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On rational approximations of the exponential function at rational points

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Let p, q, u, and v be any four positive integers, and let further δ be a number in the interval $0 < \delta \le 2$. In this note an effective lower bound for q will be obtained which insures that

$$\left|e^{u/v} - \frac{p}{q}\right| > q^{-(2+\delta)} .$$

In the special case when $u = v = 1$, it was shown by J. Popken, *Math. Z.* 29 (1929), 525-541, that

$$\left|e - \frac{p}{q}\right| > q^{-\{2+(c/\log \log q)\}} \quad \text{for} \quad q \ge C .$$

Here c and C are two positive absolute constants which, however, were not determined explicity. A similarly non-effective result was given in my paper, *J. reine angew. Math.* 166 (1932), 118-150.

The method of this note depends again on the classical formulae by Hermite which I applied also *op. cit.*

1.

Denote by m and n two non-negative integers and put

$$F(w) = \frac{w^m(w-1)^n}{m!n!} \quad \text{and} \quad F(z;\, w) = \sum_{k=0}^{\infty} z^{-k-1}\left(\frac{d}{dw}\right)^k F(w) .$$

A simple calculation shows that

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$$F(z;\ 0) = \sum_{k=m}^{m+n} z^{-k-1}\binom{k}{m}\frac{(-1)^{m+n-k}}{(m+n-k)!} \quad \text{and} \quad F(z;\ 1) = \sum_{k=n}^{m+n} z^{-k-1}\binom{k}{n}\frac{1}{(m+n-k)!}\ .$$

Put therefore

$$P(z) = (m+n)!z^{m+n+1}F(z;\ 0) \quad \text{and} \quad Q(z) = (m+n)!z^{m+n+1}F(z;\ 1)\ .$$

Then

$$P(z) = \sum_{k=m}^{m+n} k!(-1)^{m+n-k}\binom{k}{m}\binom{m+n}{k}z^{m+n-k}\ , \quad Q(z) = \sum_{k=n}^{m+n} k!\binom{k}{n}\binom{m+n}{k}z^{m+n-k}\ .$$

By these formulae, $P(z)$ and $Q(z)$ are polynomials in z of the degrees n and m , and with integral coefficients divisible by $m!$ and $n!$, respectively.

It also follows from the definitions of $F(w)$ and $F(z;\ w)$ that

$$\int_0^1 F(w)e^{-zw}dw = F(z;\ 0) - F(z;\ 1)e^{-z}\ .$$

Hence, on putting

$$R(z) = (m+n)!z^{m+n+1}e^{z}\int_0^1 F(w)e^{-zw}dw\ ,$$

we obtain Hermite's identity

(1) $$P(z)e^{z} - Q(z) = R(z)\ .$$

2.

From now on denote by r a positive integer. The identity (1) will be used only in the two special cases when either

(A) $m = r - 1\ ,\quad n = r$, or

(B) $m = r\ ,\quad n = r - 1$.

Thus in either case $m + n = 2r - 1$, and the functions $P(z)$, $Q(z)$, and $R(z)$ take the following special forms.

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Case A:

$$P_A(z) = \sum_{k=r-1}^{2r-1} k!(-1)^{k-1}\binom{k}{r-1}\binom{2r-1}{k} z^{2r-k-1} ,$$

$$Q_A(z) = \sum_{k=r}^{2r-1} k!\binom{k}{r}\binom{2r-1}{k} z^{2r-k-1} ,$$

$$R_A(z) = \binom{2r-1}{r} z^{2r} e^{z} \int_0^1 w^{r-1}(w-1)^{r} e^{-zw} dw ,$$

and

Case B:

$$P_B(z) = \sum_{k=r}^{2r-1} k!(-1)^{k-1}\binom{k}{r}\binom{2r-1}{k} z^{2r-k-1} ,$$

$$Q_B(z) = \sum_{k=r-1}^{2r-1} k!\binom{k}{r-1}\binom{2r-1}{k} z^{2r-k-1} ,$$

$$R_B(z) = \binom{2r-1}{r} z^{2r} e^{z} \int_0^1 w^{r}(w-1)^{r-1} e^{-zw} dw .$$

By these formulae, $P_A(z)$ and $Q_B(z)$ have the exact degree r, and $P_B(z)$ and $Q_A(z)$ have the exact degree $r-1$; further both $R_A(z)$ and $R_B(z)$ vanish at $z = 0$ to the order $2r$. Further, by (1),

$$P_A(z)e^{z} - Q_A(z) = R_A(z) \quad \text{and} \quad P_B(z)e^{z} - Q_B(z) = R_B(z) .$$

