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SOME BOUNDARY-VALUE PROBLEMS FOR NONLINEAR (N) DIFFUSION AND PSEUDO-PLASTIC FLOW

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Abstract

In this article, exact and approximate techniques are used to obtain parameters of interest for two problems involving differential equations of power-law type. The first problem is related to non-linear steady-state diffusion, and is investigated by means of a hodograph transformation and an approximation using a path-independent integral. The second problem involves Poiseuille flow of a pseudo-plastic fluid, and a path-independent integral is derived which yields an exact result for the geometry under consideration.

1. Introduction

A large collection of results exists for special solutions of various non-linear diffusion equations. For the most part these are fairly simple similarity solutions or somewhat similar solutions obtained by the method of group transformations. While these are useful, they apply to rather special cases and do not easily give information about more complicated situations. Many of these similarity solutions have been reviewed by Hill [10]. Atkinson and Jones [6] considered special similarity solutions for the non-linear diffusion equation

$$\nabla \cdot [D(C)\nabla C] = \frac{\partial C}{\partial t}, \tag{1.1}$$

and showed that for $D(C) = C^m$ it could be reduced by phase plane analysis

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to a non-linear ordinary differential equation of a type considered earlier by Jones [11]. Similar procedures were applied to a generalised diffusion equation (called N-diffusion by Philip [12]; α equivalent to N here)

$$\nabla \cdot [|\nabla C|^N \nabla C] = \frac{\partial C}{\partial t}, \tag{1.2}$$

and it was also noticed by Atkinson and Jones [6] that boundary-layer flow of a power-law non-Newtonian fluid could also be treated by the same techniques. One can prove existence and use various comparison theorems to extend the usefulness of these similarity solutions (see e.g. Atkinson and Peletier [7] for such results for (1.1) and Atkinson and Bouillet [3], Bouillet and Atkinson [8] for a generalised diffusion equation which includes equations (1.1) and (1.2) as special cases). However, the problems amenable to this treatment are still a small subclass of situations that might arise in practice.

A somewhat different problem has been tackled recently by Atkinson [2], who uses the method of matched asymptotic expansions to determine the steady temperature field, in an appropriate limit, associated with (1.1) ($\partial C/\partial t$ being replaced by $-V\partial C/\partial X$, $X = x - Vt$) for a rod moving at speed V. Progress is possible because in one asymptotic limit the problem can be reduced to solving a non-linear ordinary differential equation, whereas in the other limit the Kirchhoff transformation (i.e. substitute $\Phi = \int D(C)dC$) linearises the equation. Matching the two asymptotic expansions allows the solution to be completed.

All of the problems discussed above have some special features which enable some analytic progress to be made. These solutions are an addition to, and check on, more comprehensive numerical treatments. They can also lead to rigorous qualitative results which are a useful guide to understanding. In the spirit of the above remarks (or with that excuse!) we discuss here some special problems for which a somewhat unusual solution method is attempted. The method we use is a Legendre (or hodograph) transformation, but in circumstances in which it does not manage to linearise the equation (or at least not quite). We also consider problems associated with equations related to (1.2). These are:

PROBLEM A. Steady-state solutions of (1.2) are considered, when there is a region of constant concentration (or temperature in the analogous heat-conduction case) moving with constant speed V on the boundary of a strip, with fixed concentration (or temperature) on the opposite side of the strip. Thus, in moving co-ordinates $X = X_1 - Vt$, $Y = Y_1$, we have the boundary conditions

$$\begin{aligned} C &= 0 \quad \text{on } Y = h,\ -\infty < X < \infty, \\ C &= C_0 \quad \text{on } Y = 0,\ X < 0, \\ \frac{\partial C}{\partial Y} &= 0 \quad \text{on } Y = 0,\ X > 0. \end{aligned} \tag{1.3}$$

With the steady-state assumption that C depends on time only through $X = X_1 - Vt$, (1.2) becomes

$$\nabla \cdot (|\nabla C|^{\alpha} \nabla C) + V\frac{\partial C}{\partial X} = 0. \tag{1.4}$$

