; however, $\langle (Q/P)^n, 1 + Q/P\rangle = R_P$. Hence $\langle d, c\rangle = R_P$, and so $\langle d^s, c\rangle = R_P$. Now Lemma 4.5 applies. ■

This result can be applied when (P, x^n) is asymptotic with $cP(1 - 1/n)/n$ for some positive constant c and $(P, x^n) \geq Kn^{-\delta}$ for some positive K and $\delta < 1$, and it is perturbed by A such that $|(A, x^n)| = \mathbf{O}\big((P, x^n)/n^\gamma\big)$ for some $\gamma > 0$. (There are plenty of functions P with the indicated properties, occurring in connection with variations of the Hardy-Littlewood-Karamata theorem.) Then provided $Q := A + P$

has no negative coefficients, the ideal generated by P/Q and $1/Q$ in R_Q is improper (this takes some work, and will not be presented here). Thus if $P^k f$ has no negative coefficients for some f with polynomial growth on its coefficients, then $Q^m f$ also has no negative coefficients for some m. In particular, if some power of P has increasing coefficients, then so does some power of Q (set $f = 1 - x$).

The n^γ condition cannot be weakened to the corresponding condition involving $\ln n$ (or any power of $\ln n$)—consider $P = (1-x)^{-1}$ and set $g = \sum_{n\geq 1} x^n/\ln(n+1)$; then let $A = g(x^2)$ and let Q be either of $P \pm A$. It is not difficult to show that no power of Q has increasing coefficients, so that with $f = 1 - x$, we have $f \in R_P^+$ (obviously) but $f \notin R_Q^+$, even though Q is a modest perturbation of P. This improves on Example 3.2 (although in that example, the coefficients of $(1-x)Q^n$ are eventually alternating for every n).

If in Lemma 4.5, we assume that T itself belongs to R_P, even with relatively nice choices of P, the result is no longer true (this can be constructed out of Example 3.2, with T a polynomial). However, when we restrict to $P = (1-x)^{-1}$ and T is absolutely summable, the result holds.

Proposition 4.7 *Suppose that $Q = \sum a_i x^i$ with $a_i \geq 0$ and $\sum a_i < \infty$. Set $P = (1-x)^{-1}$. Let f be a Maclaurin series whose coefficients grow at most polynomially (i.e., $|(f, x^k)| = \mathbf{O}(k^d)$ for some d). If $Q^s f$ has no negative coefficients for some integer s, then there exists m such that $P^m f$ has no negative coefficients.*

Proof Here $q = T = Q$ (notation from Lemma 4.5), and $c = 1/P = 1-x$. It suffices to show $\langle Q, 1-x\rangle = R_P$. Since the coefficients of Q are absolutely summable, there exists N such that $A := \sum_{i\leq N} a_i > \sum_{i>N+1} a_i$. Since the coefficients of Q are all positive, it follows that $\sup_{|z|<1} |\sum_{i>N+1} a_i z^i| < A$ for any z in the open unit disk. We may obviously assume that $A = 1$; write $Q = q_N + h$, where q_N is the truncation to the first N coefficients. Then $(1+h)^{-1}$ is defined (via $\sum(-h)^j$) and bounded above on the unit disk, hence belongs to R_P. We also note that since polynomials belong to R_P, and Q does, so does $1 + h$. In particular, $1 + h$ is invertible.

Now apply the division algorithm to q_N and $1 - x$; we obtain $q_N = (1-x)p_N + q_N(1)$, where p_N is a polynomial (hence an element of R_P), and $q_N(1) = \sum_{i\leq N} a_i$. Thus we can write $Q - (1-x)h_N = 1 + h$, an equation in R_P, with the right side invertible. Hence $\langle Q, 1-x\rangle = R_P$. ■

The growth condition on the target function f is crucial. As a simple example, consider the case wherein $Q = 1 + 2x$, and $f = (1+2x)^{-1}$. The coefficients of the latter's Maclaurin series grow exponentially. As $Qf = 1$, Q certainly renders f nonnegative; however, no power of $(1-x)^{-1}$ will, by Lemma 3.4. On the other hand, Proposition 4.6 asserts that if some power of $1+2x$ renders f nonnegative, and f has polynomial growth (*i.e.*, $f \prec P^t$), then for some n, $(1-x)^{-n} f$ will have no negative coefficients.

A much more interesting example is available, where both the function and its inverse have radius of convergence one.

Example 4.8 A power series $Q \prec (1-x)^{-1}$ with Q^{-1} having subexponential growth on its coefficients (hence the radius of convergence of Q^{-1} is one) such that for no k does $(1-x)^{-k}Q$ have no negative coefficients, yet there exists R with no negative coefficients (specifically, Q^{-1}) such that RQ has no negative coefficients:

Form the infinite product $Q = \prod_{i=0}^{\infty}(1-x^{2^i})$. This converges on the open unit disk, and has no zeroes there. Moreover, $(1-x)^{-1}Q$ has bounded coefficients (in fact, its coefficients consist of ± 1 and 0). The function Q^{-1} is analytic on the unit disk, and is expressible as the product $Q^{-1} = \prod_{i=0}^{\infty}(1-x^{2^i})^{-1}$ which has positive coefficients. Obviously, Q can be multiplied by a power series with no negative coefficients and with radius of convergence one so that the product has no negative coefficients—multiply it by Q^{-1}. However, we can show that if R is any real power series whose coefficients grow at most polynomially, then RQ must have negative coefficients; in particular, $(1-x)^{-k}Q$ has negative coefficients for all k, even though the growth of the coefficients of Q is polynomial.

The underlying idea is very simple. Note the functional equation $Q(x) = (1-x)Q(x^2)$; an easy induction argument yields $P^nQ \prec P$ for all positive integers n, where $P = (1-x)^{-1}$. Thus Q belongs to $\bigcap_n \bigl((1-x)^n \cdot R_P\bigr)$ (in particular, the latter ideal is not zero!), and P^nQ belongs to R_P for all positive integers n. If R were a power series with polynomial growth, then there would exist k such that $R \prec P^k$, in which case $a := R/P^k$ belongs to R_P. If RQ has no negative coefficients, then P^2RQ has unbounded coefficients, all of which are nonnegative, so does not belong to R_P (evaluate it at positive real points tending to 1). However, $P^2RQ = (P^{k+2}Q) \cdot (R/P^k)$ expresses it as a product of elements of R_P, a contradiction.

The coefficients of $Q^{-1} = \sum c(j)x^j$ satisfy the interesting recurrence $c(n) = c(n-2) + c([n/2])$ (easily derivable from the functional equation). It can be shown by discrete techniques that

$$\frac{\ln_2 n}{2} - \ln_2 \ln_2 n \leq \frac{\ln_2 c(n)}{\ln_2 n} \leq \frac{\ln_2 n}{2} - \ln_2 \ln_2 n + \mathbf{O}(1).$$

Computations with Maple reveal that as n increases to 100000, the difference between the first and second terms increases *up to* .27. This will be re-examined at the end of Section 8.

More perturbation results can be obtained if we exploit the appearance of the (real) Banach algebra $B := H^\infty(D)$ (the algebra of bounded analytic functions on D, here restricted to those with real Maclaurin series coefficients, equipped with the supremum norm as functions on D) inside suitable R_P. For example, by Proposition 3.7, B is contained in R_P if $P = (1-x)^{-1}$. We recall that the point evaluations on B given by $\psi_z(b) = b(z)$ for z in D and b in B, form a weakly dense subset of the maximal ideal space of B.

We define the following condition on a function, b, analytic on D:

(@) there exist $\delta, \epsilon > 0$ such that for all z in D, $|z-1| < \epsilon$ implies $|b(z)| > \delta$.

We define the following condition on the power series (with only positive coefficients) P:

(#) P has radius of convergence one, and whenever the sequence $\{z_n\}$ of elements of D converges to an element z in $\overline{D}$ and $|P(z_n)| \to \infty$, then $z = 1$.

The condition (#) is related to the condition that $\Psi(P) = D \cup \{1\}$ of Appendix B. Obviously this is satisfied by $P = (1 - x)^{-1}$ and a lot of variants.

Corollary 4.9 *Suppose that $B \subset R_P$, P satisfies (#), $1/P$ is a bounded function on D, and b is an element of B satisfying (@). Then $\langle b, 1/P\rangle = R_P$.*

Proof Let J be the ideal $bB + (1/P)B$. If $J \neq B$, there exists a maximal ideal M containing J; necessarily M is closed, so that B/M is of dimension either one or two as a real vector space (being a real commutative Banach algebra and a division ring). Thus there exists a homomorphism (necessarily continuous, and even of norm 1) $\phi \colon B \to \mathbf{C}$ such that $\ker \phi = M$. As $1/P$ belongs to M, $\phi(1/P) = 0$. There exist $z(n)$ in D such that $\{\psi_{z(n)}\}$ (the sequence of the corresponding point evaluations) converge weakly to ϕ; by taking a suitable subsequence, we may assume that the sequence $\{z(n)\}$ converges to the number z in the closed unit disk. Now $0 = \phi(1/P) = \lim \psi_{z(n)}(1/P) = \lim 1/P\bigl(z(n)\bigr)$. By (#), this forces $z = 1$. However, this now yields $0 = \phi(b) = \lim \psi_{z(n)}(b) = b\bigl(z(n)\bigr)$, which contradicts $\bigl|\, b\bigl(z(n)\bigr) \bigr| > \delta$ for all sufficiently large n. Hence $bB + (1/P)B = B$, and all the more so $\langle b, 1/P\rangle = R_P$. ∎

For example, if $P = (1 - x)^{-1}$, and Q (not necessarily with positive coefficients) satisfies $|Q(z)| < (1 - \epsilon)/|1 - z|$ on D, then for f with polynomial growth on its coefficients, $(P+Q)f$ having no negative coefficients implies that $P^m f$ has no negative coefficients for some positive integer m. Just note that $c := Q/P = (1 - x)Q$ is bounded on the unit disk, hence belongs to B, and $b := (P + Q)/P = 1 + c$ satisfies (@), so that Corollary 4.9 applies.

A slightly different example (still with $P = (1 - x)^{-1}$) arises from $P + axP/\ln P$, where a is any real number (which can be replaced by any series bounded on D). We note that $c = ax/\ln P$ is bounded on D, and as z in D approaches 1, $c(z) \to 0$. Hence $b = 1 + c$ satisfies (@). Similar considerations apply to $P + a \ln P/x$.

Another example occurs with the same P but $c = \sum a_i x^i$ has absolutely summable coefficients and $c(1) \neq -1$, and $Q = c \cdot P$; then $(P+Q)/P = 1 + c$, and this obviously satisfies (@).

5 Log Concavity and Fast Growing Series

In order to show that more interesting Maclaurin series satisfy $(*)$, *e.g.*, $P(x) = \sum \exp(\sqrt{n})x^n$, we first require fairly accurate estimates for (P^k, x^n) (large n) and $P(r)$ (r close to 1) in terms of the coefficients of P. This becomes a project in itself, with some interesting consequences. Via reverse engineering, we also obtain Maclaurin series expansions for functions such as $\exp\bigl((1 - x)^{-\alpha}\bigr)$ for real positive α. The estimates below require a certain type of long range smoothness on the coefficients, and we obtain that $(*)$ holds. However, the smoothness is not an essential condition, since we already know that if $P \sim Q$ and P satisfies $(*)$, then so does Q.

A function $\phi\colon \mathbf{R}^{++} \to \mathbf{R}$ is *concave with long range approximability* (or LRA for short) if it satisfies the following conditions:

(a) ϕ is C^2;
(b) $\phi''(t) < 0$ for all t and $\lim_{t\to\infty} \phi''(t) = 0$;
(c) there exists a nonnegative function $F\colon \mathbf{R}^{++} \to \mathbf{R}^{+}$ such that F is bounded and for all real s and positive t with $|s| < t/2$,

$$|\phi''(t+s) - \phi''(t)| \leq |s|\, F(t)\, |\phi''(t)|^{3/2}.$$

If additionally,

(i) the function F satisfies $\lim_{t\to\infty} F(t) = 0$,
(ii) $-t^2\phi''(t) \to \infty$ as $t \to \infty$,

then we say ϕ is *concave with fine long range approximability* (or FLRA for short). If the conditions apply only for all sufficiently large t, we say put *eventually* in front of the property.

Condition (ii) is not superfluous in the presence of (i) and (a)–(c); if $\phi(t) = 1-1/(t+1)^2$, the conclusions of Theorem 5.2 and Proposition 5.4 are false. Examples of FLRA functions include $t \mapsto t^\alpha$ for $0 < \alpha < 1$, $t \mapsto (\ln t)^\beta$ for $\beta > 1$ and $t > e$, and $t \mapsto t/\ln t$. These have the property that $\phi(t)/t \to 0$, corresponding to radius of convergence one for $P := \sum e^{\phi(n)} x^n$. Corresponding to entire P are FLRA such as $\phi(t) = -1/\ln\bigl(\Gamma(t+1)^\alpha\bigr)$ for positive real α. On the other hand, $\phi(t) = -t^2$ fails to be even LRA because $\phi'' = -2$, and similarly $\phi(t) = -t^3$ fails as $\phi''(t) \to -\infty$. The latter condition is relatively easy to deal with separately (Section 7). Examples of LRA functions that are not FLRA include $t \mapsto \alpha \ln t$ for $\alpha > 0$ (in these cases, both (i) and (ii) are violated). More functions, both FLRA and LRA, can be constructed using sums and differences.

We note a few consequences of the definitions. Suppose ϕ is C^3 and LRA (FLRA). We deduce $|\phi'''(t)| = \mathbf{O}\bigl(|\phi''(t)|^{3/2}\bigr)$ (respectively, $\mathbf{o}\bigl(|\phi''(t)|^{3/2}\bigr)$), obtained by permitting $s \to 0$. Whether this condition on ϕ''' is enough to guarantee (c) in the presence of (a) and (b) is unknown, but it would be useful. The remainder term in the second order Taylor series expansion for ϕ about t, $R_2(t,s)$, satisfies $|R_2(t,s)| = |s|^2\mathbf{O}\bigl(|\phi''(t)|^{3/2}\bigr)$ (respectively, with $\mathbf{o}(\cdot)$).

Suppose that ϕ is FLRA and $\psi\colon \mathbf{R}^{++} \to \mathbf{R}$ is C^2 and $\lim_{t\to\infty} |\psi''(t)/\phi''(t)| = 0$. If $\psi'' < 0$, then $\phi + \psi$ is FLRA; in general, $\phi + \psi$ is eventually FLRA. For example, if $h(n) = \exp\sqrt{n}$ and $j(n) = n^\beta$ (with β real), then $\phi = \sqrt{t}$ is FLRA, and $\psi := \ln t^\beta = \beta \ln t$. Thus if $\beta > 0$, $\phi + \psi$ is FLRA, and otherwise it is at least eventually FLRA. This can be applied usefully to Maclaurin series such as $\sum n^\beta \exp(\sqrt{n}) x^n$, as we shall see later.

A function $H\colon \mathbf{N} \to \mathbf{R}^{+}$ is called *strongly unimodal* (or *log concave*) if for all nonnegative integers m, $H(m)^2 \geq H(m+1)H(m-1)$ (by convention, $H(-1) = 0$), and H has "no gaps", *i.e.*, if $m < a < n$ where m, a, and n are integers such that $h(m) \cdot h(n) \neq 0$, then $h(a) \neq 0$. If H is the restriction to $\mathbf{N}$ of a nowhere vanishing C^2 function $\Theta\colon \mathbf{R}^{++} \to \mathbf{R}^{+}$, then sufficient for H to be strongly unimodal is that the function $\theta := \ln \Theta$ be concave, *i.e.*, $\theta'' \leq 0$.

The following is an elementary but useful estimate of the integral of a strongly unimodal function. For a strongly unimodal function $H\colon \mathbf{N} \to \mathbf{R}^+$, assume that $H(n) \to 0$ as $n \to \infty$, so there exists an integer m_0 such that $H(m_0)$ attains the maximum value of H (there may be several contiguous values for m_0; any choice will do). Define the function ρ (possibly zero-valued) via

$$\rho(i) = \begin{cases} \dfrac{H(i)}{H(i+1)} & \text{if } 0 \le i < m_0 \\ \dfrac{H(i+1)}{H(i)} & \text{if } i \ge m_0. \end{cases}$$

In other words, $\rho(i) = \exp -\left|\ln\bigl(H(i)/H(i+1)\bigr)\right|$, and in particular $0 \le \rho(i) \le 1$.

Lemma 5.1 *Let H be a strongly unimodal function on $\mathbf{N}$, and let δ_r and δ_l be positive integers such that $\delta_l \le m_0$. Then we have the following.*

$$\begin{aligned}&\delta_r H(m_0+\delta_r) + \delta_l H(m_0-\delta_l) + H(m_0)\\ &\quad \le \sum H(i) \le (\delta_r+\delta_l+1)H(m_0) + \frac{H(m_0+\delta_r+1)}{1-\rho(m_0+\delta_r+1)} + \frac{H(m_0-\delta_l-1)}{1-\rho(m_0-\delta_l-1)}.\end{aligned}$$

Proof The only part of this that is possibly nontrivial is that that an initial or a terminal subsequence (*i.e.*, to the left or the right of the maximum) of a strongly unimodal sequence is bounded above by a geometric series whose ratio is the ratio of the coefficients at the point closest to m_0. ∎

Bounding the mass in the tail of a log concave sequence by a geometric series will be used repeatedly. For it to be useful, we have to choose reasonable, or even optimal selections for the δ_r and δ_l. When H is the restriction to the nonnegative integers of a function $\Theta(t) = r^t \exp\phi(t)$ where $r > 0$ and ϕ is LRA, then there is a natural choice for $\delta = \delta_r = \delta_l$ for which the upper and lower bounds are comparable.