Therefore the determinant

$$D(z) = \begin{vmatrix} P_A(z), & Q_A(z) \\ P_B(z), & Q_B(z) \end{vmatrix} = \begin{vmatrix} P_A(z), & -R_A(z) \\ P_B(z), & -R_B(z) \end{vmatrix}$$

is a polynomial in z of the exact degree $2r$ which has at $z = 0$ a zero of order $2r$. This determinant can therefore be written as

$$D(z) = dz^{2r}$$

where d is a constant distinct from zero. Thus

(2) $$D(z) \neq 0 \text{ if } z \neq 0 .$$

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3.

All four polynomials $P_A(z)$, $Q_A(z)$, $P_B(z)$, $Q_B(z)$ have integral coefficients divisible by $(r-1)!$, and they have the degrees r or $r-1$ in z. Denote by u and v two positive integers and put in the preceding formulae

$$z = u/v .$$

Let further

$$U_A = \frac{v^r}{(r-1)!} P_A(u/v) , \quad V_A = \frac{v^r}{(r-1)!} Q_A(u/v) , \quad W_A = \frac{v^r}{(r-1)!} R_A(u/v) ,$$

and

$$U_B = \frac{v^r}{(r-1)!} P_B(u/v) , \quad V_B = \frac{v^r}{(r-1)!} Q_B(u/v) , \quad W_B = \frac{v^r}{(r-1)!} R_B(u/v) .$$

Then, by (2), U_A, V_A, U_B, V_B are integers of determinant

(3) $$U_A V_B - U_B V_A \neq 0 .$$

We require upper estimates for these six quantities and therefore introduce the two maxima

$$X = \max\left(|U_A|, |V_A|, |U_B|, |V_B|\right) \quad \text{and} \quad Y = \max\left(|W_A|, |W_B|\right) .$$

In the sum

$$\sum_{k=0}^{2r-1} \binom{2r-1}{k} = 2^{2r-1}$$

the two terms with $k = r - 1$ and $k = r$ are identical and hence satisfy the inequality

$$\binom{2r-1}{r-1} = \binom{2r-1}{r} \leq 2^{2r-2} .$$

It follows that also

$$\frac{(2r-1)!}{(r-1)!} = r!\binom{2r-1}{r-1} \leq 2^{2r-2} r! .$$

Further, in the sums defining U_A, V_A, U_B, and V_B, the factors

$$k!, \binom{k}{r}, \text{ and } \binom{k}{r-1}$$

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assume their largest possible values when $k = 2r - 1$. Hence trivially,

$$X \leq \frac{(2r-1)!}{(r-1)!} \binom{2r-1}{r-1}^2 (u+v)^r ,$$

and therefore

$$X \leq 2^{6r-6} r! (u+v)^r . \tag{4}$$

Next, when w lies in the interval $0 \leq w \leq 1$,

$$\max\left(w^{r-1}(w-1)^r , w^r(w-1)^{r-1}\right) \leq \left(w(w-1)\right)^{r-1} \leq 4^{-(r-1)} \text{ and } e^{-zw} \leq 1 .$$

The integrals for W_A and W_B imply therefore that

$$Y \leq \frac{v^r}{(r-1)!} 2^{2r-2} (u/v)^{2r} e^{u/v} 4^{-(r-1)}$$

and hence that

$$Y \leq \frac{e^{u/v}}{(r-1)!} \left(u^2/v\right)^r . \tag{5}$$

4.

These upper estimates will now be applied to the rational approximations of $e^{u/v}$. For this purpose, denote by p and q any two positive integers and put

$$qe^{u/v} - p = d .$$

An explicit lower estimate for $|d|$ can be obtained by the following considerations.

Since the determinant (3) is distinct from zero, the same is true for at least one of the two determinants

$$\begin{vmatrix} U_A , & V_A \\ q , & p \end{vmatrix} \text{ and } \begin{vmatrix} U_B , & V_B \\ q , & p \end{vmatrix} .$$

Denote then by C the suffix A or B for which

$$\begin{vmatrix} U_C , & V_C \\ q , & p \end{vmatrix} \neq 0 .$$

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The two equations

$$U_C e^{u/v} - V_C = W_C \quad \text{and} \quad q e^{u/v} - p = d$$

imply that

$$pU_C - qV_C = dU_C - qW_C \ .$$

Here the left-hand side is an integer distinct from zero and is thus at least of absolute value 1 . Thus the following deduction can be made.