PROBLEM B. The second problem we consider involves an equation similar to (1.2), but is for Poiseuille flow of a pseudo-plastic fluid in a strip with a semi-infinite flat plate immersed in it. With a no-slip condition on the strip sides and on the plate, we have the boundary value problem

$$\nabla \cdot [|\nabla w|^{\alpha} \nabla w] = -G, \tag{1.5}$$

for the z-component of velocity $w(X, Y)$, where G is a constant. The boundary conditions are

$$\begin{aligned} w &= 0 \quad \text{on } Y = h,\ -\infty < X < \infty, \\ w &= 0 \quad \text{on } Y = 0,\ X > 0, \\ \frac{\partial w}{\partial Y} &= 0 \quad \text{on } Y = 0,\ X < 0, \end{aligned} \tag{1.6}$$

where the last equation is a symmetry condition replacing that part of the strip in $-h < y < 0$.

2. Non-linear steady-state diffusion (Problem A)

2.1. The Legendre transformation

With reference to (1.3) and (1.4), define

$$x = -X/h, \qquad y = Y/h, \qquad u(x, y) = (C_0 - C)/C_0. \tag{2.1}$$

Then $u(x, y)$ satisfies the equation

$$\nabla \cdot [\nabla u |\nabla u|^{\alpha}] - \varepsilon \frac{\partial u}{\partial x} = 0, \tag{2.2}$$

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together with the boundary conditions

$$\begin{aligned} u(x,1)&=1, & -\infty<x<\infty,\\ u(x,0)&=0, & x>0,\\ \frac{\partial u}{\partial y}(x,0)&=0, & x<0. \end{aligned} \tag{2.3}$$

The constant is given by $\varepsilon = Vh(h/C_0)^\alpha$, and α lies in the range $-1<\alpha<\infty$.

Define quantities ξ, η and ρ by

$$\xi=\frac{\partial u}{\partial x}, \qquad \eta=\frac{\partial u}{\partial y}, \qquad \rho=(\xi^2+\eta^2)^{1/2}. \tag{2.4}$$

Equation (2.2) may now be written as

$$\nabla\cdot[\boldsymbol{\rho}\rho^\alpha]=\varepsilon\xi \tag{2.5}$$

with $\boldsymbol{\rho}=(\xi,\eta)$. Now construct a function

$$\psi(\xi,\eta)=x\xi+y\eta-u. \tag{2.6}$$

The function ψ has the property that

$$\psi_\xi=x, \qquad \psi_\eta=y, \tag{2.7}$$

and defines a mapping from the real (x,y) plane to the (ξ,η) plane—a Legendre transform (sometimes called a hodograph transform). This transformation has been used by various authors (e.g. Atkinson and Champion [4] Amazigo [1]) to solve problems in power-law elasticity.

In terms of derivatives with respect to ξ, and η, the operators $\partial/\partial x$ and $\partial/\partial y$ are

$$\begin{aligned} \frac{\partial}{\partial x}&=\frac{1}{J}\left[\psi_{\eta\eta}\frac{\partial}{\partial\xi}-\psi_{\xi\eta}\frac{\partial}{\partial\eta}\right],\\ \frac{\partial}{\partial y}&=\frac{1}{J}\left[-\psi_{\xi\eta}\frac{\partial}{\partial\xi}+\psi_{\xi\xi}\frac{\partial}{\partial\eta}\right], \end{aligned} \tag{2.8}$$

where a suffix denotes partial differentiation, and the operator $J(\psi)$ is

$$J(\psi)=\psi_{\xi\xi}\psi_{\eta\eta}-\psi_{\xi\eta}^2, \tag{2.9}$$

and $J\neq 0$ for a 1-1 mapping.