First, the maximum of Θ occurs where $\phi'(t) = -\ln r$; that there is exactly one solution, $t_0 > 0$, when H is summable (or Θ is in L^1) is immediate from the concavity of θ. We notice that m_0 is either the greatest integer less than t_0, or the least integer exceeding it, *i.e.*, $|t_0 - m_0| < 1$. We will select δ to be the closest integer to $\sqrt{-1/\phi''(t_0)}/2$.

First, we calculate $\rho(m_0+n)$ for $n \ge -m_0$; this is one of $\exp\pm\bigl(\phi(m_0+n+1) - \phi(m_0+n) + \ln r\bigr)$ depending on the sign of n. We expand each of $\phi(m_0+n+1)$ and $\phi(m_0+n)$ first about m_0+n, and then about t_0, using the mean value theorem (second order).

We note that $\phi(m_0+n+1) - \phi(m_0+n) = \phi'(m_0+n) + \phi''(m_0+n_1)$ where $n \le n_1 + n + 1$. Then $|\phi''(m_0+n_1) - \phi''(m_0+n)| < F(m_0+n)|\phi''(m_0+n)|^{3/2}$. We may write $\phi'(m_0+n) = \phi'(t_0) + (m_0+n-t_0)\phi''(t_1)$ with t_1 lying between t_0 and m_0+n. Then $|\phi''(t_1) - \phi(t_0)| < F(t_0)|m_0+n-t_0|\,|\phi''(t_0)|^{3/2}$. Combining these, we deduce

$$\begin{aligned}
&|\ln r + \phi(m_0+n+1) - \phi(m_0+n) - (m_0+n+1-t_0)\phi''(t_0)| \\
&\qquad \le F(m_0+n)|\phi''(m_0+n)|^{3/2} + F(t_0)(m_0+n-t_0)^2|\phi''(t_0)|^{3/2} \\
&\qquad \le \big(F(t_0)+F(m_0+n)\big)\,(m_0+n-t_0)^2|\phi''(t_0)|^{3/2} \\
&\qquad\qquad + 2F(m_0+n)F(t_0)|m_0+n-t_0|\phi''(t_0)^2 \\
&\qquad \le \big(F(t_0)+F(m_0+n)\big)\,(n+1)^2|\phi''(t_0)|^{3/2} + 2F(m_0+n)F(t_0)(n+1)\phi''(t_0)^2.
\end{aligned}$$

The quadratic term in the second line was obtained from

$$a^{3/2} - b^{3/2} = (a-b)\big(\sqrt{a}+\sqrt{b}+\sqrt{ab}/(\sqrt{a}+\sqrt{b})\big),$$

applied with $a = |\phi''(m_0+n-t_0)|$ and $b = |\phi''(t_0)|$.

As a consequence, if F is bounded (LRA) and $|n|$ is small in comparison with $|\phi''(t_0)|^{-1/2}$, or if F tends to zero (FLRA) and $|n| \le |\phi''(t_0)|^{-1/2}$, the right side, the ratio of the error term to $(m_0+n-t_0)\phi''(t_0)$, goes to zero as $t_0 \to \infty$. Since $t_0 \to \infty$ as r approaches the radius of convergence of the original sequence, for r sufficiently near the radius of convergence, we obtain an approximation for the ratio ρ as $\big(1+\mathbf{o}(1)\big)\exp(m_0+n+1-t_0)\phi''(t_0)$. (Recall that $\phi'' < 0$ and very small in absolute value, so ρ is quite close to 1.)

Using $a - a^2/2 < 1 - e^{-a} < a$ for $0 < a < 1$ and setting n to be one of approximately $\delta := \sqrt{-1/\phi''(t_0)}$ or its negative, we obtain that

$$1 - \rho > -\big(1-\mathbf{o}(1)\big)\,1/\delta.$$

Now we estimate $H(m_0+n)/H(t_0)$; its logarithm is $(m_0+n-t_0)\ln r + \phi(m_0+n) - \phi(t_0)$. As before, $\phi(m_0+n) = \phi(t_0) + (m_0+n-t_0)\phi'(t_0) + 1/2(m_0+n-t_0)^2\phi''(t_1)$, and the right side differs from $\phi(t_0) + (m_0+n-t_0)\phi'(t_0) + 1/2(m_0+n-t_0)^2\phi''(t_0)$ by at most $\frac{1}{2}F(t_0)|m_0+n-t_0|^3\,|\phi''(t_0)|^{3/2}$. Hence

$$\ln\frac{H(m_0+n)}{H(t_0)} - \frac{1}{2}(m_0+n-t_0)^2\phi''(t_0) \le \frac{1}{2}F(t_0)|m_0+n-t_0|^3\,|\phi''(t_0)|^{3/2}.$$

Thus if ϕ satisfies FLRA, $H(m_0 \pm \delta)/H(t_0)$ is approximately

$$\big(1+\mathbf{o}(1)\big) \times \exp\frac{1}{2}(m_0+\delta-t_0)^2\phi''(t_0),$$

and this is close to $e^{-1/2}$ (what is important is that it be bounded below). The upshot is that for r sufficiently close to the radius of convergence of $P := \sum h(n)x^n$, it follows that $P(r)/\delta\exp\big(\phi(t_0) - t_0\phi'(t_0)\big)$ is squeezed between 2 and 4; in this argument, we require FLRA. We can obtain much finer results by elaborating somewhat on the method; what we have just done can be viewed as a crude but motivating computation, and so is done rather informally.

Now we obtain a finer result by being less cavalier with the estimates in the strong unimodality result. This is basically the method of Laplace, the precursor to "steepest descent".

Fix positive real r less than the radius of convergence of $P := \sum \exp\phi(n)x^n$, where $\phi\colon \mathbf{R}^{++} \to \mathbf{R}^{+}$ is LRA. There exists a unique solution t_0 to $\phi'(t_0) = -\ln r$; of course the real function $t \mapsto r^t \exp\phi(t)$ has a unimodal graph with maximum at t_0. Set $\delta = \sqrt{-1/\phi''(t_0)}$. Pick a positive integer κ; we shall estimate

$$\sum_{\{n\in\mathbf{N} \mid |n-t_0|<\kappa\delta\}} \exp\bigl(\phi(n) + n\ln r\bigr),$$

and subsequently deal with the tail, summing over $|n - t_0| > \kappa\delta$.

Expand the individual term $\phi(n) + n\ln r = \phi(n) - n\phi'(t_0)$ about t_0. We obtain $\phi(t_0) + \bigl((n - t_0) - n\bigr)\phi'(t_0) + 1/2(n - t_0)^2\phi''(t_1)$ (where t_1 lies between t_0 and n); using the LRA hypothesis, this is $\phi(t_0) - t_0\phi'(t_0) + 1/2(n - t_0)^2\phi''(t_0)$ to within $1/2(n-t_0)^3F(t_0)|\phi''(t_0)|^{3/2}$ (since t_1 is between t_0 and n, $|t_1 - t_0| \leq |n - t_0|$); the latter is at most $E := F(t_0)\kappa^3/2$ (since $\delta^2 = -1/\phi''(t_0)$). (This requires $\kappa\delta < t_0/2$; this can easily be arranged for all the relevant κ, by permitting r to be sufficiently close to the radius of convergence of P—we shall verify this later.) Obviously, E depends on t_0 and κ, but not on n; since $\phi''(t) \to 0$, E also goes to zero for κ fixed, but with r converging to the radius of convergence of P.

Thus

$$e^{-E} < \frac{\exp\bigl(\phi(n) + n\ln r\bigr)}{\exp\bigl(\phi(t_0) - t_0\phi'(t_0) + 1/2(n - t_0)^2\phi''(t_0)\bigr)} < e^{E}.$$

Now

$$\sum_{\{n\in\mathbf{N} \mid |n-t_0|<\kappa\delta\}} \exp\Bigl(\phi(t_0) - t_0\phi'(t_0) + \frac{1}{2}(n - t_0)^2\phi''(t_0)\Bigr) = \exp\bigl(\phi(t_0) - t_0\phi'(t_0)\bigr) \sum_{\{j\in\mathbf{Z} \mid |j|<\kappa\delta\}} e^{-(j/\delta)^2/2}$$

The sum on the right, $S(\kappa,\delta) := \sum_{|j|<\kappa\delta} e^{-(j/\delta)^2/2}$, is of course known with exquisite accuracy (approximately $\delta\sqrt{\pi}$ for large δ). Combining the two most recent expressions, we obtain

$$e^{-E} < \frac{\sum_{\{n\in\mathbf{N} \mid |n-t_0|<\kappa\delta\}} \exp\bigl(\phi(n) + n\ln r\bigr)}{\exp\bigl(\phi(t_0) - t_0\phi'(t_0)\bigr)\, S(\kappa,\delta)} < e^{E}.$$

Now we bound the tail, $\sum_{\{n\in\mathbf{N} \mid |n-t_0|>\kappa\delta\}} \exp\bigl(\phi(n) + n\ln r\bigr)$, by the geometric series argument employed in the elementary strong unimodal argument above. The log of the ratio at $n \sim t_0 \pm \kappa\delta$, *i.e.*, $\ln r + \phi(n+1) - \phi(n)$ is $(n - t_0 + 1/2)\phi''(t_0) \sim -\kappa/\delta$ (provided $\kappa^2F(t_0) < \delta$) with error at most $F(t_0)\kappa^3$, as in the previous estimate of the ratio (again assuming that $\kappa\delta < t_0/2$). Hence if ϕ satisfies FLRA, the tail sum is bounded above by $\delta \exp\big(\phi(t_0) - t_0\phi'(t_0)\big)\,\big(1 + F_0(t)\big)\,\exp\big(2F_0(t)\kappa^2/\delta\big)\,/\,2\kappa$. With FLRA, this simplifies to an upper bound of

$$(\sqrt{\pi}/2\kappa)\big(1 + \mathbf{o}(1)\big)\,\exp\big(\phi(t_0) - t_0\phi'(t_0)\big)\,S(\kappa, \delta).$$

Provided we can permit κ to increase to infinity as r tends to the radius of convergence, the tails contribute an arbitrarily small ratio of the mass.

Now we address the applicability of the FLRA hypotheses. We require $\kappa\delta < t_0/2$; here δ is a function of t_0, and we have some choice for κ, provided it goes to infinity as t_0 does. Condition (ii) in the definition of FLRA asserts that $\delta/t_0 \to 0$ as $t_0 \to \infty$; hence we can choose $\kappa \equiv \kappa(t_0) \to \infty$ so that $\kappa\delta/t_0 \to 0$, which is more than enough. The other requirements that we ran into along the way, $\kappa^3 F(t_0) \to 0$ and $\kappa^2 F(t_0) < \delta$ are easy enough to arrange at the same time.

Let $S(\delta) = \sum_{j\in\mathbf{Z}} \exp -(j/\delta)^2$ (asymptotically, this is $\delta\sqrt{\pi}$ for large δ), and in the following, $\delta^2(t_0) = -1/\phi''(t_0)$. We have deduced the following.

Theorem 5.2 *Let $\phi\colon \mathbf{R}^{++} \to \mathbf{R}^{+}$ be a function satisfying* FLRA*, such that $P := \sum \exp\big(\phi(n)\big)\,x^n$ has radius of convergence ρ in $\mathbf{R}^{++} \cup \{\infty\}$. For $0 < r < \rho$, let t_0 be the unique solution to $\phi'(t_0) = -\ln r$. Then as $r \uparrow \rho$,*

$$P(r) = \big(1 + \mathbf{o}(1)\big)\,S\big(\delta(t_0)\big)\,\exp\big(\phi(t_0) - t_0\phi'(t_0)\big)\,.$$

This is essentially $\sqrt{\pi}\delta(t_0)\exp\big(\phi(t_0) - t_0\phi'(t_0)\big)$. Note that the radius of convergence is given by $\rho = \lim_{n\to\infty} \exp\big(\phi(n) - \phi(n+1)\big)$ (concavity of ϕ assures that the limit exists), and this is simply $\exp\big(-\lim_{t\to\infty}\phi'(t)\big)$.

For $\phi = t^\alpha$ with $0 < \alpha < 1$, the radius of convergence is 1, and t_0, expressed as a function of r is $\big(\alpha/(-\ln r)\big)^{1/(1-\alpha)}$. If we write $r = 1 - 1/N$ ("N" suggests a large integer, but it need not be such), then $-1/\ln r \sim N$ and we see that

$$P\left(1 - \frac{1}{N}\right) \sim \sqrt{\frac{\pi}{1-\alpha}}\,\frac{1}{\alpha^{1/(1-\alpha)}} N^{\frac{1}{2}(1+1/(1-\alpha))} \exp\big((\alpha N)^{\alpha/(1-\alpha)}(1-\alpha)\big)\,.$$

In the special case that $\alpha = 1/2$, up to a scalar multiple, this is $N^{3/2}e^{N/4}$.

If $\phi = (\ln t)^\beta$ with $\beta > 1$ (which means that the function $\exp\phi(n)$ has growth infinitely slower than that of the previous example, but is not polynomial), the radius of convergence is again 1, and t_0 can be approximated by $\big(\ln\big(\ln(1/r)\big)\big)^{\beta-1}\beta/\ln(1/r)$ (this requires proof; it will be supplied in Section 7).

Suppose that $a = \big(a(i)\big)$ and $b = \big(b(i)\big)$ are strongly unimodal sequences. Their *Hadamard product* is the sequence $\big(a \circ b(i)\big) = \big(a(i)\cdot b(i)\big)$, and it is straightforward to verify that this is also strongly unimodal. More subtly, their convolution product $(c(i)) := \sum_j \big(a(j)\cdot b(i-j)\big)$ is also strongly unimodal. The corresponding results for FLRA would be that if both ϕ and ψ are FLRA, then $\phi + \psi$ is, and so would be $\ln(\exp\phi * \exp\psi)$. The first of these is true as is evident from the definitions (only property (c) presents any problems), and as we have observed earlier, it is also true under fairly weak hypotheses on second derivatives' growth at infinity. In contrast, although the convolution $\exp\phi * \exp\psi$ is log concave, property (c) (for the log) is problematic.

Thus, if we wish to determine the behaviour of high convolution powers of $\exp\phi(n)$—*i.e.*, if $P = \sum \exp\big(\phi(n)\big)\, x^n$, we wish to find the coefficients of P^k in its Maclaurin expansion—the natural induction argument to reduce to convolution of two power series will run into difficulties. Instead, we can deal with such powers directly. First, we consider a straightforward convolution of two different series.

Proposition 5.3 *Suppose that ϕ is* FLRA, *ψ is eventually* FLRA, *$|\phi'(t)| + |\psi'(t)| \to 0$ as $t \to \infty$, and ϕ' and ψ' are eventually positive. Set $\tau = \ln\big((\exp\phi) * (\exp\psi)\big)$. Then as $N \to \infty$,*

$$\exp\tau(N) = \big(1 + \mathbf{o}(1)\big)\sqrt{\pi}\,\frac{\exp\big(\phi(t_0) + \psi(N - t_0)\big)}{\big(-\phi''(t_0) - \psi''(N - t_0)\big)^{1/2}},$$

where t_0 is the unique solution of $\phi'(t) = \psi'(N-t)$, and both $t_0 \to \infty$ and $N - t_0 \to \infty$ as N does. In particular, if $\phi = \psi$, then $\exp\tau(N) \sim \sqrt{2\pi}\exp\big(2\phi(N/2)\big)\,\delta(N/2)$.

Proof We expand $\exp\big(\tau(N)\big) = \sum_i \exp\big(\phi(i) + \psi(N - i)\big)$ for N fixed. The first observation is that the sequence $\big(\exp\big(\phi(i) + \psi(N - i)\big)\big)_{0\le i\le N}$ is itself strongly unimodal (as it is a Hadamard product of two strongly unimodal sequences, $\big(\exp\phi(i)\big)_{0\le i\le N}$ and the reversal, $\big(\exp\psi(N - i)\big)_{0\le i\le N}$). The function $t \mapsto \phi(t) + \psi(N - t)$ (for $0 < t < N$) is smooth and concave on $(0, N)$. We note that ϕ' is decreasing ($\phi'' < 0$) while $\psi'(N - t)$ is increasing on $(0, N)$, and since $\phi'(t)$ and $\psi'(t)$ both go to zero, it follows that for all sufficiently large N, $\phi'(t) = \psi'(N - t)$ has a zero in $(0, N)$. By concavity, the function has a unique maximum occurring at $t_0 \equiv t_0(N)$, the only solution to $\phi'(t) = \psi'(N - t)$, and $0 < t_0 < N$.

If there exists an unbounded choice for N such that the corresponding $t_0(N)$s are bounded, say by K, then $\phi'(k) < \psi'(N - k)$ for these N, which forces $\phi'(k) \le 0$, and it follows that $\phi'(t) \le 0$ for $t \ge k$ contradicting the hypothesis. Similarly, $N - t_0 \to \infty$ (not just unbounded).