LEMMA. *If the integer* r *can be chosen such that*

(6) $$|2qW_C| \le 1 ,$$

then also

(7) $$|2dU_C| \ge 1 .$$

5.

By the definition of Y and by its upper estimate (5),

$$|2qW_C| \le \frac{2e^{u/v}}{(r-1)!} \left(u^2/v\right)^r . q \ .$$

Assume now that r satisfies the inequality

$$(r-1)! \ge 2e^{u/v} \left(u^2/v\right)^r q \ ,$$

or, equivalently, the inequality

(8) $$e^r r! \ge 2re^{u/v} \left(eu^2/v\right)^r q \ .$$

Then the condition (6) holds, and it follows from (4) and (7) that

(9) $$|d| \ge \left(2^{6r-5} r! (u+v)^r\right)^{-1} .$$

To simplify these formulae, denote by ε a constant in the interval

$$0 < \varepsilon \le \frac{1}{2} ,$$

so that

$$(1+2\varepsilon)\left(1 - \frac{\varepsilon}{2}\right) = 1 + \frac{3\varepsilon}{2} - \varepsilon^2 = 1 + \varepsilon + \varepsilon\left(\frac{1}{2} - \varepsilon\right) \ge 1 - \varepsilon \ .$$

Further assume from now on that both

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$$(10) \qquad q \ge \left(2re^{u/v}(eu^2/v)^r\right)^{1/\varepsilon}$$

and

$$(11) \qquad e^r r! \ge q^{1+\varepsilon} .$$

Then the inequality (8) likewise is satisfied. Now almost trivially,

$$r! > r^r e^{-r} .$$

The hypothesis (11) may then be replaced by the following stronger one,

$$(12) \qquad r^r \ge q^{1+\varepsilon} .$$

In order to satisy this condition, assume that q , in addition to the condition (10), also has the property that

$$(13) \qquad \log\log\log q \le \frac{\varepsilon}{2}\log\log q ,$$

and then define r as a function of q by the equation

$$(14) \qquad r = \left[\frac{(1+2\varepsilon)\log q}{\log\log q}\right] + 1 .$$

Then

$$r > \frac{(1+2\varepsilon)\log q}{\log\log q}$$

and therefore

$$\log r > \log\log q - \log\log\log q$$

because $\log(1+2\varepsilon)$ is positive. It follows that

$$r\log r > \frac{(1+2\varepsilon)\log q}{\log\log q}\left(1 - \frac{\varepsilon}{2}\right)\log\log q \ge (1+\varepsilon)\log q .$$

This shows that (12) is a consequence of the two formulae (13) and (14).

6.

Also the condition (13) may be replaced by a simpler one. If for the moment c is any positive constant and x a positive variable, the function

$$x^{-c}\log x$$

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assumes its maximum at $x = e^{1/c}$ so that for all x ,

$$\log x \leq (ce)^{-1}x^{c} \;;$$

hence for $c = 1/2$,

$$\log x \leq (2/e)x^{1/2} \;. \tag{15}$$

On putting $x = \log\log q$ in (13), this inequality takes the form

$$\log x \leq \frac{\varepsilon}{2}\, x$$

and is thus certainly satisfied if

$$(2/e)x^{1/2} \leq \frac{\varepsilon}{2}\, x \;, \text{ that is, if } \; x \geq \left(\frac{4}{e\varepsilon}\right)^{2} .$$

The condition (13) is thus a consequence of the simpler condition

$$q \geq e^{e^{(4/e\varepsilon)^{2}}} . \tag{16}$$

We have just replaced the condition (13) for q by (16). As a next simplification, the other condition (10) for q will now be replaced by two conditions in which the integer r no longer occurs.