The strip in Figure 1 maps onto the region in the (ξ,η) plane show in Figure 2. From (2.5), (2.8) and (2.7), the equation satisfied by ψ is

$$\begin{aligned} \psi_{\xi\xi}+\psi_{\eta\eta}&+\frac{\alpha}{\rho^2}[\xi^2\psi_{\eta\eta}-2\xi\eta\psi_{\xi\eta}+\eta^2\psi_{\xi\xi}]\\ &=\varepsilon\xi(\psi_{\xi\xi}\psi_{\eta\eta}-\psi_{\xi\eta}^2), \end{aligned} \tag{2.10}$$

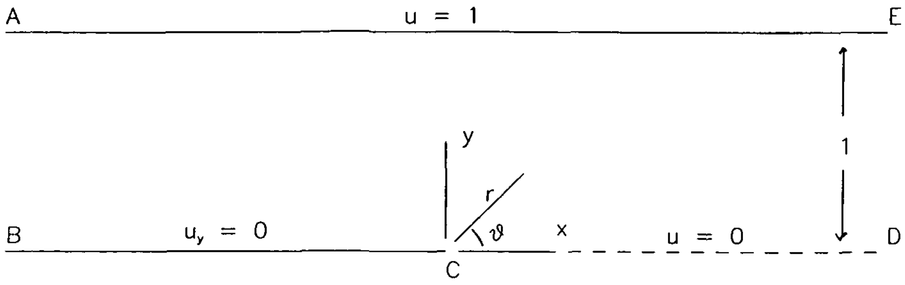

FIGURE 1

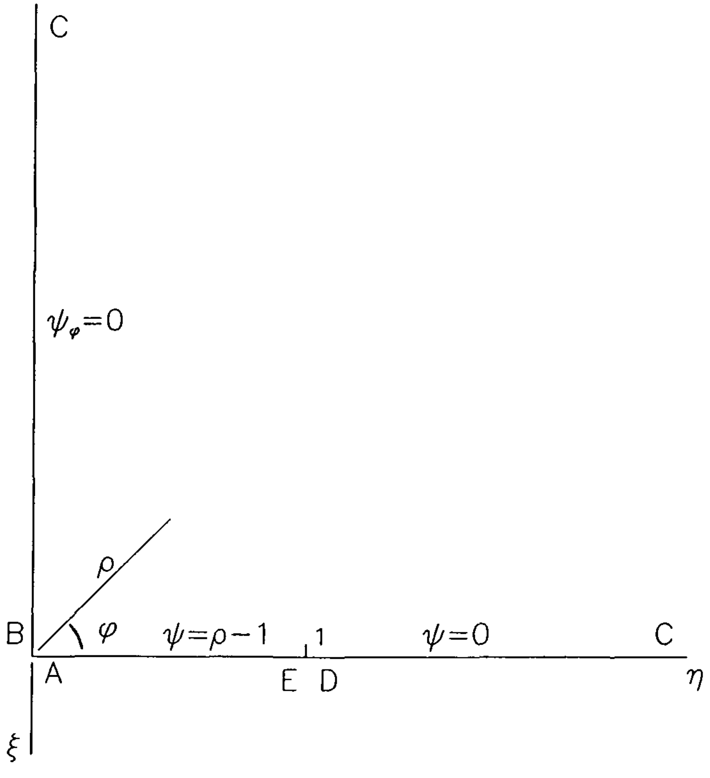

FIGURE 2

together with the boundary conditions

$$\begin{aligned} \psi_\eta(\xi, 0) &= 0, \xi < 0; \\ \psi(0, \eta) &= \eta - 1, 0 < \eta < 1, \\ &= 0, 1 < \eta < \infty. \end{aligned} \tag{2.11}$$

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If we define polar co-ordinates (ρ, ϕ) by

$$\xi = -\rho\sin\phi, \; \eta = \rho\cos\phi, \tag{2.12}$$

then the equation satisfied by ψ becomes

$$n\psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \frac{1}{\rho^2}\psi_{\phi\phi} = -\varepsilon n\rho^{-1/n}\sin\phi\Delta(\psi), \tag{2.13}$$

where the operator Δ is given by

$$\Delta(\psi) = (\psi_{\phi\phi} + \rho\psi_\rho)\psi_{\rho\rho} - \left(\psi_{\rho\phi} - \frac{1}{\rho}\psi_\phi\right)^2 \tag{2.14}$$

and

$$n = 1/(1+\alpha), \qquad 0 < n < \infty. \tag{2.15}$$

(Note that when $n = 1$, the problem is linear). The boundary conditions (2.11) become

$$\psi_\phi\left(\rho, \frac{\pi}{2}\right) = 0; \qquad \begin{aligned} \psi(\rho, 0) &= \rho - 1, \qquad 0 < \rho < 1, \\ &= 0, \qquad 1 < \rho < \infty. \end{aligned} \tag{2.16}$$

We also require that $\psi \to 0$ as $\rho \to \infty$, and that ψ_η is bounded as $\rho \to 0$ and as $(\rho, \phi) \to (1, 0)$.