Finally, we can expand around t_0, and by increasing N as required, we ensure that there are "enough" terms for which the long range approximation can be used. For $|i - t_0| < t_0/2, (N - t_0)/2$, we have

$$\begin{aligned}
&\phi(i) + \psi(N - i)\\
&\qquad = \phi(t_0) + \psi(N - t_0) + (i - t_0)\big(\phi'(t_0) - \psi'(N - t_0)\big)\\
&\qquad\qquad + \frac{(i - t_0)^2}{2}\big(\phi''(t_1) + \psi''(N - t_2)\big)\\
&\qquad = \phi(t_0) + \psi(N - t_0) + (i - t_0)\cdot 0 + \frac{(i - t_0)^2}{2}\big(\phi''(t_1) + \psi''(N - t_2)\big).
\end{aligned}$$

The expression $\phi''(t_1) + \psi''(N - t_2)$ is approximately $\phi''(t_0) + \psi''(N - t_0)$ with error at most

$$|i - t_0|\big(F(t_0)|\phi''(t_0)|^{3/2} + G(N - t_0)|\psi''(t_0)|^{3/2}\big)$$

https://doi.org/10.4153/CJM-2003-041-7 Published online by Cambridge University Press

(where G is the function corresponding to F for $\psi(t)$). Hence with error at most

$$|i-t_0|^3\big(F(t_0)|\phi''(t_0)|^{3/2}+G(N-t_0)|\psi''(t_0)|^{3/2}\big),$$

we can approximate $\phi(i)+\psi(N-t_0)$ by

$$\phi(t_0)+\psi(N-t_0)+\frac{(i-t_0)^2}{2}\big(\phi''(t_0)+\psi''(N-t_0)\big).$$

As $N\to\infty$, both t_0 and $N-t_0$ also tend to infinity, so that $F(t_0)\to 0$ and $G(N-t_0)\to 0$. If we restrict i to vary over $|i-t_0|<\kappa\delta$ where

$$\delta^2 := \sqrt{-1/\big(\phi''(t_0)+\psi''(N-t_0)\big)},$$

then the error terms tend to 0 as well. Hence

$$\exp\big(\phi(i)+\psi(N-i)\big) = \big(1+\mathbf{o}(1)\big)\exp\big(\phi(t_0)+\psi(N-t_0)\big)\exp\Big(-\big((i-t_0)/\delta\big)^2/2\Big).$$

Summing these over suitably restricted i as we did previously, we obtain

$$\begin{aligned}\sum_{|i-t_0|<\kappa\delta}&\exp\big(\phi(i)+\psi(N-i)\big)\\ &=\big(1+\mathbf{o}(1)\big)\exp\big(\phi(t_0)+\psi(N-t_0)\big)\sum_{|i-t_0|<\kappa\delta}\exp\Big(-\big((i-t_0)\delta\big)^2/2\Big)\\ &=\big(1+\mathbf{o}(1)\big)\exp\big(\phi(t_0)+\psi(N-t_0)\big)S(\kappa,\delta),\end{aligned}$$

where $S(\kappa,\delta)$ was defined previously, and is approximately $\sqrt{\pi}\delta$. The justification, that by increasing N, we can correspondingly increase κ so that the long range approximation is applicable to all the terms in the sum is, as previously, a consequence of (ii) of the definition of FLRA.

The estimate of the tail mass is very similar to the earlier one, exploiting the strong unimodularity of Hadamard product and estimating the ratio as before.

If $\phi=\psi$, then obviously $t_0=N/2$. ∎

If in the preceding $\phi=\psi$, a more accurate estimate can be made—the odd order terms in the expansion cancel, and so if we assume that ϕ is C^4, an error estimate involving the fourth derivative results.

For a convolution power, say the kth, we have a slightly more complicated problem, resolved by summing over a ball in $\mathbf{Z}^{k-1}$, this time with centre at N/k instead of $N/2$.

Proposition 5.4 *Let ϕ be FLRA and let k be a positive integer. Let $\big(\exp\tau(n)\big)$ denote the integer distribution obtained by convolving $\big(\exp\phi(n)\big)$ with itself k times. Then as $N\to\infty$*

$$\exp\tau(N)=\big(1+\mathbf{o}(1)\big)\sqrt{\frac{\pi^{k-1}}{k}}\,\frac{\exp\big(k\phi(\frac{N}{k})\big)}{\big(-\phi''(\frac{N}{k})\big)^{(k-1)/2}}.$$

https://doi.org/10.4153/CJM-2003-041-7 Published online by Cambridge University Press

Proof The answer suggests the method. The Nth term in the convolution is the sum over all terms of the form $\exp\sum_{j=1}^{k}\phi\big(u(j)\big)$, where $\sum_j u(j) = N$. Set $v(j) = u(j) - N/k$, so that $\big(kv(j)\big)$ is in $\mathbf{Z}^k$ and $\sum v(j) = 0$ (so $\big(kv(j)\big)$ is a lattice point in the obvious hyperplane). Restrict ourselves to all such points in the ball defined by $\sum v(j)^2 < (\kappa\delta)^2$, where $\delta^2 = -1/\phi''(N/k)$.

Expanding the terms in $\sum\phi\big(v(j)+N/k\big)$ about N/k, we obtain

$$k\phi(N/k) + 0\cdot\phi'(N/k) + \frac{1}{2}\Big(\sum v(j)^2\Big)\,\phi''(N/k)$$

(the zero term comes from $\sum v(j) = 0$), with error at most

$$\frac{1}{2}\Big(\sum v(j)^2\Big)\,F(t_0)\sum|v(j)|\,|\phi''(N/k)|^{3/2}$$

(assuming that $\kappa\delta < N/2k$). From the definition of δ, the error term is bounded above by $2\times 1/2F(t_0) = F(t_0)$. Thus, up to a factor of $\exp\pm F(t_0)$,

$$\sum_{\substack{\{v(j)\in\mathbf{Z}^k+N/k\,|\,\sum_j v(j)=0,\\ \sum_j v(j)^2<\kappa\delta^2\}}}\exp\Big(\sum\phi\big(v(j)+N/k\big)\Big)$$

simplifies to

$$\exp\big(k\phi(N/k)\big)\sum\exp\Big(-\frac{1}{2}\sum v(j)^2/\delta^2\Big).$$

Once again, the exponential sum on the right is well known, and is asymptotic with $\delta^{k-1}\sqrt{1/k}\pi^{(k-1)/2}$ (the corresponding quadratic form is

$$2\sum_{j\le k-1}v(j)^2 + 2\sum_{i<j\le k-1}v(i)v(j)\,).$$

The argument that this is applicable for all sufficiently large κ (as N increases) is the same as that of the earlier arguments. ■

6 FLRA and $(*)$

We wish to use the preceding section to obtain a much larger class of Maclaurin series that satisfy $(*)$. We strengthen slightly the condition on ϕ; in addition to being FLRA, it is also required to be C^3. It is possible to prove this result without assuming C^3, but then we get bogged down in difference operators, and it is altogether too cumbersome. We will prove the following.

Theorem 6.1 *Let $P = \sum\exp\phi(n)x^n$ where ϕ is C^3 and FLRA, and the radius of convergence of P is the positive number (or infinity) ρ. Suppose that $0\le f\le P^k$ and $\liminf_N(fP,x^N)/(P^{k+1},x^N) = 0$. Then there exist $r\equiv r(N) > 0$ increasing up to ρ such that $f\big(r(N)\big)\big/P^k\big(r(N)\big)\to 0$.*

In particular, the set of point evaluation traces is dense in the pure trace space of R_P.

Note we are not assuming that $\lim_{j\to\infty}(fP^k,x^j)/(P^{k+1},x^j) = 0$, simply that there is a subsequence along which the ratios go to zero. (If the limit were zero, it would

https://doi.org/10.4153/CJM-2003-041-7 Published online by Cambridge University Press

be too easy.) The conclusion is of the same type, namely that $\liminf_{r\uparrow c} f(r)/P^k(r) = 0$, *not* $\lim_{r\uparrow c} f(r)/P^k(r) = 0$ which would be far too strong—the function f/P^k, although bounded on $[0,c)$, can oscillate wildly as the variable approaches c from the left.

Pick $\epsilon > 0$; there exist infinitely many choices of positive integers N for which $(fP^k, x^N) \leq \epsilon(P^{k+1}, x^N)$, so the choice of such $N \equiv N(\epsilon)$ can be made arbitrarily large. We shall have to choose the $\kappa \equiv \kappa(\epsilon)$ to go to ∞ as $\epsilon \to 0$. We shall have further growth restrictions, *e.g.*, $e^{\kappa^2(\epsilon)}\epsilon \to 0$, *i.e.*, $\kappa(\epsilon) = \mathbf{o}(\sqrt{\ln 1/\epsilon})$, a weird condition.

For s a positive integer, define ϕ_s via $\phi_s(t) = \ln(P^s, x^t)$ (for integer values of t); by the earlier convolution result, Proposition 5.4, we may expand $\exp\phi_s(t)$ as indicated there. Write $f = \sum \lambda(n)\exp\phi_k(n)x^n$ where $0 \leq \lambda(n) \leq 1$ for all n. Then $(Pf, x^N) = \sum_{i=0}^N \lambda(i)\exp\big(\phi_k(i) + \phi(N-i)\big)$. We expand this around $t_0 = Nk/(k+1)$. For κ to be determined later and $\delta = \big(-\phi''\big(N/(k+1)\big)\big)^{-1/2}$, we have

$$
\begin{aligned}
(Pf, x^N) &\geq \sum_{\{i\leq N \,|\, |i-\frac{Nk}{k+1}|<\kappa\delta\}} \lambda(i)\exp\big(\phi_k(i) + \phi(N-i)\big)\\
&= \sum_{\{j-\frac{Nk}{k+1}\in\mathbf{Z} \,|\, |j|<\kappa\delta\}} \lambda\Big(j + \frac{Nk}{k+1}\Big)\exp\Big(\phi_k\Big(j+\frac{Nk}{k+1}\Big) + \phi\Big(\frac{N}{k+1} - j\Big)\Big)\\
&= \big(1+\mathbf{o}(1)\big)K \sum_{\{j-\frac{Nk}{k+1}\in\mathbf{Z} \,|\, |j|<\kappa\delta\}} \lambda\Big(j+\frac{Nk}{k+1}\Big)\\
&\qquad \times \exp\Big(k\phi\Big(\frac{j}{k} + \frac{N}{k+1}\Big) + \phi\Big(\frac{N}{k+1} - j\Big)\\
&\qquad\qquad + (k-1)\ln\delta\Big(\frac{j}{k} + \frac{N}{k+1}\Big)\Big)\\
&= \big(1+\mathbf{o}(1)\big)K \sum_{\{j-\frac{Nk}{k+1}\in\mathbf{Z} \,|\, |j|<\kappa\delta\}} \lambda\Big(j+\frac{Nk}{k+1}\Big)\delta^{k-1}\Big(\frac{j}{k}+\frac{N}{k+1}\Big)\\
&\qquad \times \exp\Big((k+1)\phi\Big(\frac{N}{k+1}\Big) + \phi\Big(\frac{N}{k+1} - j\Big)\\
&\qquad\qquad + (j^2/2)\Big(1+\frac{1}{k}\Big)\phi''\Big(\frac{N}{k+1}\Big)\Big)\\
&= \big(1+\mathbf{o}(1)\big)\exp\Big((k+1)\phi\Big(\frac{N}{k+1}\Big)\Big)K\\
&\qquad \times \sum_{\{j-\frac{Nk}{k+1}\in\mathbf{Z} \,|\, |j|<\kappa\delta\}} \lambda\Big(j+\frac{Nk}{k+1}\Big)\delta^{k-1}\Big(\frac{j}{k}+\frac{N}{k+1}\Big)\\
&\qquad\qquad \exp\Big(\frac{j^2}{2}\Big(1+\frac{1}{k}\Big)\phi''\Big(\frac{N}{k+1}\Big)\Big).
\end{aligned}
$$

We already have that

$$(P^{k+1},x^N) = \big(1+\mathbf{o}(1)\big)\,C''\exp\Big((k+1)\phi\Big(\frac{N}{k+1}\Big)\Big)\,\delta^k\Big(\frac{N}{k+1}\Big),$$

so that

$$\epsilon \geq \frac{(Pf,x^N)}{(P^{k+1},x^N)} \geq \big(1+\mathbf{o}(1)\big)\times$$
$$\times\, C''' \frac{\sum_{\{j-\frac{Nk}{k+1}\in\mathbf{Z}\,|\,|j|<\kappa\delta\}} \lambda(j+\frac{Nk}{k+1})\delta^{k-1}(\frac{j}{k}+\frac{N}{k+1})\exp\big(\frac{j^2}{2}(1+\frac{1}{k})\phi''(\frac{N}{k+1})\big)}{\delta^k(\frac{N}{k+1})}$$

Similarly, we obtain an estimate for $f(r)$, where r is determined by $\phi'(N/(k+1)) = -\ln r$.

$$\begin{aligned}
f(r) &\geq \big(\mathbf{o}(1)\big)\,P^k(r) + \sum_{|i-\frac{Nk}{n+1}|<\kappa\delta} \lambda(i)\exp\Big(\phi_k(i) - i\phi'\Big(\frac{N}{k+1}\Big)\Big)\\
&= \big(\mathbf{o}(1)\big)\,P^k(r) + \sum_{|j|<\kappa\delta}\lambda\Big(j+\frac{Nk}{k+1}\Big)\\
&\qquad\times\exp\Big(\phi_k\Big(j+\frac{Nk}{k+1}\Big) - \Big(j+\frac{Nk}{k+1}\Big)\phi'\Big(\frac{N}{k+1}\Big)\Big)\\
&= \big(\mathbf{o}(1)\big)\,P^k(r) + \big(1+\mathbf{o}(1)\big)\,C\sum_{|j|<\kappa\delta}\lambda\Big(j+\frac{Nk}{k+1}\Big)\,\delta^{k-1}\Big(\frac{j}{k}+\frac{N}{k+1}\Big)\\
&\qquad\times\exp\Big(k\phi\Big(\frac{j}{k}+\frac{N}{k+1}\Big) - \Big(j+\frac{Nk}{k+1}\Big)\phi'\Big(\frac{N}{k+1}\Big)\Big)\\
&= \big(\mathbf{o}(1)\big)\,P^k(r) + \big(1+\mathbf{o}(1)\big)\,C\sum_{|j|<\kappa\delta}\lambda\Big(j+\frac{Nk}{k+1}\Big)\,\delta^{k-1}\Big(\frac{j}{k}+\frac{N}{k+1}\Big)\\
&\qquad\times\exp\Big(k\phi\Big(\frac{N}{k+1}\Big) - \frac{Nk}{k+1}\phi'\Big(\frac{N}{k+1}\Big) + (j^2/2k)\phi''\Big(\frac{N}{k+1}\Big)\Big)\\
&= \big(\mathbf{o}(1)\big)\,P^k(r) + \big(1+\mathbf{o}(1)\big)\,C\exp\Big(k\phi\Big(\frac{N}{k+1}\Big) - \frac{Nk}{k+1}\phi'\Big(\frac{N}{k+1}\Big)\Big)\\
&\qquad\times\sum_{|j|<\kappa\delta}\lambda\Big(j+\frac{Nk}{k+1}\Big)\,\delta^{k-1}\Big(\frac{j}{k}+\frac{N}{k+1}\Big)\,\exp\Big(\frac{j^2}{2k}\phi''\Big(\frac{N}{k+1}\Big)\Big).
\end{aligned}$$

The first line (discarding the tail) comes from assuming $\lambda(i) = 1$ for i outside the range there, so that we are only considering an estimate for the tail of P^{k+1}, which we can obtain directly from the tail of P. (Unfortunately, although we know the distribution of P^{k+1} is log concave, so that tails can be estimated by ratios, our estimate for

https://doi.org/10.4153/CJM-2003-041-7 Published online by Cambridge University Press

(P^{k+1}, x^m) for large m is not sufficiently accurate to estimate the ratio of consecutive coefficients.) Rather than write $1 - \mathbf{o}(1)$, we of course permit the values of the $\mathbf{o}(1)$ term to be negative.

The error in the third line resulting from the approximation in the second order term yields a factor of less than $\exp\bigl(F\bigl(N/(k+1)\bigr)\, j^3/2k^2 \bigl|\phi''\bigl(N/(k+1)\bigr)\bigr|^{3/2}\bigr)$, *i.e.*, less than $\exp\bigl(\kappa^3 F\bigl(N/(k+1)\bigr)/2k^2\bigr)$ (so we shall need to force $\kappa^3 F\bigl(N/(k+1)\bigr) \to 0$ as $N \to \infty$).

From our previous expression for

$$P(r) = \bigl(1 + \mathbf{o}(1)\bigr)\, \pi^{1/2} \exp\Bigl(\phi\Bigl(\frac{N}{(k+1)}\Bigr) - \Bigl(\frac{N}{(k+1)}\Bigr)\phi'\Bigl(\frac{N}{(k+1)}\Bigr)\Bigr)\, \delta\Bigl(\frac{N}{k+1}\Bigr),$$

we have that

$$P^k(r) = \bigl(1 + \mathbf{o}(1)\bigr)\, \pi^{k/2} \exp\Bigl(k\phi\Bigl(\frac{N}{(k+1)}\Bigr) - \Bigl(\frac{Nk}{(k+1)}\Bigr)\phi'\Bigl(\frac{N}{(k+1)}\Bigr)\Bigr)\, \delta^k\Bigl(\frac{N}{k+1}\Bigr).$$

Thus

$$\frac{f(r)}{P^k(r)} \le \mathbf{o}(1) + \bigl(1 + \mathbf{o}(1)\bigr)\, C' \frac{\sum_{|j| < \kappa\delta} \lambda\bigl(\frac{j+Nk}{(k+1)}\bigr)\, \delta^{k-1}\bigl(\frac{j}{k} + \frac{N}{(k+1)}\bigr) \exp\Bigl(\bigl(\frac{j^2}{2k}\bigr)\phi''\bigl(\frac{N}{(k+1)}\bigr)\Bigr)}{\delta^k\bigl(\frac{N}{k+1}\bigr)}.$$

Normally it would be hopeless to expect the expression for $f(r)/P^k(r)$ to be even remotely comparable to that for $(Pf, x^N)/(P^{k+1}, x^N)$, because

$$\exp\Bigl((j^2/2k)\phi''\bigl(N/(k+1)\bigr)\Bigr)$$

is a lot bigger than $\exp\bigl((j^2/2)(1 + 1/k)\phi''\bigl(N/(k+1)\bigr)\bigr)$ (recalling that $\phi'' < 0$). However, because there is some freedom available with the choice of κ (it is forced to go to infinity, but it doesn't matter how slowly), we can make use of a gross overestimate.

