Canad. Math. Bull. Vol. **14** (4), 1971

PERIODIC SOLUTIONS FOR $\dot{x}=Ax+g(x,t)+\epsilon p(t)$

BY
PETER J. PONZO

We wish to establish the existence of a periodic solution to

$$\dot{x} = Ax+g(x,t)+\epsilon p(t), \quad (\dot{} \equiv d/dt) \tag{1}$$

where x, g and p are n-vectors, A is an $n\times n$ constant matrix, and ϵ is a small scalar parameter. We assume that g and p are locally Lipschitz in x and continuous and T-periodic in t, and that the origin is a point of asymptotically stable equilibrium, when $\epsilon=0$.

Although the result below is not new ([1], [2]), the proof is simple and of some interest and provides an explicit bound on ϵ which will guarantee the existence of a T-periodic solution. It also gives a bound on the norm of the periodic solution.

In what follows, $\|.\|$ denotes the Euclidean norm.

THEOREM. *If*

(i) $p(t+T)=p(t)$ *and* $\|p(t)\|\leq 1$ *for all* t,

(ii) $g(x,t+T)=g(x,t)$ *and* $\|g(x,t)\|=o(\|x\|)$ *uniformly in* t,

(iii) *the eigenvalues of* A *have negative real part, then* (1) *possesses a* T*-periodic solution for* ϵ *sufficiently small.*

Proof. We wish to select the constant $c>0$ such that the surface $V(x)=x^TBx=c^2$ confines interior trajectories, where B is the unique, real, symmetric, positive definite matrix which satisfies $A^TB+BA=-I$ (I, the unit matrix). Assuming the existence of such a constant, we may apply Brouwer's fixed point theorem to the region $V(x)\leq c^2$ and conclude that a fixed point exists for the transformation $x(t_0)\to x(t_0+T)$, where $x(t_0)$ is in $V\leq c^2$ and is the initial condition for a solution $x(t)$ which remains in $V\leq c^2$. Since (1) is invariant with respect to a time translation of amount T, this fixed point will generate a T-periodic solution to (1).

We show that a constant c does exist.

Let $\lambda>0$ and Λ be the smallest and largest eigenvalues of B, respectively. Then $\lambda\|x\|^2\leq x^TBx\leq \Lambda\|x\|^2$ so that $V(x)=c^2$ lies in

$$c/\sqrt{\Lambda} \leq \|x\| \leq c/\sqrt{\lambda} \tag{2}$$

In order that the surface $V=c^2$ confine interior trajectories, it is sufficient to have $dV/dt<0$ everywhere on the surface.

We have

$$dV/dt = -x^Tx+2(g^T+\epsilon p^T)Bx.$$

https://doi.org/10.4153/CMB-1971-105-4 Published online by Cambridge University Press

Now $\|Bx\| \leq \Lambda \|x\|$, so that

$$dV/dt \leq -\|x\|^2 + 2(\|g\| + \epsilon)\Lambda \|x\|.$$

Hence $dV/dt < 0$ provided

$$\|x\| > 2\Lambda[\|g(x, t)\| + \epsilon]. \tag{3}$$

We now show that, for ϵ sufficiently small, a c exists such that (2) implies (3). It will then follow that $V = c^2$ lies in a region where $dV/dt < 0$, hence will confine interior trajectories.

Since $\|g\| = o(\|x\|)$, then, for any $0 < k < 1$, $\Delta(k)$ exists such that $2\Lambda \|g\| < k\|x\|$ for $\|x\| < \Delta$. Hence, if x is restricted to the region

$$2\epsilon\Lambda/(1-k) \leq \|x\| < \Delta \tag{4}$$

we will have $\|x\| \geq k\|x\| + 2\epsilon\Lambda > 2\Lambda[\|g\| + \epsilon]$ and (3) will be satisfied.

Note that (4) imposes an upper bound on ϵ, namely $\epsilon < (1-k)\Delta/2\Lambda$.

We now select c such that $V = c^2$ lies entirely in the region defined by (4), where $dV/dt < 0$. Using (2), we choose

$$2\epsilon\Lambda\sqrt{\Lambda}/(1-k) \leq c < \Delta\sqrt{\lambda}. \tag{5}$$

This is possible if

$$\epsilon < [(1-k)\Delta/2\Lambda]\sqrt{\lambda/\Lambda}. \tag{6}$$

Consequently, if ϵ satisfies (6), for some $o < k < 1$, then $V = c^2$ confines interior trajectories, where c is any number satisfying (5). Q.E.D.

Note that $V = c^2$ confines interior trajectories regardless of whether or not $p(t)$ and $g(x, t)$ are periodic. Indeed we can have $p = p(x, t)$, with $\|p\| \leq 1$ for all x and t. Also, if $x = \phi(t)$ is the periodic solution, then $\phi^T B \phi \leq c^2$, which gives an upper bound on the amplitude, namely $\|\phi\| \leq c/\sqrt{\lambda}$.

Further, for $\epsilon = 0$, (3) gives an estimate of the region of asymptotic stability for the null solution of $\dot{x} = Ax + g$. Also, for the linear system $\dot{x} = Ax + \epsilon p(t)$, $\dot{V} < 0$ for $\|x\| > 2\epsilon\Lambda$ from (3) and all solutions eventually enter the interior of $x^T Bx = c^2$ with $c > 2\epsilon\Lambda\sqrt{\Lambda}$.

Note that

$$\dot{x} = B(t)x + g(x, t) + \epsilon p(t), \quad \text{with } B(t+T) = B(t),$$

reduces to the form (1) under the transformation $x = Q(t)y$ where $Q(t+T) = Q(t)$ is obtained from the principal matrix solution for $\dot{X} = B(t)X$ (i.e. $X = Q\, e^{At}$).

One further point of interest is that the number k is arbitrary, in (0, 1). Note that the upper bound on ϵ, from (6), vanishes at $k = 1$ and $k = 0$ (in the latter case, $\Delta = 0$). Presumably k might be chosen to maximize $(1-k)\Delta(k)$.

https://doi.org/10.4153/CMB-1971-105-4 Published online by Cambridge University Press

For example, the forced van der Pol equation,

$$\dot{x} = y, \dot{y} = \mu(x^2-1)y-x+\epsilon p(t)$$

has

$$\|g\| = \mu x^2|y| = \mu r^3 \cos^2\theta\,|\sin\theta| \le \frac{2\mu r^3}{3\sqrt{3}},$$

where $r=\sqrt{x^2+y^2}$. Hence $\|g\|\le kr/2\Lambda$ for $r<\Delta=\sqrt{3\sqrt{3}/4\mu\Lambda}\,k^{1/2}$, and $(1-k)\Delta$ is maximized for $k=\frac{1}{3}$.

References

1. L. Cesari, *Asymptotic behaviour and stability problems in ordinary differential equations*, Springer-Verlag, Berlin (1963), 115–182.

2. H. I. Freedman, *Estimates on the existence region for periodic solutions of equations involving a small parameter*, SIAM J. Appl. Math. **16** (1968), 1341–1349.

University of Waterloo,
Waterloo, Ontario

https://doi.org/10.4153/CMB-1971-105-4 Published online by Cambridge University Press