The solution of (2.13) with $\varepsilon = 0$ with boundary conditions (2.16) has been obtained by Atkinson and Champion [4] using a Mellin transform. If we denote this solution by $\psi^{(0)}(\rho, \phi)$, then

$$\begin{aligned} \psi^{(0)}(\rho, \phi) = &- 1 - \frac{4}{\pi(n+1)}\rho\log_e\rho\sin\phi \\ &+ \frac{\rho}{\pi(n+1)^2}\left[(3 + 6n - n^2)\sin\phi + 2(n+1)^2\left(\frac{\pi}{2} - \phi\right)\cos\phi\right] \\ &+ \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(2k+1)\sin(2k+1)\phi}{S_k^-(S_k^- + 1)(2nS_k^- + n - 1)}\rho^{-S_k^-}, \; 0 < \rho < 1, \\ \psi^{(0)}(\rho, \phi) = &- \frac{4n^2}{\pi(n+1)^2}\rho^{-1/n}\sin\phi \\ &- \frac{4}{\pi}\sum_{k=1}^{\infty}\frac{(2k+1)\sin(2k+1)\phi}{S_k^+(S_k^+ + 1)(2nS_k^+ + n - 1)}\rho^{-S_k^+}, \qquad 1 < \rho < \infty, \end{aligned} \tag{2.17}$$

where

$$S_k^\pm = \frac{1}{2}\left[\frac{1}{n} - 1 \pm \left\{\left(\frac{1}{n} - 1\right)^2 + 4\frac{(2k+1)^2}{n}\right\}^{1/2}\right]. \tag{2.18}$$

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In particular, the behaviour of $\psi^{(0)}$ as $\rho \to \infty$ is

$$\psi^{(0)} \sim -\frac{4}{\pi}\left(\frac{n}{n+1}\right)^2 \rho^{-1/n} \sin\phi, \tag{2.19}$$

which corresponds to the behaviour of $u(x, y)$ as $r \to 0$ in the real plane.

If we require the far-field behaviour for $\varepsilon \neq 0$, we may still write

$$\psi \sim Q\rho^{-1/n} \sin\phi, \qquad \rho \to \infty, \tag{2.20}$$

because the left-hand side of (2.13) dominates for large ρ. However, obtaining Q is not a simple matter—in the next section we show how an approximation to Q may be obtained when ε is small.

2.2. The reciprocal theorem

Consider the equation

$$n\psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \frac{1}{\rho^2}\psi_{\phi\phi} = F(\psi), \tag{2.21}$$

where $F(\psi)$ is a nonlinear operator. In the case $F(\psi) \equiv 0$, Atkinson and Champion [5] have derived an integral relationship which is useful for relating far-field behaviour to integrals over inner boundaries, and also for formulating the problem as an integral equation. Here we modify this technique for non-zero $F(\psi)$.

Define operators $L(\psi)$ and $M(v)$ by

$$L(\psi) = n\psi_{\rho\rho} + \frac{1}{\rho}\psi_\rho + \frac{1}{\rho^2}\psi_{\phi\phi}, \tag{2.22}$$

and

$$M(v) = nv_{\rho\rho} + \frac{(2n-1)}{\rho}v_\rho + \frac{1}{\rho^2}v_{\phi\phi}.$$