For $|j| \le \kappa\delta\bigl(N/(k+1)\bigr)$,

$$\begin{aligned}\exp\Bigl((j^2/2k)\phi''\bigl(N/(k+1)\bigr)\Bigr) &= \exp\Bigl(-(j^2/2)\phi''\bigl(N/(k+1)\bigr)\Bigr)\\ &\quad\times \exp\Bigl((j^2/2)(1 + 1/k)\phi''\bigl(N/(k+1)\bigr)\Bigr)\\ &\le \exp(\kappa^2/2) \exp\Bigl((j^2/2)(1 + 1/k)\phi''\bigl(N/(k+1)\bigr)\Bigr).\end{aligned}$$

Provided we can choose $\kappa \equiv \kappa(\epsilon)$ so that $\exp(\kappa^2/2)\epsilon \to 0$ as $\epsilon \to 0$ (and κ obeys all the earlier constraints, the most interesting being that it becomes arbitrarily large), *e.g.*, $\kappa \le \sqrt{\ln 1/\epsilon}$ is more than sufficient, we will deduce that $f(r)/P(r)$

https://doi.org/10.4153/CJM-2003-041-7 Published online by Cambridge University Press

will tend to zero (along our choices for r, $\exp -\big(N/(k+1)\big)$). This is possible—for each ϵ there are infinitely many choices for the integer N, so that N and consequently $N/(k+1)$ can be made arbitrarily large at the outset, and so $F\big(N/(k+1)\big)$ can be made arbitrarily small. Then we require that $\kappa(\epsilon)$ be at most

$$\inf\Big\{ (\ln 1/\epsilon)^{1/2}, F^{-1/4}\big(N/(k+1)\big), \mathbf{o}\Big(\big(N/(k+1)\big) \big| N/(k+1)\phi''\big(N/(k+1)\big) \big| \Big) \Big\}.$$

Each of the three terms can be made arbitrarily large as $\epsilon \to 0$, so the choices can be made. ■

7 Spiky Distributions

"Spiky" refers to the behaviour of the distribution $n \mapsto r^n e^{\phi(n)}$—for almost all sufficiently large $r > 0$, there exists a single integer, $N \equiv N(r)$, such that $r^N e^{\phi(N)}$ exceeds $\sum_{n \neq N} r^n e^{\phi(n)}$ and the ratio tends to infinity. (This is not a rigourous definition, but merely explains what is meant by spiky.) For example, this occurs if $\phi = -t^3$; more generally, we consider the situation that $\phi'' < 0$, but $-\phi''(t) \to \infty$. It turns out that $(*)$ fails very generally, but a weakened version of $(*)$, sufficient to prove the density result, still holds.

The definition of "δ" becomes problematic. However, spikiness is relatively easy to deal with, since the corresponding distributions $\exp\big(\phi(n)\big)\, r^n$ eventually decrease so quickly that the tails begin one or two terms away from of t_0 (where $\phi'(t_0) = -\ln r$).

Set $h(t) = \exp \phi(t)$, and for $r > 0$ less than the radius of convergence of $P := \sum h(n)x^n$, set $H(t) = h(t)r^t$. Then $H'(t) = \big(\phi'(t) - \ln r\big) H(t)$ and $H''(t) = \big(\phi''(t) + \big(\phi'(t) - \ln r\big)^2\big) H(t)$. So as before, if $\phi'(t) = -\ln r$ has a solution with $t > 0$, then H attains a unique maximum at t_0. If $r \leq 1$ this may fail, in which case H is strictly decreasing; on the other hand, if r can be chosen sufficiently large (*e.g.*, if $\phi(t)/t \to -\infty$—since the distribution is strongly unimodal, this is equivalent to $\phi'(t) \to -\infty$—so that P is entire), then since ϕ' is decreasing to $-\infty$, then there will be a solution.

Hence either H attains its one maximum at $t = 0$ (so the distribution is monotone decreasing), or at a point $t_0 > 0$ for which $\phi'(t_0) = -\ln r$. In the former case, we can make a crude estimate using the tail beyond two (using the ratio method for strongly unimodal distributions), specifically $\exp \phi(0) + r \exp \phi(1) + r^2 \phi(2)/|\phi'(2) + \ln r|$ (really only effective if the denominator is fairly small), but this is not very interesting. On the other hand, if $t_0(r) \to \infty$, we can get very sharp estimates for $P(r)$.

Select the integer n such that $n \leq t_0 \leq n+1$; the initial claim is that $P(r)$ is very well estimated by $h(n)r^n + h(n+1)r^{n+1}$ (this will be fine-tuned, so that for most values of r only one of the two summands is required). Consider the ratio, $h(n+2)r^{n+2}/h(n+1)r^{n+1}$; its log is $\phi(n+2) - \phi(n+1) - \phi'(t_0)$. This is $\phi'(s_0+1) - \phi'(t_0)$ (where $n \leq s_0 \leq n+1$, so $s_0 + 1 > t_0$), and this expands to $(s_0 + 1 - t_0)\phi''(s_1)$, where $t_0 \leq s_1 \leq n+2$.

As $\phi'' \to -\infty$, the original ratio tends to zero. This implies that as $t_0 \to \infty$, the mass of the tail to the right of n in $\sum h(i)r^i$ is $\mathbf{o}(1) \cdot h(n)r^n$. Similarly, the mass to the left of $n-1$ is $\mathbf{o}(1) \cdot h(n-1)r^{n-1}$. ■

Notice that $h(t_0)r^{t_0}$ itself may be much too large as an approximation (this is particularly true if $2t_0$ is close to an odd integer).

Now we investigate the ratio of the two summands, using the same notation. The log of $h(n+1)r^{n+1}/h(n)r^n$ is simply $\phi(n+1) - \phi(n) - \phi'(t_0)$. This expands to $\phi'(s_0) - \phi'(t_0)$, where $n \leq s_0, t_0 \leq n+1$, and expanding this, we obtain $(s_0 - t_0)\phi''(s_1)$; unfortunately, this is not sufficient for what we have in mind, so we assume ϕ is C^3, and expand directly about t_0. Set $Y = t_0 - n$ (the fractional part of t_0). We obtain $(1-2Y)\phi''(t_0)/2 + \big((Y+1)^3 - Y^3\big)\phi'''(s_0)/6$, where $n \leq s_0 \leq n+1$. As Y varies over $[0,1)$, $(Y+1)^3 - Y^3 > 0$ and the sign of $1-2Y$ changes only at $Y = 1/2$.

We are required to assume that ϕ''' is bounded above (this avoids very zigzaggy behaviour). Since we are assuming that $\phi'' \to -\infty$, if $Y < 1/2$, the sign of the expression is negative and (provided $|Y - 1/2|$ is not too small), the expression itself goes to $-\infty$, which means that almost all the mass is tied up in the nth term. If $Y > 1/2$, depending on the behaviour of the ratio ϕ''/ϕ''', the mass is essentially all in one or the other piece. Note that if $\phi''' \to -\infty$ faster than ϕ'' does, the expression will be negative for every value of Y, and almost all the mass is in the nth term regardless of the sign of $1-2Y$. (This occurs, for example, if $\phi = -e^{t^2}$.)

On the other hand, if $Y = 1/2$ and ϕ''' is bounded above and below, then the n-th and $n+1$ st terms have bounded ratios and both are required to approximate $P(r)$. ■

To estimate the convolution of $\{h(i)\}$ with itself, we observe that a similar phenomenon occurs—for N large enough, $h(N/2)^2$ (if N is even) or $2h\big((N+1)/2\big) \cdot h\big((N-1)/2\big)$ (if N is odd) is remarkably accurate. We recall that for any positive integer N, the finite sequence $\{h(i)h(N-i)\}_{0 \leq i \leq N}$ is itself strongly unimodal, with maxima at $i = N/2$ if N is even, or at $i = (N \pm 1)/2$.

If $N = 2l$, estimate the ratio at $i = l \pm 1$ to that at $i = l$—its logarithm is $\phi(l+1) + \phi(l-1) - 2\phi(l)$; expanding about l, we obtain $0 \cdot \phi(l) + 0\phi'(l) + \big(\phi''(l_1) + \phi''(l_2)\big)/2$, which goes to $-\infty$ as N does (since l_1 and l_2 are near $N/2$). So both tails go to zero compared with the middle term, $h(l)^2$.

If $N = 2l+1$, we estimate the ratio of $h(l-1)h(l+2)$ to $h(l)h(l+1)$; again, taking the logarithms, we obtain $\phi(l-1) + \phi(l+2) - \phi(l) - \phi(l+1)$. We have $\phi(l-1) - \phi(l) = -\phi(l_0)$,where $l-1 \leq l_0 \leq l$, and $\phi(l+2) - \phi(l+1) = \phi(l_1)$, where $l+1 \leq l_1 \leq l+2$. Thus the log of the expression simplifies to $(l_1 - l_0)\phi''(l_2)$ where $l-1 \leq l_2 \leq l+2$ and $l_1 - l_0 \geq 1$. Hence the ratio tends to zero (very quickly). Since there are two terms of the form $h(l)h(l+1)$, we obtain the estimate $2h(l)h(l+1)$. Again, $\exp 2\phi(N/2)$ is almost always much too large.

This extends, tediously but with no new ideas, to the case of k-fold convolutions. Write $N = qk+m$ where q and r are integers, and $0 \leq m < k$. The optimal product of the $h(i)$ comes from the partition of N given as q (with multiplicity $k-m$) and $q+1$ (with multiplicity m). It follows as in the arguments above that $h(q)^{k-m}h(q+1)^m\binom{k}{m}$ is an accurate approximation of $h_k(N)$, up to a factor of $1 + \mathbf{o}(1)$. If k does not divide N, then $\exp k\phi(N/k)$ is asymptotically much too large, as above. ■

Now we require an elementary result about order units in R_P. As before, for P and

https://doi.org/10.4153/CJM-2003-041-7 Published online by Cambridge University Press

Q convergent power series, $P \circ Q$ denotes their *Hadamard* product, *i.e.*, $(P \circ Q, x^j) = (P, x^j) \cdot (Q, x^j)$.

Lemma 7.1 *Suppose $\phi\colon \mathbf{R}^+ \to \mathbf{R}$ is C^3, $\phi'''(t)$ is bounded above, and $\phi''(t) \to -\infty$. Set $P = \sum \exp\bigl(\phi(n)\bigr)\, x^n$, let k be a positive integer, and define $\lambda\colon \mathbf{N} \to \{0,1\}$ via $\lambda(m) = 1$ if and only if $m = ks$ for some integer s. Then $f := \lambda \circ P^k$ satisfies $fP^k \sim P^{2k}$.*

Proof By the previous result, for all sufficiently large $N = (2k)q + m$ $(0 \le m < 2k)$, (P^{2k}, x^N) is $\bigl(1+\mathbf{o}(1)\bigr)\binom{2k}{m}\exp\bigl((2k-m)\phi(q)+m\phi(q+1)\bigr)$; of course $\binom{2k}{m}$ is bounded (as N varies), say by C. If $m < k$, then (fP^k, x^N) is at least

$$\bigl(1 + \mathbf{o}(1)\bigr)\,(P^k, x^{kq})(P^k, x^{kq+m}) \ge \exp\bigl(k\phi(q) + (k-m)\phi(q) + m\phi(q+1)\bigr),$$

hence $(fP^k, x^N) \ge (P^{2k}, x^N)/C\bigl(1 + \mathbf{o}(1)\bigr)$. If $k \le m < 2k$, the same thing applies to $(P^k, x^{k(q+1)}) \cdot (P^k, x^{kq+m-k})$. Thus the ratio is bounded for all sufficiently large N; since this excludes only finitely many terms, and all the coefficients of fP^k are strictly positive (*e.g.*, because $(f, x^0) = (P^k, x^0) > 0$), the ratio of coefficients is bounded. ∎

Lemma 7.2 *Same hypotheses as Lemma 7.1. If $r > 0$ and $\ln r = -\phi'(t_0)$ for t_0 an integer, then $P^k(r) = \bigl(1 + \mathbf{o}(1)\bigr)\, r^{t_0 k} e^{k\phi(t_0)}$ (as $r \to \infty$, but restricted so that t_0 is an integer).*

Proof Set $t_0 = n$. By the first result of this section, $P(r) = \bigl(1 + \mathbf{o}(1)\bigr)\bigl(r^n \exp\phi(n) + r^{n+1}\exp\phi(n+1)\bigr)$, but it is an easy estimate (with $Y = 0$) to show the second term goes to zero compared with the first. ∎

We define a weaker version of $(*)$:

$(*/2)$ If $0 \le (f, x^j) \le (P^k, x^j)$ for all j, and $\liminf_{n\to\infty}(fP^{3k}, x^n)/(P^{4k}, x^n) = 0$, then

$$\liminf_{r\uparrow\rho(P)} f(r)/P^k(r) = 0.$$

It is trivial that this condition is weaker than $(*)$, and routine to see that $(*/2)$ is sufficient for density of the point evaluation traces. *None* of the power series that we obtain from the functions considered in this section satisfy $(*)$.

Example 7.3 Let ϕ satisfy the conditions above, and set $P = \sum \exp\bigl(\phi(j)\bigr)\, x^j$. Set $k = 4$, and $f = \lambda \circ P^4$. Then $\liminf (fP, x^n)/(P^{k+1}, x^n) = 0$, but $\inf_{r\ge 0} f/P^4(r) > 0$. In particular, $(*)$ fails.

By an earlier lemma, $fP^4 \sim P^8$, so that f/P^4 is an order unit, and thus the last property holds. On the other hand, $(fP, x^{5s+2})/(P^5, x^{5s+2}) \to 0$, as follows from the estimates above for (P^4, x^n). ∎

Theorem 7.4 *If ϕ is C^3 and satisfies $\phi'' \to -\infty$ and $\phi'''(t)$ is bounded above, then $P := \sum \exp\bigl(\phi(n)\bigr)\, x^n$ satisfies $(*/2)$. In particular, the set of point evaluation traces of R_P is dense in the pure trace space.*

Proof We show that if f satisfies $0 \leq (f, x^n) \leq (P^k, x^n)$ and $f/P^k\big|[0,\infty) > \delta > 0$, then $\lambda \circ P^k \prec f$, and it follows that $(fP^k, x^n)/(P^{2k}, x^n)$ is bounded below (away from zero). In fact, we do not use all of the hypotheses. For each integer l, set $r_l = e^{-\phi'(l)}$, so that $\ln r_l = -\phi'(l)$. Hence as $l \to \infty$, $P(r_l) = \bigl(1 + \mathbf{o}(1)\bigr)\, r^l e^{\phi(l)}$, so that $P^k(r_l) = \bigl(1 + \mathbf{o}(1)\bigr)\, r^{kl} e^{k\phi(l)}$. Thus $f(r_l) > \delta/\bigl(1 + \mathbf{o}(1)\bigr)\, r^{kl} e^{k\phi(l)}$. On the other hand, $f = (f, x^{kl}) x^{kl} \exp\bigl(\phi(kl)\bigr)$ + other terms; the sum of the values of the other terms at r_l is $\mathbf{o}(1)$ compared with the value for the kl term (since they involve $\phi(l+i)$ for some integer $i \neq 0$). Hence $(f, x^{kl}) > (P^k, x^{kl})\delta/2$ for all sufficiently large l. It follows easily that $(fP^k, x^{2kl}) > (P^{2k}, x^{kl})$ for all l and $(fP^k, x^{2kl}) > (P^{2k}, x^{kl})\delta/2$ for all sufficiently large l. Hence there exists δ_0 such that $(fP^k, x^{2kl}) > (P^{2k}, x^{kl})\delta_1$ for all l. Thus $fP^k \geq \lambda \circ P^{2k}$ (λ defined with k replaced by $2k$), so by Lemma 7.1, $fP^{3k} \sim P^{4k}$. This verifies $(*/2)$. ■

8 Some Examples

In this section, we will discuss various special cases of our previous estimates. In so doing, we run into problems about how accurate an estimate is needed for the solution t_0 to the equation $\phi'(t_0) = -\ln r$ when $r > 0$ (but less than the radius of convergence). In many cases of interest, finding t_0 exactly is hopeless, but finding a reasonable approximation is routine. Then we want to know if this approximation is good enough for the estimates to hold.

It is frequently awkward to calculate exactly the value of t_0, the solution to $\phi'(t_0) = -\ln r$. However, it is often easier to approximate it by a solution, t_1, to a simpler equation. Here we want to determine how close the approximation must be in order that we obtain asymptotically the same results. Explicitly, we have $P(r) \sim C \exp\bigl(\phi(t_0) + t_0 \ln r\bigr) \big/ \bigl(-\phi''(t_0)\bigr)^{1/2}$. We compare this to the corresponding expression with t_0 replaced by t_1. Interestingly, while it would be desirable to replace $\ln r$ by $-\phi'(t_1)$, it turns out that this is not feasible in most cases.

We expand $\phi(t_0) + t_0 \ln r - \bigl(\phi(t_1) + t_1 \ln r\bigr)$ about t_0 as usual, and obtain (from $\phi'(t_0) = -\ln r$) the expression $-(t_0 - t_1)^2 \phi''(t_0)/2$, with error term at most $F(t_0)|t_1 - t_0|^3\, |\phi''(t_0)|^{3/2}$. Hence if $|t_1 - t_0| = \mathbf{o}(\delta)$, both the expression and the error will go to zero, so that the exponential of it will result in multiplication by $1 + \mathbf{o}(1)$. The ratio $\bigl(\phi''(t_1)/\phi''(t_0)\bigr)^{1/2}$ can be estimated first by squaring, then using the crucial property (c) of FLRA functions; we obtain that the square of the ratio is $1 + \mathbf{o}\bigl(|t_1 - t_0|\bigl(-\phi''(t_0)\bigr)^{1/2}\bigr)$. Hence $|t_1 - t_0| = \mathbf{o}\bigl(\delta(t_0)\bigr)$ is sufficient for the asymptotic expression for $P(r)$ to remain valid.