Evidently,

$$2r \leq e^{r}$$

for all positive integers r . Hence the inequality (10) is certainly satisfied if

$$q^{\varepsilon} \geq e^{u/v}\left(e^{2}u/v\right)^{r} . \tag{17}$$

Here, by (14),

$$r \leq \frac{(1+2\varepsilon)\log q}{\log\log q} + 1 \;.$$

Further

$$0 < \varepsilon \leq \tfrac{1}{2} \;, \quad 1 + 4\varepsilon \leq 3 \;, \quad \left(\frac{4}{e\varepsilon}\right)^{2} > 8 \;.$$

Since

$$\log\log q \geq \left(\frac{4}{e\varepsilon}\right)^2 ,$$

it follows therefore easily that

$$r < \frac{3\log q}{\log\log q} ,$$

and so also

$$\left(e^2u/v\right)^r < q^{\frac{3\log\left(e^2u/v\right)}{\log\log q}}$$

Hence the single inequality (17) for q may be replaced by the pair of conditions

$$(18) \qquad q \geq e^{\left(e^2u/v\right)^{6/\varepsilon}} \quad \text{and} \quad q \geq e^{2(u/v)/\varepsilon} .$$

It depends on the size of u/v which of these two conditions is the stronger one.

7.

By what so far has been proved, the three conditions (16) and (18) for q, together with the definition (14) of r as a function of q, imply the inequalitiy (9) for d. This inequality still contains the parameter r, which will now be eliminated from it.

We found already that

$$r \leq \frac{(1+2\varepsilon)\log q}{\log\log q} + 1 .$$

Here $\log(1+x) < x$ for positive x, so that

$$\log r \leq \log\log q + \log(1+2\varepsilon) - \log\log\log q + \log\left(1 + \frac{\log\log q}{(1+2\varepsilon)\log q}\right) \leq$$

$$\leq \log\log q - \log\log\log q + 2\varepsilon + \frac{\log\log q}{\log q} .$$

Here, by (16),

$$\log\log\log q \geq 2\log\left(\frac{4}{e\varepsilon}\right) \geq 2\log\frac{8}{e} \geq 2 \geq 4\varepsilon$$

and

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$$\frac{\log\,\log q}{\log q} \le \left(\frac{4}{e\varepsilon}\right)^2 e^{-(4/e\varepsilon)^2} \le \left(\frac{4}{e\varepsilon}\right)^2 \left(\frac{1}{6}\left(\frac{4}{e\varepsilon}\right)^6\right)^{-1} \le 6\left(\frac{e\varepsilon}{4}\right)^4 < \varepsilon\ .$$

It follows that

$$\varepsilon\,\frac{\log q}{\log\,\log q} > 1\ ,$$

whence

$$(19)\qquad r < \frac{(1+3\varepsilon)\log q}{\log\,\log q}\ .$$

Further

$$\log r \le \log\,\log q - 4\varepsilon + 2\varepsilon - \varepsilon < \log\,\log q\ ,$$

so that

$$r\log r < (1+3\varepsilon)\log q$$

and finally

$$(20)\qquad r! \le r^r < q^{1+3\varepsilon}\ .$$

On substituting this lower estimate for $r!$ in (9), it follows that

$$|d| > 32\left(\{64(u+v)\}^r q^{1+3\varepsilon}\right)^{-1}\ .$$

Here, by (19),

$$\{64(u+v)\}^r < q^{\frac{(1+3\varepsilon)\log\{64(u+v)\}}{\log\,\log q}} \le q^{\frac{5\log\{64(u+v)\}}{2\log\,\log q}}\ .$$

In order to simplify this estimate, add to the previous conditions (16) and (18) for q the following new one,

$$(21)\qquad q \ge e^{\{64(u+v)\}^{5/(2\varepsilon)}}\ .$$

The lower bound for $|d|$ takes then the simple form

$$|d| > 32q^{-(1+4\varepsilon)}\ .$$

Here it is convenient to put

$$4\varepsilon = \delta\ .$$

Then the result just obtained may be formulated as follows.

THEOREM. *Let* δ *be a constant in the interval* $0 < \delta \le 2$ *, and let*

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p, q, u, and v be four positive integers where q is restricted by the conditions

$$q \geq e^{e^{(1/e\delta)^2}} , \quad q \geq e^{(e^2 u/v)^{24/\delta}} , \quad q \geq e^{8(u/v)/\delta} , \quad q \geq e^{\{64(u+v)\}^{10/\delta}} .$$

Then

$$\left| e^{u/v} - \frac{p}{q} \right| > q^{-(2+\delta)} .$$

The conditions for q in this theorem are stronger than necessary, and it would in particular be possible to replace the first condition by a weaker one. However, such a change would probably complicate the proof and the final result. My aim was to establish an effective lower bound for $\left| e^{u/v} - \frac{p}{q} \right|$ which does not contain any unknown constants.

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