It may be shown that

$$vL(\psi) - \psi M(v) = \nabla \cdot \mathbf{P}(v, \psi), \tag{2.23}$$

where $\mathbf{P} = (P_1, P_2)$, with

$$P_1(v, \psi) = \left[n\left(\psi v_\rho - v\psi_\rho\right) + \frac{(n-1)}{\rho}\psi v\right]\sin\phi + \frac{1}{\rho}(\psi v_\phi - v\psi_\phi)\cos\phi, \tag{2.24}$$

and

$$P_2(v, \psi) = -\left[n(\psi v_\rho - v\psi_\rho) + \frac{(n-1)}{\rho}\psi v\right]\cos\phi + \frac{1}{\rho}(\psi v_\phi - v\psi_\phi)\sin\varphi.$$

Use of (2.23), together with the divergence theorem, gives

$$\iint_A [vL(\psi) - \psi M(v)]\, dA = \int_\Gamma P_i\, ds_i\,, \tag{2.25}$$

where Γ is a curve enclosing the area A, and $ds_i = n_i\, ds$, , where n_i is the outward normal to Γ, and ds is an element of arc length of Γ. Use of (2.25) and (2.21) gives

$$\int_\Gamma P_i\, ds_i = \iint_A vF(\psi)\, dA\,, \tag{2.26}$$

where we have chosen v to satisfy

$$M(v) = nv_{\rho\rho} + \frac{(2n-1)}{\rho} v_\rho + \frac{1}{\rho^2} v_{\phi\phi} = 0\,. \tag{2.27}$$

As discussed in the previous sections, we know that

$$\psi \sim Q\rho^{-1/n} \sin\phi\,, \qquad \rho \to \infty\,. \tag{2.28}$$

If we consider that part of the Γ integral which corresponds to a large quarter-circle of radius ρ^* then, denoting Γ_∞ by the limit of the integral around this curve as $\rho^* \to \infty$, we have

$$\Gamma_\infty = -\frac{\pi}{4}(n+1)Q \tag{2.29}$$

if $v(\rho\,, \phi)$ is chosen to be

$$v(\rho\,, \phi) = \rho^{1/n} \sin\phi\,. \tag{2.30}$$

The contributions to Γ from the boundaries $\phi = \pi/2$ $(0 < \rho < \infty)$ and $\phi = 0$ $(1 < \rho < \infty)$ are both zero, and hence

$$\frac{1}{4}\pi(n+1)Q = \int_0^\infty \int_0^{\pi/2} \rho^{1/n} \sin\phi F(\psi)\rho\, d\rho\, d\phi + \int_0^1 [P_1(v\,, \psi)]_{\phi=0}\, d\rho\,, \tag{2.31}$$

with v given in (2.30). If we choose the specific function $F(\psi)$ for our problem, namely

$$F(\psi) = -\varepsilon n\rho^{-1/n} \sin\phi\Delta(\psi)\,, \tag{2.32}$$

where $\Delta(\psi)$ is given in (2.14), then (2.31) becomes

$$\frac{1}{4}\pi(n+1)Q = -\varepsilon n \int_0^{\pi/2} \int_0^\infty \sin^2\phi\Delta(\psi)\rho\, d\rho\, d\phi - \frac{n^2}{n+1}\,. \tag{2.33}$$

It is not possible to obtain Q directly from (2.33) because ψ is, as yet, unknown. However, for small ε, we may pose

$$\psi(\rho\,, \phi) \sim \psi^{(0)}(\rho\,, \phi) + \varepsilon\psi^{(1)}(\rho\,, \phi) + \cdots \tag{2.34}$$

where, from (2.21) and (2.32),

$$\begin{aligned} L(\psi^{(0)}) &= 0\,, \\ L(\psi^{(1)}) &= -n\rho^{-1/n}\sin\phi\Delta(\psi^{(0)}) \text{ etc.} \end{aligned} \tag{2.35}$$

The boundary conditions for $\psi^{(0)}$ are

$$\psi^{(0)}_\phi\left(\rho\,,\frac{\pi}{2}\right) = 0; \qquad \psi^{(0)}(\rho\,,0) = \rho - 1\,, \qquad 0 < \rho < 1\,,$$
$$= 0\,, \qquad 1 < \rho < \infty \tag{2.36}$$

and for $\psi^{(i)}$ $(i = 1\,, 2\,, \dots)$

$$\psi^{(i)}_\phi\left(\rho\,,\frac{\pi}{2}\right) = \psi^{(i)}(\rho\,,0) = 0\,. \tag{2.37}$$