A specific problem is the following. Let k be a positive real number (initially an integer), and form the function $P = \exp k/(1-x)$. This is defined everywhere, except at 1, where it has an essential singularity. Its Maclaurin series has radius of convergence one, and the coefficients are all positive; explicitly, $(P, x^N) = \sum_{j=0}^{\infty} k^j \binom{N+j-1}{j-1} / j!$. This looks like one of those horrible expressions that turns up from time to time in

https://doi.org/10.4153/CJM-2003-041-7 Published online by Cambridge University Press

papers on combinatorics or Bessel functions (to which this problem is related). We would like to get an asymptotic estimate for this (for large N and fixed k) using the preceding methods.

This is essentially the reverse of the earlier problems. We look for an eventually FLRA function ϕ of the form $\phi(t) = \alpha \ln t + \kappa t^{\beta}$, where the Greek letters are to be determined, and $0 < \beta < 1$ and $\kappa > 0$, and set $Q(x) = \sum \exp \phi(n) x^n$. For t sufficiently large, $\phi''(t) < 0$, and it follows (earlier lemma) that ϕ is eventually FLRA when $\alpha < 0$; it is FLRA if $\alpha > 0$. Since $\phi(t)/t \to 0$, Q has radius of convergence one. We wish to find conditions on the Greek letters so that $Q(r)/P(r) \to 1$ as $r \uparrow 1$.

Let r be a positive real number less than one, and define t_0 via $\phi'(t_0) = -\ln r$. Except when $\beta = 1/2$, finding an exact expression for t_0 is problematic. Instead, we attempt to use an approximation. We have $\alpha/t_0 + \kappa\beta/t_0^{1-\beta} = -\ln r$. Set t_1 to be the solution obtained by deleting the α/t_0 term; *i.e.*, $\kappa\beta/t_1^{1-\beta} = -\ln r$. Multiplying the expressions by t_0 and t_1 respectively, and taking the difference, we obtain the equation, $(t_1 - t_0)(-\ln r) - (t_1^{\beta} - t_0^{\beta})\kappa\beta = \alpha$. As $r \uparrow 1$, $-\ln r \downarrow 0$; from $|t_1 - t_0| \le (1/\beta)\,|t_1^{1/\beta} - t_0^{1/\beta}|$, we obtain $t_1^{\beta} - t_0^{\beta} \to \alpha/\kappa\beta$. Hence $|(t_1^{\beta}/t_0^{\beta}) - 1| = \mathbf{O}(t_0^{-\beta})$. Of course, as $r \to 1$, $t_0 \to \infty$. In particular, $|(t_1/t_0) - 1| \le (1/\beta)\,|(t_1^{\beta}/t_0^{\beta}) - 1| = \mathbf{O}(t_0^{-\beta})$.

Now $Q(r)$ is asymptotically $H(t_0) := C\delta(t_0)\exp\bigl(\phi(t_0) - t_0\phi'(t_0)\bigr)$ (recall that $\delta = (\phi'')^{-1/2}$); set $J(t_1) = C\delta(t_1)\exp\bigl(\phi(t_1) - t_1\phi'(t_0)\bigr)$ (note: $\phi'(t_0)$, not $\phi'(t_1)$; the latter would be more desirable, but does not yield a close enough approximation). We consider the ratio $J(t_1)/H(t_0)$. We first observe that the ratio of the exponential terms is simply the exponential of $\phi(t_0) - \phi'(t_1) - t_0\phi'(t_0) + t_1\phi'(t_0)$. Using the FLRA property, provided $|t_1 - t_0| < t_0/2$ (which of course it is for all r sufficiently close to 1, by the estimate in the preceding paragraph), this simplifies to at most $|(t_1 - t_0)^2\phi''(t_0)|$. The ratio $\delta(t_1)/\delta(t_0)$ is estimated by squaring, and we see that $\bigl(\delta(t_0)/\delta(t_1)\bigr)^2 - 1 = \mathbf{O}\bigl(|t_1 - t_0|/\delta(t_0)\bigr)$. Hence sufficient for $J(t_1)/H(t_0) \to 1$ is that $|t_1 - t_0| = \mathbf{o}\bigl(\delta(t_0)\bigr)$. However, $|\phi''(t)|$ is up to a scalar, $t^{\beta-2}$, so $\delta(t_0)$ is asymptotic with $t_0^{1-\beta/2}$. We already obtained that $|t_1 - t_0| = \mathbf{O}(t_0^{1-\beta})$, hence $|t_1 - t_0|/\delta(t_0) = \mathbf{O}(t^{-\beta/2})$. Thus $J(t_1)/H(t_0) \to 1$.

Thus $Q(r)$ is asymptotic with $C\exp\bigl(\phi(t_1) + t_1\ln r\bigr)\,\delta(t_1)$. Observing that $t_1 \ln r = -t_1^{\beta}\kappa\beta$, this expands to

$$C\exp\bigl(\kappa t_1^{\beta}(1-\beta)\bigr)\cdot\bigl(\kappa\beta/(-\ln r)\bigr)^{\alpha/(1-\beta)}\cdot t_1^{1-\beta/2}/\bigl(\beta(1-\beta)\bigr).$$

using that $-1/\ln r$ is approximately $(1-r)^{-1}$, then t_1 is replaceable by

$$\bigl((1-r)^{-1}\kappa\beta\bigr)^{1/1-\beta}.$$

We obtain the approximate expansion

$$C\exp\Bigl(\kappa^{\frac{1}{1-\beta}}\beta^{\frac{\beta}{1-\beta}}\Bigl(\frac{1}{1-r}\Bigr)^{\beta/(1-\beta)}(1-\beta)\Bigr)$$
$$\cdot\Bigl(\frac{\kappa\beta}{1-r}\Bigr)^{\frac{\alpha}{1-\beta}}\Bigl(\frac{\kappa\beta}{1-r}\Bigr)^{(1-\beta/2)/1-\beta}\Big/\beta(1-\beta)$$

https://doi.org/10.4153/CJM-2003-041-7 Published online by Cambridge University Press

$$= C \exp\Big(\frac{1}{1-r}\Big)^{\beta/1-\beta} \kappa^{1/(1-\beta)} \beta^{\beta/(1-\beta)}$$

$$\cdot \Big(\frac{1}{1-r}\Big)^{\frac{1+\alpha-\beta/2}{1-\beta}} \cdot (\kappa\beta)^{\frac{1+\alpha-\beta/2}{1-\beta}} \cdot \frac{1}{\beta(1-\beta)}.$$

Equating this to $\exp k/(1-r)$, we first obtain $\beta = 1-\beta$, *i.e.*, $\beta = 1/2$. Hence the term inside the exponential yields $\kappa^2/2 = k$, *i.e.*, $\kappa = \sqrt{k/2}$ ($\kappa > 0$). The power $1/(1-r)$ outside the exponential must be zero, whence $1+\alpha-\beta/2 = 0$, so $\alpha = -3/4$. The constant term is $2C$. We therefore obtain the corresponding $\phi(t) = -(3/4)\ln t + \sqrt{k/2}t^{1/2} - \ln(2C)$. Hence it is a good guess that

$$\sum_{j=0}^{\infty} \frac{k^j}{j!} \binom{N+j-1}{j-1}$$

is asymptotic (as $N \to \infty$ and fixed k) with $\exp\phi(N) = N^{-3/4}e^{\sqrt{kN/2}}/2C$. In fact, this can be shown by using the strong unimodularity of the sequence

$$\Big\{\frac{k^j}{j!}\binom{N+j-1}{j-1}\Big\}_{j=1}^{\infty}.$$

Variations on this are easily obtainable; suppose for instance that $P(x) = \exp k/(1-x)^a$ for some positive number a. The direct expansion yields a similar sum (expressible using gamma functions, if necessary; $(N-1)!\big((1-x)^{-b}, x^N\big) = \prod_{j=1}^{N-1}(b+j)$), while the corresponding ϕ is attainable by comparing with the expression in (3); we see that $a = \beta/(1-\beta)$, so $\beta = a/(1+a)$, then $k = \kappa^{1+a}\big(a/(1+a)\big)^a/(1+a)$, whence $\kappa = \big(k(1+a)\big)^{1/(1+a)}(1+1/a)$. Again $\alpha = \beta/2-1$, and the scalar factor can be worked out. ■

Consider the situation in which $\phi_1(t) = -\alpha \ln\Gamma(t+1)$ where α is a positive real number, and set $P_1 = \sum \exp\phi_1(n)x^n$. Since $\phi_1(t)/t \to -\infty$, P_1 is entire. If $\alpha = 1$, then P_1 is just the exponential function. We wish to obtain asymptotic expressions for $P_1(r)$ for large positive r fixed but arbitrary α. It is computationally more convenient to replace ϕ_1 by $\phi(t) = -\alpha\big(t(\ln t - 1) + (1/2)\ln 2\pi t\big)$, its Stirling approximation; let $P = \sum \exp\phi(n)x^n$. It is easy to check that for large r, $P_1(r)/P(r)$ is close to 1.

We have $\phi'(t) = -\alpha(\ln t + 1/2t)$ and $\phi''(t) = -\alpha t^{-1}(1-1/2t)$; it is straightforward to check that ϕ is FLRA. For r a (large) positive real number, let t_0 be the solution to $\phi'(t_0) = -\ln r$, and let t_1 be the solution to the simpler equation, $-\alpha\ln t = -\ln r$, *i.e.*, $t_1 = r^{1/\alpha}$. We estimate the difference $|t_1 - t_0|$; subtracting the defining equations, we obtain $0 = \alpha\ln(t_1/t_0) - \alpha/2t_0$, so $\ln(t_1/t_0) = 1/2t_0$. Hence $t_0/t_1 = e^{-1/2t_0}$, and thus $|t_0/t_1 - 1| < 1/2t_0$. Hence $|t_0 - t_1| < 3/4$, which is more than sufficient for approximation.

Now $\delta(t) = \big(1+\mathbf{o}(1)\big)(t/\alpha)^{1/2}$, so we obtain as our approximation $C\exp\big(\phi(t_1) + t_1\ln r\big)\sqrt{t_1/\alpha}$, which expands to $C'\exp(\alpha r^{1/\alpha})r^{(1/2)(1/\alpha-1)}$. The constant C' (which

https://doi.org/10.4153/CJM-2003-041-7 Published online by Cambridge University Press

depends on α) can be determined routinely, but every time I try, I get a different answer. When $\alpha = 1$, we recover the exponential function.

Now set $\phi(t) = (\ln t)^2$; the coefficients of $P := \sum \exp \phi(n) x^n$ are thus growing subexponentially, but more than polynomially. The radius of convergence is one, since $\phi(t)/t \to 0$. We check that ϕ is FLRA, and let t_0 be the solution to $\phi'(t_0) = -\ln r$, *i.e.*, $2 \ln t / t = -\ln r$. Set $s = -1/\ln r$, so the equation becomes $t_0 / 2 \ln t_0 = s$. Let $t_1 = 2s \ln s$ be the approximation.

Then $t_1 = 2(t_0/2\ln t_0)(\ln t_0 - \ln\ln t_0 - \ln 2)$. Thus $t_1/t_0 = 1 - (\ln\ln t_0 + \ln 2)/\ln t_0$, so that $|t_1 - t_0| = t_0 \big(\ln(2\ln t_0)\big) / \ln t_0$. Since $\phi''(t) = 2(1 - \ln t)/t^2$, $\delta(t_0)$ is approximately $t_0/\sqrt{2\ln t_0}$ and thus $|t_1 - t_0| = \mathbf{o}\big(\delta(t_0)\big)$. So t_1 is a good approximation for t_0. The rest is just filling in the formula estimating $P(r)$, $C\delta(t_1)\exp\big((\ln t_1)^2 + t_1 \ln r\big)$. This simplifies on replacing s by $(1-r)^{-1}$; the outcome is a function of $1/(1-r)$.

The reader may wonder whether there are similar results when the behaviour of ϕ'' is intermediate to those of the previous sections. Specifically, FLRA entails that $\phi'' \to 0$ (among other conditions), spikiness requires $\phi'' \to -\infty$. The condition that $\phi''(t)$ converge to a nonzero (negative) number is of course more natural than either of the other two, in particular because it includes the error function. In fact, the analogous results do hold, but there is a practical restriction on their applicability.

Let $P = \sum \exp \phi(n) x^n$ where ϕ is C^2, $\phi'' < 0$, and $\phi''(t) \to -1/\sigma^2$, where σ is neither zero nor infinite. Now the function $\delta := \sqrt{-1/\phi''(t)}$ remains bounded, and we may as well replace it by the constant σ. The result analogous to Theorem 5.2 is that as $r \uparrow \rho$ (the radius of convergence of P),

$$P(r) = \big(1 + \mathbf{o}(1)\big) S(\sigma) \exp\big(\phi(t_0) - t_0 \ln r\big),$$

where t_0 is the unique solution of $\phi'(t_0) = -\ln r$. This requires only a modest alteration of the arguments in Theorem 5.2 (basically $\kappa \equiv \kappa(r)$ can be chosen to go to infinity slowly enough that $\kappa^2 \cdot \big(\phi''(t_0) + 1/\sigma^2\big) \to 0$). Note that $S(\sigma)$ is just a constant, so the asymptotic behaviour is simple to describe.

The corresponding analogue of Proposition 5.4 (convolution) also holds, specifically: if $P = \sum \exp \phi(n) x^n$, then

$$(P^k, x^n) = \big(1 + \mathbf{o}(1)\big) \sqrt{\frac{\pi^{k-1}}{k}} \sigma^{k-1} \exp\Big(k\phi\Big(\frac{N}{k}\Big)\Big).$$

The practical problem referred to earlier is in the estimate for $P(r)$. This requires finding a solution t_0 to $\phi'(t_0) = -\ln r$. This is usually nasty, so that an exact solution is rarely obtainable. When FLRA holds, the first few paragraphs of this section showed that we have some flexibility in approximating t_0, *i.e.*, provided the approximation t_1 satisfies $|t_1 - t_0| = \mathbf{o}(\delta)$, then we can take t_1 in place of t_0, and since $\delta \to \infty$, we can be somewhat sloppy in our choice. However, when $\phi'' \to -1/\sigma^2$, applying the corresponding argument (to that in the beginning of this section) reveals that our approximation must satisfy $|t_1 - t_0| \to 0$ (*i.e.*, as functions of r, as $r \to \infty$). This is a much more onerous requirement on an approximate solution.

With these two results in hand, that P satisfies $(*)$ can be derived as in the proof of Theorem 6.1 (if anything, the proof of this result and the two preceding are slightly simpler than their counterparts for FLRA).

An interesting sample computation can be derived from Example 4.8. In that weird example, $Q^{-1} = \sum c(n)x^n$ where the coefficients satisfy the recurrence relation $c(n) = c(n-2) + c([n/2])$. The continuous form of this difference equation is the scale-changing differential equation (for lack of a better description) $2C'(2r) = C(r)$, with initial condition $C(0) = 1$. My colleague Victor Leblanc pointed out that this has an entire solution, given by

$$C(x) = \sum \frac{x^n}{2^{\binom{n+1}{2}} n!}.$$

Setting

$$\phi(t) = -\frac{t(t+1)}{2}\ln 2 - t(\ln t - 1) - \frac{1}{2}\ln(2\pi t),$$

we see that the coefficients of $\sum e^{\phi(n)}x^n$ are asymptotic with those of C (the $n!$ term has been replaced by its first order Stirling approximation), and moreover ϕ is concave, and $\phi''(t) \to -\ln 2$. We can thus use the result above to estimate $C(r)$; this in turn might be useful to estimate $c(n)$ (the estimate in Example 4.8 is obtained directly via discrete techniques).

Here $\rho = \infty$, $\sigma = \sqrt{1/\ln 2}$, and $\phi' = -t\ln 2 - \ln t - (\ln 2)/2 - 1/(2t)$. We try the approximation

$$t_1 = \frac{\ln r}{\ln 2} - \frac{\ln\ln r}{\ln 2} - \left(\frac{1}{2} - \frac{\ln\ln 2}{\ln 2}\right) + \frac{\ln\ln r}{\ln r \ln 2} + \left(\frac{1}{2\ln 2 - \frac{\ln\ln 2 + 1/2}{(2\ln 2)^2}}\right)\frac{1}{\ln r}$$

(obtained by looking for a solution of the form $\alpha \ln r + \beta \ln\ln r + \gamma + \lambda \ln\ln r/\ln r + \mu/\ln r$, in turn derived from Newton's method). Using the simple-minded approximation for $\ln(a-b) = \ln a - b/a + \mathbf{O}\big((b/a)^2\big)$ when b is much smaller than a, we see that $\phi'(t_1) + \ln r = \mathbf{O}\big((\ln\ln r/\ln r)^2\big)$. Subtracting this from $\phi'(t_0) + \ln r = 0$, we obtain $\phi'(t_1) - \phi'(t_0) = \mathbf{O}\big((\ln\ln r/\ln r)^2\big)$. As $r \to \infty$, both t_1 and t_0 both go to infinity. Now $|t_1 - t_0|\phi''(t_2) = \mathbf{O}\big((\ln\ln r/\ln r)^2\big)$ for some t_2 between t_1 and t_0, which thus goes to infinity, and as $\phi'' \to -1/\sigma^2$, we obtain that $|t_1 - t_0| = \mathbf{O}\big((\ln\ln r/\ln r)^2\big)$. So our choice for t_1 can be substituted into the formula; essentially, this amounts to $\phi(t_1) + t_1 \ln r$.

We obtain that up to multiplication by the (computable) constant S, $C(r)$ is $\exp\big(\phi(t_1) + t_1\ln r\big)$; plugging this in and converting to base 2, we obtain:

$$\begin{aligned}\frac{\ln_2 C(r)}{\ln_2 r} &= \frac{\ln_2 r}{2} - \ln_2\ln_2 r + \frac{1}{\ln 2} - \frac{1}{2}\\ &\quad + \frac{(\ln\ln 2)^2}{2(\ln 2)^2} + \frac{(\ln_2\ln_2 r)^2}{2\ln_2 r} + \frac{h + \ln S - \frac{(\ln\ln 2)^2}{\ln 2}}{\ln 2\ln_2 r} + \mathbf{O}\left(\frac{(\ln\ln r)^3}{(\ln r)^2}\right)\end{aligned}$$

where h is a constant, approximately -1.764. In base e, there was a term of the form $\ln\ln r/\ln r$, but this has miraculously disappeared in base 2. Note the agreement in the first two terms with the estimate in the discrete case (Example 4.8).