The function $\psi^{(0)}$ is the solution obtained by Atkinson and Champion [4], and given in (2.17). If we write

$$Q \sim Q_0 + \varepsilon Q_1 + \cdots\,, \tag{2.38}$$

then, from (2.33),

$$Q_0 = -\frac{4}{\pi}\left(\frac{n}{n+1}\right)^2\,, \tag{2.39}$$

$$Q_1 = \frac{-4n}{\pi(n+1)}\int_0^\infty\int_0^{\pi/2}\sin^2\phi\Delta(\psi^{(0)})\rho\,d\rho\,d\phi \tag{2.40}$$

etc.

Having found an approximation to ψ as $\rho \to \infty$ in the hodograph plane, we may now transform back into the real plane to obtain the behaviour of $u(x\,, y)$ near $r = 0$, where polar co-ordinates

$$x = r\cos\theta\,, \qquad y = r\sin\theta \tag{2.41}$$

are used. It can be shown from (2.6), (2.7) and (2.28) (see e.g. Atkinson and Champion [4]) that

$$u(r\,,\theta) \sim KG(\theta)r^{\frac{1}{n+1}}\,, \qquad r \to 0\,, \tag{2.42}$$

where

$$G(\theta) = \left(\frac{\sin 2\phi}{2\sin\theta}\right)^{-\frac{1}{n+1}}\sin\phi\,, \tag{2.43}$$

and

$$2\phi = \theta + \sin^{-1}\left(\frac{n-1}{n+1}\sin\theta\right)\,. \tag{2.44}$$

The constant K is given by

$$K = -\eta\left[\left(1 + \frac{1}{n}\right)|Q|\right]^{\frac{n}{n+1}}\,, \tag{2.45}$$

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where $\eta = \text{sign}(Q) = \pm 1$, and Q is defined in (2.28). For the problem under consideration, Q is given by (2.38), (2.39) and (2.40).

The use of the expansion (2.34) to obtain the result (2.38) requires checking because, although (2.34) is valid for $\rho = 0(1)$ and $\rho \to \infty$, it may not be uniformly valid for small ρ. Let us look at (2.35) and use the result

$$\begin{aligned}\psi^{(0)} \sim & -1 - \frac{4}{\pi(n+1)}\rho \log \rho \sin\phi \\ & + \frac{\rho}{\pi(n+1)^2}\left[(3+6n-n^2)\sin\phi + 2(n+1)^2\left(\frac{\pi}{2}-\phi\right)\cos\phi\right]\end{aligned} \tag{2.46}$$

as $\rho \to 0$, which follows from (2.17).

Use of (2.14) gives

$$\Delta(\psi^{(0)}) \sim \left[\frac{4}{\pi(n+1)}\right]^2 (n\sin^2\phi + \cos^2\phi), \qquad \rho \to 0. \tag{2.47}$$

Hence the right-hand side of the second of equations (2.35) generates a solution for $\psi^{(1)}$ which is $0(\rho^{2-1/n})$. This is of higher order, for $1 < n < \infty$, than the first non-constant term in the expansion of $\psi^{(0)}$, which is $0(\rho \log \rho)$ as $\rho \to 0$. Hence we expect the expansion (2.34) to be valid everywhere if $1 < n < \infty$.

However, if $0 < n < 1$ the expansion (2.34) breaks down when ρ is small enough such that $\rho \ln \rho = 0(\varepsilon \rho^{2-1/n})$, and we have, in this case, to check the validity of the approximation (2.38). If we introduce a constant $\delta = \varepsilon^{n/(1-n)}$, then the expansion (2.34) will be uniformly valid if $\rho \geq \delta$. If the path used in (2.26) is again the boundary of Figure 2 with the origin excluded by a quarter-circle of radius δ, then, by taking the limit of the resulting equation as $\delta \to 0$, it can be shown that (2.38) remains valid (up to and including the $0(\varepsilon)$ term) in the range $0 < n < 1$.