9 Miscellany

A number of questions arise from the results in this paper.

Trace Space

It is plausible, especially in view of Proposition 3.7, that for $P = (1-x)^{-1}$, the pure trace space of R_P is naturally homeomorphic to the Stone-Čech compactification of $[0, 1)$. This is equivalent to every bounded real-valued continuous function on $[0, 1)$ being uniformly approximated by elements of R_P. Here of course, a typical element of R_P can be written in the form $(1-x)^m q$ where m is a positive integer, and $(q, x^k) = \mathbf{O}(k^{m-1})$. If the pure trace space of this R_P is $\beta[0, 1)$, then so is the pure trace space of R_Q for every Q such that some power of Q has increasing coefficients and the set of point evaluation traces of R_Q is dense in the trace space. The following is a step in this direction.

Proposition 9.1 *Let $P = (1-x)^{-1}$. Let S and T be subsets of $[0, 1)$ each of which has 1 as its only limit point, and for which both $S \setminus T$ and $T \setminus S$ are infinite. Then there exists $s(1) < s(2) < \cdots$ in S and $t(1) < t(2) < \cdots$ with $s(i) \uparrow 1$ and $t(i) \uparrow 1$, together with a Maclaurin series f in R_P such that $|f(r)| \leq 1$ for all r in $[0, 1)$ and $f\big(s(i)\big) \geq 1/2$, and $f\big(t(i)\big) \leq -1/2$ for all i.*

We require a preliminary estimate.

Let $m(1) < m(2) < \cdots$ be an increasing sequence of positive integers (typically with $m(i+1)/m(i) \geq 2$), and set $f = 1 + 2\sum_{j=1}^{\infty}(-1)^j x^{m(j)}$. We note that the coefficients of $(1-x)^{-1}f$ are all ± 1, so that $(1-x)^{-1}f \prec (1-x)^{-1}$. Hence if some power of P has all of its coefficients increasing (or merely if P is equivalent to such a power series), then f belongs to R_P, no matter what the choice of $\{m(j)\}$.

Let κ be a positive real number less than 1. Suppose that $\{r_j\}$ is a sequence of real numbers increasing up to 1, such that for all j, we have

$$(\dagger) \qquad \frac{\kappa}{m(j)} > -\ln r_j > \frac{\ln \frac{1}{\kappa}}{m(j+1)}.$$

We estimate $f(r_j)$. Exponentiating and manipulating the inequalities, we have $r_j^{m(j)} > e^{-\kappa}$ and $r_j^{m(j+1)} < \kappa$. The sequence $\{2(-1)^k r_j^{m(k)}\}$ is alternating and the terms are decreasing in absolute value. Hence $\sum_{k \geq j+1} 2(-1)^k r_j^{m(k)}$ has absolute value less than 2κ.

Now consider the initial segment of the expansion of $f(r_j)$. Write $\alpha_i = r_j^{m(i)} := 1 - \epsilon(i)$. As $\alpha_i \geq \alpha_{i+1}$, it follows that $\epsilon(i) \leq \epsilon(i+1)$. Set $S_j = \sum_{i=1}^{j} \alpha_i (-1)^{i+1}$. It is easy to check (by considering the case of j odd, then j even), that $0 \leq S_j \leq \epsilon(j)$ if j is even, and $1 \geq S_j \geq 1 - \epsilon(j)$ if j is odd. It follows that $|1 - 2S_j - (-1)^j| \leq 2\epsilon(j)$. As $1 - 2S_j$ is the initial segment of $f(r_j)$, we deduce that $|f(r_j) - (-1)^j| < 2\kappa + 2\epsilon(j)$.

Now $\epsilon(j) = 1 - r_j^{m(j)}$, and as $r^j > e^{-\kappa}$, it follows that $\epsilon(j) < 1 - e^{-\kappa}$, and as $\kappa < 1$, this is less than κ. Hence $|f(r_j) - (-1)^j| < 4\kappa$. Thus if κ is chosen less than

https://doi.org/10.4153/CJM-2003-041-7 Published online by Cambridge University Press

one quarter, we deduce f is positive on $\{r_{2s}\}$ and negative on $\{r_{2s+1}\}$; if $\kappa < 1/8$, f is at least $1/2$ on $\{r_{2s}\}$ and at most $-1/2$ on $\{r_{2s+1}\}$. These estimates can be refined (we have used fairly crude approximations), but there is no obvious reason to do so.

Proof of Proposition 9.1 We construct sequences $\{r_j\}$ and $\{m(j)\}$ satisfying (†) above (for any fixed, small κ), so that r_{2j} belong to $S \setminus T$ and r_{2j-1} belong to $T \setminus S$. Let $t(1)$ be any element of $T \setminus S$ such that $\kappa / \ln\bigl(1/t(1)\bigr)$ exceeds one (possible since $\ln(1/r) \to 0$ as $r \uparrow 1$), set $r_1 = t(1)$, and let $m(1)$ be any integer less than $\kappa / ln(1/r_1)$. Now let $m(2)$ be any integer exceeding $m(1)$ such that $m(2) > \ln(1/\kappa)/\ln(1/r_1)$. Next, there exists $s(1)$ in $S \setminus T$ such that $\ln\bigl(1/s(1)\bigr) < \kappa/m(2)$ (again since $\ln(1/r) \to 0$ as $r \uparrow 1$); set $r_2 = s(1)$. Next choose $m(3)$ to be any integer exceeding $\ln(1/\kappa)/\ln(1/r_2)$, and this process obviously can be continued, alternating between elements of $S \setminus T$ and $T \setminus S$. ∎

Laurent Power Series

If $P = \sum_{j \in \mathbf{Z}} x^j \exp \phi(j)$ has radii of convergence $r \geq 0$ and $r < R \leq \infty$, practically all the results here apply—the only modification is that we require $\phi \colon \mathbf{R} \to \mathbf{R}$ to be smooth everywhere (and the definitions of FLRA and LRA be modified accordingly). More interesting is when the coefficients of the negative exponents are given by a different expression, *e.g.*, for which the Laurent series converges at r from the right (this occurs with many log convex distributions, see [H1]). This presents some complications.

Several Variables

Some of the results extend without much difficulty, others do not extend at all. The critical condition in the definition of FLRA has to be replaced by the restrictive condition

$$\|D^2(\phi)(t+s) - D^2(\phi)(t)\| \leq \|s\| F(t) H(-\phi)(t)^{3/2},$$

where $H(f)$ is the Hessian, $\det D^2 f$ (we also require that $D^2(\phi)$ be negative definite). Let supp P be a convex (lattice) cone in $\mathbf{N}^d$. Provided v in $\mathbf{N}^d$ is "well-inside" the interior of supp P, estimates for (P^k, x^v) (using monomial notation) can be obtained by similar techniques (summing over lattice points in an ellipsoid) to those of the one variable case, as can estimates for $P(r)$, where $D\phi(v_0) = -\ln r$ ($\ln r$ is an element of $\mathbf{R}^d$). Unfortunately, near the boundary of supp P, the estimates are more complicated (right on the boundary, they are obtainable by reductions to lower dimensional situations). A weakened version of ($*$)—where we only require the exponents v_i to satisfy $v_i/\|v_i\|$ has limit points only in the interior of the convex hull of supp P—holds, but this does not appear to be sufficient to show that the set of point evaluations (corresponding to the positive real points in the domain of holomorphy) is dense in the trace space of R_P.

Curiously, if we consider Laurent power series in several variables (with supp $P = \mathbf{Z}^d$), there is no boundary (except at infinity, with which we can deal), and the results, suitably modified, carry through.

Appendix

A Quantitative Results—Fun with Chebyshev Polynomials

A particular consequence of Lemma 3.5(C) is that if $Q = (x-z)(x-\overline{z})$ for z a complex number of modulus at least one, then there exists an integer n such that $(1-x)^{-n}Q^{-1}$ has no negative Maclaurin series coefficients. How big must n be? We show here by completely elementary means that $n = 2$ is sufficient, and $n = 1$ works for a large set of zs. However, when $|z| = 1$, the set of such for which $n = 1$ is sufficient is very sparse. Throughout this section, we write $z = re^{i\theta}$ where $0 \le \theta \le \pi$.

Theorem A.1 *If $Q = (x-z)(x-\overline{z})$ where $|z| = 1$ and $(1-x)^{-1}Q^{-1}$ has no negative coefficients in its Maclaurin series, then* $\arg z = 2\pi/m$ *for some integer m.*

The converse is also true, but less interesting. It is important to observe that not all roots of unity work—only those closest to 1 among their primitive brethren of the same order.

The set of z for which $(1-x)^{-1}Q^{-1}$ has no negative Maclaurin series coefficients is a complicated but tractable set. For each positive integer N, define the function $\phi_N = \frac{\sin(N+1)\theta}{\sin\theta}$; as is well-known, $\phi_N(\theta) = \chi_N(2\cos\theta)$ where χ_N is the Chebyshev polynomial of degree N. Moreover, $\phi_N \cdot \phi_1 = \phi_{N+1} + \phi_{N-1}$ (by convention, $\phi_{-1} = 0$), and $\phi_N \cdot \phi_M = \sum_{i=0}^{N+M-|N-M|} \phi_{N+M-2i}$.

With $z = re^{i\theta}$,

$$\frac{1}{(1-x/z)(1-x/\overline{z})} = \sum \frac{\phi_j}{r^j} x^j.$$

The easiest way to see this is by multiplying it out, but it can also be obtained by partial fractions, or the two-term recursion, or by two by two matrix techniques, *etc.* The left side is r^2/Q. Hence

$$\left(\frac{1}{1-x}\frac{r^2}{Q}, x^N\right) = \sum_{j=0}^{N} \frac{\phi_j}{r^j} := \psi_N.$$

Set $q = r^2 - r\phi_1 + 1$; then $q = |1-z|^2$, hence is positive, except in the uninteresting case that $z = 1$. It is easy to check by induction that $r^N \psi_N q = r^{N+2} - r\phi_{N+1} + \phi_N$. In particular, $(1-x)^{-1}Q^{-1}$ has no negative Maclaurin coefficients if and only if $\psi_N(r,\theta) \ge 0$ for all N. This is merely a restatement of the problem—however, a surprising result simplifies it considerably.

Theorem A.2 *Suppose $z = re^{i\theta}$ with $r \ge 1$ and $2\pi/(n+2) \le \theta \le 2\pi/(n+1)$ for some integer n. Then $(1-x)^{-1}Q^{-1}$ has no negative coefficients in its Maclaurin expansion if and only if $r^{n+2} - r\phi_{n+1}(\theta) + \phi_n(\theta) \ge 0$.*

In other words, to test positivity of *all* the coefficients, we need only check the nth, where n is determined by θ (there is ambiguity if θ is of the form $2\pi/k$, but then either choice will do). When $n = 1$, we require $\pi \ge \theta \ge 2\pi/3$, and the condition is simply that $r + 2\cos\theta \ge 0$ (*i.e.*, outside the circle of radius 1 centred at $(-1,0)$);

when $n = 2$, the condition is just $r^2 + 2r\cos\theta + 4\cos^2\theta - 1 \geq 0$ and $2\pi/3 \leq \theta \leq \pi/2$ (the boundary of this resembles an arc of a slightly distorted lemniscate). [For small values of n, it is easier to work with $r^N\psi_N$ rather than $r^N\psi_N q$.] Easy consequences of Theorem A.2 include the following.

Lemma A.3 *For $z = re^{i\theta}$ with $r \geq 1$ and $0 \leq \theta \leq \pi$, sufficient for all Maclaurin series coefficients of $(1-x)^{-1}Q^{-1}$ to be nonnegative, are any of the following conditions:*

(i) $r \geq 2$;
(ii) $r \geq 2/\sqrt{3}$ *and* $\theta \leq 2\pi/3$;
(iii) $r \geq 1.09$ *and* $\theta \leq \pi/2$;
(iv) $r \geq 1.044$ *and* $\theta \leq 2\pi/5$.

It comes as no surprise that $(1-x)^{-2}Q^{-1}$ has no negative coefficients regardless of the choice of z (of modulus at least 1). Hence if $P(x)$ is a real polynomial of degree d with no roots in the open unit disk, then $(1-x)^{-d}P^{-1}$ has no negative Maclaurin coefficients. This can be sharpened in view of Lemma A.3, *e.g.*, if all roots of P have modulus at least 2, then $[(d+1)/2]$ can replace d in the exponent.

Proof of Theorem A.1 When $|z| = 1$ (*i.e.*, $r = 1$), $\psi_N \cdot q = 1 - \phi_{N+1} + \phi_N$. By the difference formula for sines, $\phi_{N+1} - \phi_N = \cos\big((N + \frac{3}{2})\theta\big) / \cos\frac{1}{2}\theta$. Hence $\psi_N \geq 0$ if and only if $\cos\theta/2 \geq \cos(2N+3)\theta/2$ (since $0 \leq \theta \leq \pi$; at $\theta = \pi$, $\phi_k = k+1$, and the original inequality holds). If θ is an irrational multiple of π, then $\{(2N+3)\theta/2\}$ is dense in $[0, 2\pi]$ modulo 2π; but $\cos\theta/2 < 1$, whence there exists N for which the inequality fails.

Otherwise, if $\theta = 2\pi a/b$ for relatively prime positive integers a and b, then $(2k+3)a/2b$ has a minimal value modulo 2 as k varies, and it is $1/2b$ if a is odd, and 0 else. In the former case, we would require $\cos\pi a/b \geq \cos\pi/b$, which forces $a = 1$. In the latter case, we obtain a contradiction immediately. ■

Proof of Converse to Theorem A.1 If $\theta = 2\pi/n$, then $\min_k(2k+3)/2n$ is $1/2n$, *etc.* ■

Corollary A.4 *Suppose $z = \exp 2\pi i a/b$ with $z \neq \pm 1$ and a and b relatively prime positive integers. Then $(1-x^b)/(1-x)Q$ has no negative Maclaurin coefficients if and only if $a = 1$; moreover, the coefficient of x^k is zero when $k \equiv -1, -2 \mod b$.*

Proof Since $Q(z) = 0$, Q divides $1 - x^b$, and we may write $1 - x^b = (1-x)Q \cdot p(x)$ (where p is another polynomial with real coefficients). Thus $1/(1-x)Q = p/(1-x^b)$. We expand the latter as $p \cdot \sum_t x^{bt}$; since the degree of p is $b - 3$, we have that p is the initial segment of $1/(1-x)Q$ (up to the degree $b-3$ term). Hence p has no negative coefficients if and only if $1/(1-x)Q$ has none. ■

The proof of Theorem A.2 is divided in three parts. The first is easy. Set $\eta_N = r^N\psi_N q = r^{N+2} - r\phi_{N+1} + \phi_N$.

Lemma A.5 *If $r_0 \geq 1$ and $0 \leq \theta_0 \leq \pi$ and for some integer N, $\psi_N(r_0, \theta_0) \geq 0$, then for all $r \geq r_0$, $\psi_N(r, \theta_0) \geq 0$.*

Proof We note that $\partial \eta_N / \partial r = (N+2) r^{N+1} - \phi_{N+1}$. As $|\phi_{N+1}(\theta)| \leq N+2$ for all θ, it follows that $\partial \eta_N / \partial r \geq 0$; obviously all higher order partials (with respect to r) are nonnegative, hence η is increasing as a function in r for fixed θ. Thus $\eta_N(r, \theta_0) \geq 0$ and so $\psi_N(r, \theta_0) \geq 0$. ■

Lemma A.6 *If $r \cos\theta \geq 1$, then all Maclaurin coefficients of $(1-x)^{-1} Q^{-1}$ are nonnegative.*

Proof It is sufficient to show $\eta_N(1/\cos\theta, \theta) \geq 0$ for all N when $\theta < \pi/2$. A simple computation yields

$$\cos^{M+2}\theta \cdot \eta_N(1/\cos\theta, \theta) = 1 - \cos^{M+1}\theta \cos(M+1)\theta \geq 0. \quad ■$$

The corresponding result for $r \sin\theta$ (the y coordinate) fails. The biggest value of y required is about 1.116.

The next result yields that for θ in the interval $I_M := [2\pi/(M+2), 2\pi/(M+1)]$, we need only consider the behaviour of η_N for $N \geq M$ (*i.e.*, for small values of N, nonnegativity of η_N comes for free if $r \geq 1$).

Lemma A.7 *If $0 \leq \theta \leq 2\pi/(N+2)$, then $1 - \phi_{N+1}(\theta) + \phi_N(\theta) \geq 0$.*

Proof As $\phi_{N+1} - \phi_N = \cos\frac{(2N+3)\theta}{2} / \cos\frac{\theta}{2}$, it is sufficient to show that either $\cos\frac{(2N+3)\theta}{2} \leq 0$ or $0 \leq \cos\frac{(2N+3)\theta}{2} \leq \cos\frac{\theta}{2}$. Sufficient for the first is $\frac{\pi}{2} \leq \frac{(2N+3)\theta}{2} \leq \frac{3\pi}{2}$, *i.e.*, $\frac{\pi}{(2N+3)} \leq \theta \leq \frac{3\pi}{(2N+3)}$. Otherwise, set $\Phi(\theta) = 1 - \phi_{N+1}(\theta) + \phi_N(\theta)$. We calculate

$$\begin{aligned}
\Phi'(\theta) &= \frac{-\sin\frac{\theta}{2} \cdot \cos\frac{2N+3}{2}\theta + (N+1)\cos\frac{\theta}{2} \cdot \sin\frac{2N+3}{2}\theta}{2\cos^2\frac{\theta}{2}} \\
&= \frac{(N+1)\sin\frac{2N+3}{2}\theta}{\cos\frac{\theta}{2}} + \frac{\sin(N+1)\theta}{2\cos^2\frac{\theta}{2}}.
\end{aligned}$$

For $0 \leq \theta \leq \pi/(2N+3)$, each of the two constituents is positive, whence $\Phi'(\theta) \geq 0$. Thus Φ is increasing on the interval, so that $\Phi(\theta) \geq \Phi(0) = 0$.