3. Pseudo-plastic flow in a strip (Problem B)

With reference to (1.5) and (1.6), define

$$x = X/h, \qquad y = Y/h, \qquad u = w/h, \qquad \alpha = \frac{1}{n} - 1. \tag{3.1}$$

Then $u(x, y)$ satisfies the equation

$$\nabla \cdot [\nabla u |\nabla u|^{1/n-1}] = -G, \tag{3.2}$$

together with the boundary conditions (see Figure 3)

$$\begin{aligned} u(x,1) = 0; \qquad u(x,0) &= 0, \qquad x > 0, \\ u_y(x,0) &= 0, \qquad x < 0, \end{aligned} \tag{3.3}$$

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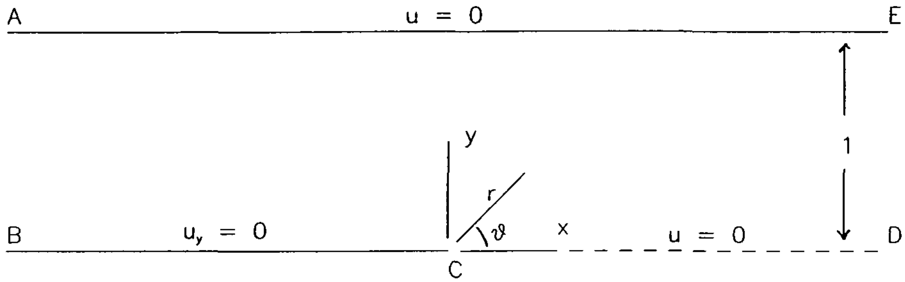

FIGURE 3

where a suffix denotes partial differentiation.

Unlike the problem in Section 2, a 1-1 mapping into the hodograph plane is not possible. However, if we are interested in the behaviour of u near $r = 0$, where $r = (x^2 + y^2)^{1/2}$, then we may use the result, simply shown, that the integral

$$I = \int_S \left[\frac{\partial W}{\partial \eta}\xi\, dx + \left(W - \frac{\partial W}{\partial \xi}\xi\right) dy\right], \tag{3.4}$$

with

$$W = \frac{n}{n+1}\rho^{1+1/n} - Gu, \tag{3.5}$$

is path independent if the curve S encloses no singularities. When $G = 0$, this reduces to the path-independent integral developed by Eshelby [9] and Rice [13] for problems in non-linear elasticity—W then corresponds to the elastic energy density. The quantities ξ, η and ρ are defined by

$$\xi = \frac{\partial u}{\partial x}, \qquad \eta = \frac{\partial u}{\partial y}, \qquad \rho = (\xi^2 + \eta^2)^{1/2}. \tag{3.6}$$

Let us consider the integral (3.4) taken around the boundary $ABCDEA$ of Figure 3, including a curve C_ε, of small characteristic dimension ε, excluding the point C. Near C, the behaviour of u will be of the form given in (2.42), and we wish to find an expression for the constant K. It can be seen that the integrand of I is $0(1/r)$ as $r \to 0$, and hence we will get a finite contribution from C_ε as $\varepsilon \to 0$. The contributions to I from BC, CD and EA are all zero and, hence, from (3.4),

$$I_c = \int_0^1 [W]_{x=+\infty}\, dy - \int_0^1 [W]_{x=-\infty}\, dy, \tag{3.7}$$

where

$$\begin{aligned} I_c = \lim_{\varepsilon\to 0}\int_{C_\varepsilon} \rho^{1/n-1}\left[\xi\eta\, dx + \left(\frac{n}{n+1}\rho^2 - \xi^2\right) dy\right] \\ + \lim_{\varepsilon\to 0}\int_{C_\varepsilon}(-Gu)\, dy, \end{aligned} \tag{3.8}$$

and the second integral of equation (3.8) tends to zero because u is bounded at C.