For $3\pi/(2N+3) \leq \theta \leq 2\pi/(N+2)$, we have $3\pi \leq (2N+3)\theta/2 \leq 4\pi$, so that each constituent of $\Phi'(\theta)$ is negative. Since $\Phi\big(3\pi/(2N+3)\big) \geq 0$ there exists at most one zero of Φ in this interval. However,

$$\Phi\Big(\frac{2\pi}{N+2}\Big) = 1 - \frac{\cos\big(\frac{2N+3}{2} \cdot \frac{2\pi}{N+2}\big)}{\cos\frac{\pi}{N+2}} = 1 - \frac{\cos(2\pi - \frac{\pi}{N+2})}{\cos\frac{\pi}{N+2}} = 0.$$

Hence Φ is nonnegative on this interval, and thus on $[0, 2\pi/(N+2)]$. ■

Lemma A.8 *For $r \geq 1$ and $0 \leq \theta \leq 2\pi/(N+2)$, $\eta_N(r, \theta) \geq 0$.*

https://doi.org/10.4153/CJM-2003-041-7 Published online by Cambridge University Press

Proof By Lemma A.7, $\eta_N(1,\theta) \geq 0$, and the result follows from Lemma A.5. ■

The following weird identity underlies the surprising aspect of the results here, namely that for a specific value of θ, only one of the coefficients has to be checked for nonnegativity. Set $R_N = r^N \psi_N$ (this clears the denominators of ψ_N, resulting in a monic polynomial of degree N with constant term ϕ_N).

Lemma A.9 *For $N \geq M+2$,*

$$R_N = R_M \cdot \phi_{N-M} + (r^{M+1} - R_{M-1})R_{N-M-1} + rR_{M-1}R_{N-M-2}.$$

Proof The proof is by induction, and is no more difficult than any of the other induction arguments, but is far more prone to bookkeeping errors, so is included. First, when $N = M+2$, the right side is

$$\begin{aligned} &R_M\phi_2 + (r^{M+1} - R_{M-1})R_1 + rR_{M-1} \cdot 1 \\ &\qquad = R_M\phi_2 + r^{M+1}(r+\phi_1) + R_{M-1} \cdot \bigl(r - (r+\phi_1)\bigr) \\ &\qquad = R_M\phi_2 + r^{M+2} + r^{M+1}\phi_1 - R_{M-1}\phi_1. \end{aligned}$$

Now calculate the coefficient of r^k in this; for $k = M+2$, $M+1$, and M, respectively the outcomes are $1, \phi_1, \phi_2$, and for general $k = M-i$, the outcome is $\phi_2\phi_i - \phi_1\phi_{i+1} = \phi_{i+2}$. Hence the expression is R_{M+2}, as desired.

Now we suppose that $N \geq M+2$, and verify the induction. We repeatedly exploit the identity $R_k = rR_{k-1} + \phi_k$.

$$\begin{aligned} &R_{N+1} - R_M\phi_{N-M+1} - (r^{M+1} - R_{M-1})R_{N-M} - rR_{M-1}R_{N-M-1} \\ &\quad = rR_N + \phi_{N+1} - R_M\phi_{N-M+1} - (r^{M+1} - R_{M-1})(rR_{N-M-1} + \phi_{N-M}) \\ &\qquad - rR_{M-1}(rR_{N-M-2} + \phi_{N-M-1}) \\ &\quad = r\bigl(R_N - (r^{M+1} - R_{M-1})R_{N-M-1} - rR_{M-1}R_{N-M-2}\bigr) + \phi_{N+1} - R_M\phi_{N-M+1} \\ &\qquad - (r^{M+1} - R_{M-1})\phi_{N-M} - rR_{M-1}\phi_{N-M-1} \\ &\quad = rR_M\phi_{N-M} - R_M \cdot \phi_{N-M+1} - r\phi_{N-M-1}R_{M-1} \\ &\qquad + R_{M-1}\phi_{N-M} + \phi_{N+1} - r^{M+1}\phi_{N-M} \\ &\quad = R_{M-1} \cdot \bigl(r^2\phi_{N-M} - r(\phi_{N-M+1} + \phi_{N-M-1}) + \phi_{N-M}\bigr) \\ &\qquad + r\phi_{N-M}\phi_M - \phi_M\phi_{N-M+1} + \phi_{N+1} - r^{M+1}\phi_{N-M} \\ &\quad = \phi_{N-M}R_{M-1} \cdot (r^2 - r\phi_1 + 1) + \phi_{N+1} - r^{M+1}\phi_{N-M} \\ &\qquad + r\phi_{N-M}\phi_M - \phi_M\phi_{N-M+1} \end{aligned}$$

$$
\begin{aligned}
&= \phi_{N-M} \cdot (r^{M+1} - r\phi_M + \phi_{M-1}) - r^{M+1}\phi_{N-M} \\
&\qquad + r\phi_{N-M}\phi_M - \phi_M\phi_{N-M+1} + \phi_{N+1} \\
&= \phi_{N+1} + \phi_{M-1}\phi_{N-M} - \phi_M\phi_{N-M+1} \\
&= 0.
\end{aligned}
$$

(The induction hypothesis is used in going from the second equality to the third.) ■

The corresponding relation for the ψ_N is

$$\psi_N = \psi_M \cdot \frac{\phi_{N-M}}{r^{N-M}} + \left(1 - \frac{\psi_{M-1}}{r^2}\right)\psi_{N-M-1} + \frac{\psi_{M-1}\psi_{N-M-1}}{r^2}.$$

Proof of Theorem A.2 For M a positive integer, let C_M denote the curve in $\mathbf{R}^2$ given by $\eta_M(r,\theta) = 0$ restricted to θ in I_M and $r \geq 1$. We note that $\eta_M(1,\theta) < 0$ on this interval except at the endpoints, where it is zero (Theorem A.1). Moreover, since $\partial^i\eta_M/\partial r^i(r,\theta) \geq 0$ for all $i \geq 1$ and $r \geq 1$, for each θ_0 in I_M, there exists unique $r_0 \geq 1$ such that $\eta_M(r_0,\theta_0) = 0$. It is sufficient by Lemma A.5 to show that $\eta_N|C_M \geq 0$ for all integers N.

If $N \leq M-1$, by Lemma A.8, η_N is nonnegative on $r \geq 1$ and $0 \leq \theta \leq 2\pi/N+2$; as $2\pi/(N+2) \geq 2\pi/(M+1)$, this region includes C_M. Obviously $\eta_M|C_M = 0$. If $N = M+1$, we note that $\psi_{M+1} = \psi_M + \phi_{M+1}/r^{M+1}$; restricted to C_M, this is $\sin(M+2)\theta/r^{M+1}\sin\theta$, but this is clearly positive for θ in $I_M = \left[\frac{2\pi}{M+2}, \frac{2\pi}{M+1}\right]$.

If $N \geq M+2$, we apply the identity of Lemma A.9 (recalling that ψ, η, R are all in positive ratios to each other). Inductively, R_{N-M-1} and R_{N-M-2} are nonnegative on C_M, and $r^{M+1} - R_{M-1} = r^{M+2} + \phi_M - R_M$, so is also nonnegative on C_M (as R_M restricts to zero, and ϕ_M is nonnegative there), so by Lemma A.9, R_N is nonnegative on C_M. Thus η_N (and ψ_N) are nonnegative on C_M. ■

With $M = 1$, $2\pi/3 \leq \theta \leq \pi$, then necessary and sufficient for *all* the Maclaurin coefficients of $(1-x)^{-1}Q^{-1}$ to be nonnegative is simply that $R_1(r,\theta) \geq 0$ (we can move freely between η_N, R_N, and ψ_N), and this is simply $r + 2\cos\theta \geq 0$, which describes the outside of the disk of radius one centred at $(-1,0)$. With $M = 2$ (so $\pi/2 \leq \theta \leq 2\pi/3$), the condition is $(r+\cos\theta)^2 \geq 1 - 3\cos^2\theta$.

For each interval I_M, we wish to find the minimal value of r, denoted $r(M)$, such that $R_M(r,\theta) \geq 0$ for all θ in I_M. Of course, $r(M)$ is the maximal value of r on the curve C_M (the zero set of η_M, and also of R_M and of ψ_M). By Theorem A.2, if $r \geq r(M)$ and θ belongs to I_M, then the Maclaurin coefficients of $(1-x)^{-1}Q^{-1}$ are all nonnegative. We shall actually improve this—by means of very fine estimates for $r(M)$, we shall show that $\{r(M)\}$ is strictly decreasing (and then by Lemma A.6, it follows that the limit is 1). We will thereby obtain that if $0 \leq \theta \leq 2\pi/(M+1)$ and $r \geq r(M)$ for some value of M, then the Maclaurin coefficients of $(1-x)^{-1}Q^{-1}$ are nonnegative.

A simple-minded way to determine $r(M)$ is to regard r as defined implicitly on I_M via $\eta_M(r,\theta) = 0$, and maximize $r(\theta)$; the critical point (θ_M) occurs for which

$r = \frac{\partial \phi_M}{\partial \theta} / \frac{\partial \phi_{M+1}}{\partial \theta}$, and in principal this can be substituted back into $\eta_M = 0$ to solve for $r(M)$. Unfortunately, this method is effective only for very small values of M, and then using R_M instead of η_M. For example, when $M = 1$, we obtain the obvious solution, $r(1) = 2$; when $M = 2$, $r(2) = 2/\sqrt{3}$ (and $\theta_2 = \arccos(-1/\sqrt{12})$, whatever that is).

By using elementary approximation techniques, we can show $r(M)-1 = O(M^{-3})$, but sharper results are needed to show $r(M) > r(M+1)$. To obtain an upper bound for $r(M)$, we rewrite

$$R_M = r^{M+2} - (r-1)\phi_{M+1} - \frac{\cos \frac{2M+3}{2}\theta}{\cos \frac{\theta}{2}}$$

(obtained by replacing ϕ_M by ϕ_{M+1} plus the difference, and using the difference formula for sines). On I_M, $\phi_{M+1} = \bigl(\sin(M+2)\theta\bigr) / \sin\theta$ varies from zero to one; on the other hand, the right most term in the displayed expression hardly varies at all. Now we observe that for $M \geq 2$,

$$\frac{\cos \frac{2M+3}{2}\theta}{\cos\theta/2} \leq \frac{1}{\cos\theta/2} \leq \frac{1}{\cos\frac{\pi}{M+1}} \leq \frac{1}{1-\pi^2/2(M+1)^2}$$

(This is fairly crude for small values of M; the lower bound we obtain later will be closer to the actual value.)

We claim that if $r_1 = 1 + \pi^2/(M+1)^2(2M+1)$, then $\eta_M(r_1,\theta) > 0$ for all θ in I_M, and this entails $r(M) \leq r_1$. For ease of notation, let $\epsilon = r_1 - 1$ and $\theta_1 = \pi/(M+1)$. Then for θ in I_M,

$$\eta_M(r,\theta) \geq r^{M+2} - (r-1) - \frac{1}{\cos\theta_1}, \quad \text{whence}$$

$$\begin{aligned}
\cos\theta_1 \cdot \eta_M(1+\epsilon,\theta) &\geq \cos\theta_1 \cdot \bigl(r_1^{M+2} - (r_1-1) - 1\bigr) \\
&= \cos\theta_1 \cdot \bigl((1+\epsilon)^{M+2} - \epsilon\bigr) - 1 \\
&\geq \left(1 - \frac{\theta_1^2}{2}\right)\left(1 + (M+2)\epsilon + \binom{M+2}{2}\frac{\epsilon^2}{2} - \epsilon\right) - 1 \\
&= \frac{\epsilon}{2}\left(1 - \epsilon(M^2-1) - \epsilon^2\frac{(2M+1)(M+2)(M+1)}{4}\right) \\
&= \frac{\epsilon}{2}\left(1 - \frac{\pi^2(M-1)}{(M+1)(2M+1)} - \frac{\pi^4(M+2)}{4(M+1)^3(2M+1)}\right).
\end{aligned}$$

The last expression is nonnegative if $M \geq 3$ ($\pi^2 \sim 10$); the case of $M = 2$ has already been calculated.

Next we claim that if $\theta_0 = 4\pi/(2M+3)$ (almost, but not quite, the midpoint of I_M), and $r_0 = 1 + (2\pi)^2/(2M+3)^3$, then $\eta_M(r_0,\theta_0) < 0$, from which it would follow that $r(M) > r_0$. Again, write $\epsilon = r_0 - 1$. A simple computation reveals that $\phi_{M+1}(\theta_0) = 1/2\cos\theta_0/2$ (of course, θ_0 was chosen for this purpose), so that

$\eta_M(r_0, \theta_0) = (1+\epsilon)^{M+2} - (1+\epsilon/2)/\cos\theta_0/2$. As $-(1+\epsilon/2) < -(1+\epsilon)^{1/2}$, we deduce that

$$\eta_M(r_0,\theta_0) < \left((1+\epsilon)^{\frac{2M+3}{2}} - \frac{1}{\cos\theta_0/2}\right)(1+\epsilon)^{1/2}.$$

It suffices to show $(1+\epsilon)^{-(2M+3)/2} > \cos\theta_0/2$.

The expansions (regrettably involving more than the usual numbers of terms)

$$\begin{aligned}(1+\epsilon)^{-\frac{2M+3}{2}} &> 1 - \frac{2M+3}{2}\epsilon + \frac{2M+3}{2}\left(\frac{2M+1}{2}\right)^2\frac{\epsilon^2}{2} \\ &\quad - \frac{2M+3}{2}\,\frac{2M+1}{2}\,\frac{2M-1}{2}\,\frac{\epsilon^3}{6} \\ \cos\theta_0/2 &\le 1 - \frac{\theta_0^2}{8} + \frac{\theta_0^4}{16\cdot 24},\end{aligned}$$

yield (on subtracting the second from the first)

$$(1+\epsilon)^{-\frac{2M+3}{2}} - \cos\theta_0/2 > \epsilon^2(2M+3)\left(\frac{M}{6} - \frac{\epsilon(4M^2-1)}{48}\right).$$

Sufficient for this to be positive is $8M > (2\pi)^2(4M^2-1)/(2M+3)^3$, which holds for $M \ge 2$.

Proposition A.10 *For $M \ge 3$, we have*

$$\frac{(2\pi)^2}{(2M+3)^3} \le r(M) - 1 \le \frac{\pi^2}{(M+1)^2(2M+1)}.$$

Moreover, $r(M) > r(M+1)$.

Proof The inequalities on the top line were proved above; the latter one is an immediate consequence. ∎

A more æsthetic upper bound for $r(M) - 1$ would be $\pi^2/2(M+1)^3$, but this cannot be obtained from the crude methods above. The lower bound is much closer to $r(M) - 1$ than the upper bound, largely because the specific θ_0 is very close to the critical value.

Now we can show that for any choice of z outside the open unit disc, $(1-x)^{-2}Q^{-1}$ has no negative Maclaurin coefficients. In view of the preceding, it should be fairly easy—however, this section of the appendix required far more time than all the rest of it.

As $q \cdot \big((1-x)^{-1}Q^{-1}, x^k\big) = r^2 - \phi_{k+1}/r^{k-1} + \phi_k/r^k$, we find

$$\begin{aligned}
F_N(r,\theta) &:= q \cdot \big((1-x)^{-2}Q^{-1}, x^N\big) \\
&= q \cdot \sum_{j=0}^{N} \big((1-x)^{-1}Q^{-1}, x^j\big) \\
&= (N+1)r^2 - (r\phi_1 + \phi_2 + \cdots + \phi_{j+1}/r^{j-1}) + (\phi_0 + \phi_1/r + \cdots + \phi_j/r^j) \\
&= (N+1)r^2 - r^2(\psi_{N+1} - 1) + \psi_N \\
&= r^2(N+2) - (r^2-1)\psi_{N+1} - \frac{\phi_{N+1}}{r^{N+1}}.
\end{aligned}$$

From the last line, we see that $F_N(1,\theta) = N + 2 - \phi_{N+1}(\theta) \geq 0$. Set $S_N = r^N F_N$, which equals the polynomial $(N+2)r^{N+2} - rR_{N+1} + R_N$.

Now we show that $\frac{\partial S_N}{\partial r}(1,\theta) \geq 0$.

$$\begin{aligned}
\frac{\partial S_N}{\partial r}(1,\theta) &= (N+2)^2 + \left(\frac{\partial R_N}{\partial r} - \frac{\partial R_{N+1}}{\partial r} - R_{N+1}\right)\Bigg|_{r=1} \\
&= (N+2)^2 - \big(R_N(1,\theta) + R_{N+1}(1,\theta)\big)
\end{aligned}$$

Multiply this last expression by $q(1,\theta) = 2 - 2\cos\theta = 4\sin^2\theta/2$. This yields

$$\begin{aligned}
&4(N+2)\sin^2\theta/2 - (2 - \phi_{N+2} + \phi_N) \\
&\quad = 4(N+2)\sin^2\theta/2 - 2 + 2\cos(N+2)\theta \\
&\quad = 4(N+2)\sin^2\theta/2 - 4\sin^2\frac{N+2}{2}\theta \\
&\quad = 4(N+2)\sin^2\theta/2\left(1 - \frac{\sin^2\frac{N+2}{2}\theta}{(N+2)^2\sin^2\theta/2}\right).
\end{aligned}$$

This is nonnegative since $|\frac{\sin k\theta}{k\sin\theta}| \leq 1$ for all integers k. Hence $\frac{\partial S_N}{\partial r}(1,\theta) \geq 0$.

Set $T_N = S_N \cdot q$; this is $r^{N+2}\big((N+1)r^2 - (N+2)\phi_1 r + N + 3\big) + r^2\phi_{N+2} - 2\phi_{N+1}r + \phi_N$. We wish to show that $\frac{\partial^2 T_N}{\partial r^2}(1,\theta) \geq 0$. We make the following observation, based on simple inequalities from the Taylor expansions:

If $0 \leq K\theta \leq 6$, $K \geq 2$, and $\theta^2 \leq 6$, then $\sin K\theta/K\sin\theta \geq 1 - (K\theta)^2/6$.

Now,

$$\begin{aligned}
\frac{\partial^2 T_N}{\partial r^2}(1,\theta) &= (N+4)(N+3)(N+1) - (N+3)(N+2)^2\phi_1 \\
&\qquad + (N+3)(N+2)(N+1) + 2\phi_{N+2} \\
&= (N+3)\big(4(N+2)^2\sin^2\theta/2\big) - 2(N+3) + \frac{2\sin(N+3)\theta}{\sin\theta}
\end{aligned}$$

https://doi.org/10.4153/CJM-2003-041-7 Published online by Cambridge University Press

If $(N+3)\theta \leq 6$ and $\theta^2 \leq 6$, then this last expression is at least as large as

$$\begin{aligned}(N+3)\big(4(N+2)^2\sin^2\theta/2\big) - 2(N+3) + 2(N+3) - \frac{2\big((N+3)\theta\big)^2(N+3)}{6}\\ = (N+3)\Big(4(N+2)^2\sin^2\theta/2 - \frac{(N+3)^2\theta^2}{3}\Big)\\ \geq (N+3)\Big((N+2)^2\Big(\theta^2 - \frac{\theta^4}{12}\Big) - \frac{(N+3)^2\theta^2}{3}\Big)\\ = (N+3)\Big(\theta^2\Big(2N^2/3 + 2N + 1 - (N+2)^2\frac{\theta^2}{12}\Big)\Big) > 0.\end{aligned}$$

On the other hand, if θ or $(N+3)\theta$ is "large" (so that the observation may not be applied), it is easy to see directly that the second derivative is nonnegative. Now we check the signs of $\frac{\partial^i T_N}{\partial r^i}(1,\theta)$ for $i \neq 2$. If $i \geq 3$, positivity is obvious. For $i = 1$, $\frac{\partial T_N}{\partial r} = \frac{\partial S_N}{\partial r} \cdot q + S_N \cdot \frac{\partial q}{\partial r}$, and at $r = 1$, both summands are nonnegative. Thus T_N is increasing in r for $r \geq 1$, and since $T_N = qr^N F_N$, it follows that $T_N(1,\theta) \geq 0$. Hence $T_N(r,\theta) \geq 0$ for all θ and $r \geq 1$, and of course this implies that the Maclaurin coefficients of $(1-x)^{-2}Q^{-1}$ are nonnegative.

Proposition A.11 *If $Q = (x-z)(x-\overline{z})$ and $|z| \geq 1$, then all Maclaurin series coefficients of $(1-x)^{-2}Q^{-1}$ are nonnegative.*

If $P(x)$ is a real polynomial of degree d with no roots of modulus less than 1, then A.11 implies that $(1-x)^{-d}P^{-1}$ has no negative Maclaurin series coefficients. This can be improved when we know roughly the positions or moduli of the roots. For example, if all the roots have modulus 2 or more, then the exponent d can be replaced by $[d+1/2]$; or if k roots have modulus at least 1.09 and positive real part, then the exponent of $(1-x)^{-1}$ can be reduced to $d-k$. (On the other hand, $P = (1+x)^d$ requires the exponent to be d.)

B Non-Positive Homomorphisms

Here we investigate in more detail the phenomenon illustrated in Example 3.2, which amounts to the failure of order unit cancellation in R_P for fairly simple choices of P. It turns out that this can be detected by the presence of special ring homomorphisms $\phi\colon R_P \to \mathbf{C}$. These are *not* generally order-preserving.

For this appendix, we define a *complex homomorphism* on R_P to be a real linear ring homomorphism $\phi\colon R_P \to \mathbf{C}$. The pure traces are examples, but others exist. For example, if P has radius of convergence 1 and no zeroes in the open unit disc D, then for every z in D, the point evaluation map $\phi_z := a \mapsto a(z)$ is a complex homomorphism. If P has radius of convergence 1 but has a zero, say at z_0 in D, then ϕ_{z_0} need not be defined—*e.g.*, if $P = (1+2x)(1-x)^{-1}$, there is no complex homomorphism ϕ sending the element x of R_P to $-1/2$. On the other hand, ϕ_1 is not defined when $P = (1-x)^{-1}$, but there does exist a complex homomorphism such that $\phi(x) = 1$ (a weak limit of the pure traces coming from ϕ_t as $t \uparrow 1$).

Recall that the monomial x belongs to R_P if (for example) $\max_k\{(P, x^k)/(P, x^{k+1})\} < \infty$. This forces the radius of convergence of P to be finite. When this is the case, define an invariant of P, called $\Psi(P)$, via

$$\Psi(P) = \{\phi(x) \mid \phi \text{ is a complex homomorphism of } R_P\}.$$

By the limit argument above, ρ itself (the radius of convergence of P) always belongs to $\Psi(P)$. We note that if $R_P \subseteq R_Q$, and x belongs to R_P, then $\Psi(Q) \subseteq \Psi(P)$ (and the inclusion can be strict, as examples below will show).

Next, $\Psi(P)$ must be contained in the *closed* disk of radius equal to the radius of convergence, ρ, of P: if z is a complex number of modulus exceeding ρ, define $f = (1 - x/z)^{-1}$ if z is real and $\big((1 - x/z)(1 - x/\overline{z})\big)^{-1}$ if not (the same function that appears in the first appendix). In the first case, $f = \sum x^k/z^k$, and by [H1, Proposition 10], belongs to R_P; since f^{-1} is a polynomial, both f and f^{-1} belong to R_P. Hence for any complex homomorphism ϕ of R_P, $\phi(f^{-1})$ is not zero, whence $\phi(x) \neq z$. In the second case, the power series expansion (studied to death in the first appendix) has larger radius of convergence than ρ, so again f and f^{-1} belong to R_P. Thus if ϕ is a complex homomorphism, $\phi(x)^2 - 2\phi(x)r\cos\theta + r^2 \neq 0$, so that $\phi(x)$ is not a root of polynomial f^{-1}, hence cannot be z.

Lemma B.1

(i) *If* $P = (1 - x)^{-1}$, *then* $\Psi(P) = D \cup \{1\}$;
(ii) *If* $P = (1 + 2x)(1 - x)^{-1}$, *then* $\Psi(P) = D \cup \{1\} \setminus \{-1/2\}$;
(iii) *If* $P = (2 + x)(1 - x^2)^{-1}$, *then* $\Psi(P) = D \cup \{\pm 1\}$;
(iv) *If* $P = (1 + 2x)(1 - x^2)^{-1}$, *then* $\Psi(P) = D \cup \{\pm 1\} \setminus \{-1/2\}$;
(v) *If* $P = \sum x^k/(k + 1)^2$, *then* $\Psi(P) = \overline{D}$.

Proof (i) We need only show that if $|z| = 1$, but if $z \neq 1$, then z does not belong to $\Psi(P)$. We note that the argument above involving $f = \big((1 - x/z)(1 - x/\overline{z})\big)^{-1}$ applies (we have already shown that even when $|z| = 1$, f belongs to R_P).

(ii) As in (i), we can exclude points on the unit circle, and $-1/2$ is also excluded by the argument above, whereas other points of D are easily checked to be included. The fact that -1 belongs to the set in cases (iii) and (iv) will be deferred to a later argument. Case (v) admits an obvious argument. ■

This invariant is motivated by the next result. Order unit cancellation was defined just prior to Example 3.2.

Proposition B.2 *Suppose that* $P \sim (1 - x)^{-t}$ *for some positive integer* t, *and there exists a complex homomorphism*, ϕ, *of* R_P *such that* $\phi(x) = \xi$ *where* $|\xi| = 1$ *and* $\xi \neq 1$. *Set* $f = \big((1 - x/\xi)(1 - x/\overline{\xi})\big)^{-1}$. *Then*

(a) f *does not belong to* R_P;
(b) $P^m f$ *has a negative Maclaurin coefficient for every integer* m;
(c) *order unit cancellation fails in* R_P;
(d) *no power of* P *has increasing Maclaurin coefficients.*

https://doi.org/10.4153/CJM-2003-041-7 Published online by Cambridge University Press

Proof We note that f^{-1} is a monic real polynomial with roots $\xi^{\pm 1}$ (if $\xi = -1$, the arguments still apply), and in particular, both f and f^{-1} are strictly positive on the closed unit interval.

If f belongs to R_P, then both $f^{\pm 1}$ would belong (polynomials belong since $P \sim (1-x)^{-t}$). This (again) forces $\phi(f^{-1}) \neq 0$, which yields $f^{-1}(\xi) \neq 0$, a contradiction.

Suppose that $P^l f$ has no negative coefficients. Set $P_0 = (1-x)^{-1}$. Now f is an order unit in R_{P_0}, so that $(1-x)^{-l'} f$ has no negative coefficients and $(1-x)^{-l'} f \prec (1-x)^{-l'}$. Multiply this by P^l; we obtain $(1-x)^{-l'}(P^l f) \prec (1-x)^{-l'} P^l$. From the observation that if R, S, and T have no negative coefficients and $R \sim S$, then $RT \sim ST$, we obtain $P^{l'}(P^l f) \prec P^{l+l'}$. This forces f to belong to R_P, a contradiction. Hence $P^l f$ has negative Maclaurin series coefficient for all l.

We note that f belongs to R_{P_0}, and in fact $f^{-1} \prec P_0 \prec P$. Thus $a := 1/(f^{-1}P)$ belongs to R_P. Also, $f^{-1}a = 1/P$, which belongs to R_P^+. We claim that a is not positive in R_P.

There exists real $K > 0$ such that $0 \leq f^{-1} \leq K \cdot 1$ in R_P, so if a were positive in R_P, we would obtain $1/P \leq Ka$ in R_P. This translates to $P^l \leq KP^l f$ (coefficient-wise) for some positive integer l, in particular, $P^l f$ would have positive coefficients, a contradiction.

Obviously $(P_0)^{l't}P^{-l'}a$ is in R_P^+, and by hypothesis, $(P_0)^{l't}P^{-l'}$ is an order unit of R_P. Hence order unit cancellation fails.

Finally, the coefficients of P^m are increasing if and only if $(1-x)P^m$ has no negative coefficients. This would mean $P^m P_0^{-1}$ has no negative coefficients, so that if $(P_0)^k f$ had no negative coefficients, then the same would be true of $P^{mk} f$, contradicting (b). ∎

Now we give a criterion for a point to be in $\Psi(P)$. First, a routine observation.

Lemma B.3 *Let R be a commutative unital $\mathbf{R}$-algebra, and let S be a countably infinite collection of $\mathbf{R}$-algebra homomorphisms $\psi \colon R \to \mathbf{C}$. Suppose that for all r in R, $\sup_{\psi \in S} |\psi(r)| < \infty$. Then S has a limit point in the topology of pointwise convergence.*

Proof Define a pseudo-norm on R via $\|r\| = \sup_{\psi \in S} |\psi(r)|$, and set $I = \{r \in R \mid \|r\| = 0\}$. Then I is an ideal, and $\|\cdot\|$ induces a submultiplicative norm on the quotient $R_0 = R/I$. Complete R_0 to a Banach algebra, R_1, and it is trivial that each ψ in S extends uniquely to a multiplicative (real) linear functional (*i.e.*, a complex homomorphism) on R_1. Any such is continuous, so the maximal ideals of R correspond to the complex homomorphisms, and since the algebra is unital, the set of all such is a weakly compact set. Any limit point of the image of S therein will pull back to a complex homomorphism on the original R. ∎

Lemma B.4 *Suppose P is a Maclaurin series with no negative coefficients and radius of convergence 1, and in addition, the monomial x belongs to R_P. Suppose that ξ is a complex number of modulus 1 such that for all r in R_P, $\limsup_{t \uparrow 1} |r(t\xi)| < \infty$. Then there exists a complex homomorphism of R_P whose value at x is ξ.*

Proof Set $R = R_P$, and select a sequence of positive real numbers $t_n \uparrow 1$. Let S be $\{\phi_{t_n\xi}\}$ (point evaluations). By Lemma B.3, S has a weak limit point. Any weak limit point of S will send x to $\lim t_n\xi = \xi$. ∎

Proposition B.5 *Suppose that P has no negative Maclaurin series coefficients, has radius of convergence* 1*, and x belongs to R_P. Suppose that $\{z_n\}$ is a subset of D such that for some positive integer K, $P(|z_n|) \leq K|P(z_n)|$ for all n. Then any limit point of $\{z_n\}$ belongs to $\Psi(P)$.*

Proof By Lemma B.3, it is sufficient to show that for all r in R_P, $\sup_n |r(z_n)| < \infty$, and to show this, it is sufficient to do so with r in R_P^+. For such an r, there exists Q with no negative coefficients and a positive integer k such that $Q \prec P^k$ and $r = Q/P^k$. Hence there exists positive real L such that $0 \leq (Q, x^m) \leq L(P, x^m)$. Since Q has no negative coefficients,

$$|Q(z_n)| \leq Q(|z_n|) \leq LP^k(|z_n|) \leq LK^k|P^k(z_n)|.$$

Thus $|r(z_n)| \leq LK^k$. ∎

This can be applied fairly easily in the following situation.

Corollary B.6 *Suppose that P has positive coefficients and radius of convergence* 1*, and extends to an analytic function on deleted neighbourhoods of* 1 *and ξ, where $\xi \neq 1$ has modulus* 1*, and suppose P has poles of equal order at* 1 *and ξ. Then ξ belongs to $\Psi(P)$.*

Proof As $t \uparrow 1$, $|P(t\xi)|$ behaves as $(1-t)^{-k}$ up to a constant multiple, and this is also the behaviour of $P(t)$. Taking a sequence $t_n \uparrow 1$, Proposition B.5 applies, and we deduce that ξ belongs to $\Psi(P)$. ∎

This applies to $P = (2+x)(1-x^2)^{-1} = 2 + x + 2x^2 + x^3 + \cdots$ and $P = (1+2x)(1-x^2)^{-1} = 1 + 2x + x^2 + 2x^3 + \cdots$, with $\xi = -1$, yielding cases (iii) and (iv) of Lemma B.1.

The condition on the equality of orders of the poles is important—if $P = \big((1+x)(1-x)^2\big)^{-1}$, then each of P and $P_0 = (1-x)^{-1}$ positively divide a power of each other, so that $\Psi(P) = \Psi(P_0) = D \cup \{1\}$.

In the opposite direction, it is easy to see that if P has radius of convergence one and there exists r in R_P that is rational (or merely meromorphic on a neighbourhood of the closed unit disk), together with a sequence $\{z_n\} \subset D$ converging to $\xi \in \overline{D}$ such that $\{|r(z_n)|\}$ is unbounded, then $\xi \notin \Psi(P)$.

Results in this appendix would be stronger if we knew that $R_{(1-x)^{-1}}$ itself had order unit cancellation. This is connected to the absence of weird ring homomorphisms, as follows. Set $P = (1-x)^{-1}$; suppose that R_P satisfies the following property.

(#) If F is a field and $\phi\colon R_P \to F$ is an onto homomorphism of rings such that $\phi(x) = 1$, then ϕ is a trace.

https://doi.org/10.4153/CJM-2003-041-7 Published online by Cambridge University Press

Then R_P satisfies order unit cancellation.

To see this, suppose that u is an order unit of R_P and a is an element thereof such that ua is positive. Form the ideal, I, generated by u and $1/P$. If $I \neq R_P$, then I is a proper ideal, so there exists a maximal ideal M containing I. Then the induced map $\phi\colon R_P \to R_P/M = F$ is a ring homomorphism sending $1/P$ to zero. As $1/P = 1 - x$, we obtain that $\phi(x) = 1$. By hypothesis, ϕ is a trace, and therefore $\phi(u) > 0$, a contradiction. Hence $I = R_P$. By Corollary 4.3, a is in R_P^+.

This can be modified to give a criterion for more general R_P to have order unit cancellation, but in either case, the unwelcome presence of an arbitrary field image (F) makes it unwieldy. (The modification is that $\phi(1/P) = 0$, rather than $\phi(x) = 1$, imply that ϕ is a trace—this yields that if $P^2 \prec P$ [H1], then order unit cancellation holds.)

The presence or absence of a complex homomorphism has a bearing on possible inclusions between R_P and R_Q. For example, if $\Psi(Q)$ contains a point not in $\Psi(P)$, then there is no inclusion of rings (let alone as ordered rings) $R_P \subseteq R_Q$.

References

[B] R. P. Boas, *Invitation to complex analysis.* Random House, New York, 1987.

[GH] K. R. Goodearl and D. E. Handelman, *Rank functions and K_0 of regular rings.* J. Pure Appl. Algebra **7**(1976), 195–216.

[H1] D. Handelman, *Eventual positivity for analytic functions.* Math. Ann. **304**(1996), 315–338.

[H2] ———, *Characters of compact Lie groups.* J. Algebra **173**(1995), 67–96.

[H3] ———, *Positive polynomials, convex integral polytopes and a random walk problem.* Lecture Notes in Math. **1282**, Springer-Verlag, 1987.

[H4] ———, *Positive polynomials and product type actions of compact groups.* Mem. Amer. Math. Soc. (320) **54**(1985).

[R] W. Rudin, *Real and complex analysis.* McGraw-Hill, 1966.

[T] E. C. Titchmarsh, *The theory of functions.* 2nd edition, Oxford, 1952.

Mathematics Department
University of Ottawa
Ottawa, Ontario
K1N 6N5
email: dehsg@uottawa.ca