To calculate I_c it is convenient to work with the function ψ given by (2.6). As stated in the previous section (see (2.20), (2.42) and (2.45)), the function ψ has the form

$$\psi \sim Q\rho^{-1/n}\sin\phi, \qquad \rho \to \infty\ (r \to 0), \tag{3.9}$$

where

$$\xi = -\rho\sin\phi, \qquad \eta = \rho\cos\phi, \tag{3.10}$$

and

$$Q = -\lambda\frac{n}{n+1}|K|^{1+1/n} \tag{3.11}$$

where K is the constant to be found, and $\lambda = \operatorname{sign}(K) = \pm 1$.

From the path independence of I we may consider curves C_ε such that ρ is constant, and then let $\rho \to \infty$. From (2.7) and (3.10) we have

$$\begin{aligned} x &= -\psi_\rho\sin\phi - \frac{\psi_\phi}{\rho}\cos\phi, \\ y &= \psi_\rho\cos\phi - \frac{\psi_\phi}{\rho}\sin\phi, \end{aligned} \tag{3.12}$$

which implies, from (3.9), that

$$\begin{aligned} x &\sim Q\rho^{-(1+1/n)}\left(\frac{1}{n}\sin^2\phi - \cos^2\phi\right), \\ y &\sim -Q\rho^{-(1+1/n)}\left(1+\frac{1}{n}\right)\sin\phi\cos\phi, \end{aligned} \tag{3.13}$$

as $\rho \to \infty$. Hence, for fixed ρ,

$$\begin{aligned} dx &\sim Q\left(1+\frac{1}{n}\right)\rho^{-(1+1/n)}\sin 2\phi\, d\phi, \\ dy &\sim -Q\left(1+\frac{1}{n}\right)\rho^{-(1+1/n)}\cos 2\phi\, d\phi, \end{aligned} \tag{3.14}$$

and these results, together with (3.8), give

$$\begin{aligned} I_c &= -\left(1+\frac{1}{n}\right)Q\int_0^{\pi/2}\left[\sin\phi\cos\phi\sin 2\phi + \left(\frac{n}{n+1} - \sin^2\phi\right)\cos 2\phi\right] d\varphi \\ &= -\frac{\pi Q(n+1)}{4n}. \end{aligned} \tag{3.15}$$

Hence, from equation (3.11), we have

$$I_c = -\frac{\pi\lambda}{4}|K|^{1+1/n}, \tag{3.16}$$

where $\lambda = \operatorname{sign}(K) = \pm 1$.

With reference to equation (3.7), calculation of K requires knowledge of the behaviour of W as $x \to \pm\infty$. For $x \to +\infty$, we expect that $u(x, y) \sim u^+(y)$, with

$$\frac{d}{dy}\left(\frac{du^+}{dy}\left|\frac{du^+}{dy}\right|^{1/n-1}\right) = -G, \tag{3.17}$$

and

$$u^+(1) = u^+(0) = 0. \tag{3.18}$$

The solution for $u^+(y)$ gives

$$|\eta^+| = \left|\frac{du^+}{dy}\right| = G^n\left|\frac{1}{2} - y\right|^n, \tag{3.19}$$

which implies that

$$\int_0^1 [W]_{x=+\infty}\, dy = \frac{n}{(n+1)(n+2)}\left(\frac{1}{2}G\right)^{n+1}. \tag{3.20}$$

For $x \to -\infty$, $u \sim u^-(y)$ where $u^-(y)$ satisfies (3.17) together with

$$u^-(1) = \frac{du^-}{dy}(0) = 0. \tag{3.21}$$

The solution for $u^-(y)$ gives

$$|\eta^-| = \left|\frac{du^-}{dy}\right| = G^n y^n, \tag{3.22}$$

and hence

$$\int_0^1 [W]_{x=-\infty}\, dy = \frac{n}{(n+1)(n+2)}G^{n+1}. \tag{3.23}$$

Finally, combination of (3.7), (3.16), (3.20) and (3.23) gives, for the constant K,

$$K = -\left[\frac{4n(1-(\frac{1}{2})^{n+1})}{\pi(n+1)(n+2)}\right]^{n/(n+1)} G^n. \tag{3.24}$$

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