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The General Definition of the Complex Monge–Ampère Operator on Compact Kähler Manifolds

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Abstract. We introduce a wide subclass $\mathcal{F}(X,\omega)$ of quasi-plurisubharmonic functions in a compact Kähler manifold, on which the complex Monge-Ampère operator is well defined and the convergence theorem is valid. We also prove that $\mathcal{F}(X,\omega)$ is a convex cone and includes all quasi-plurisubharmonic functions that are in the Cegrell class.

1 Introduction

Let X be a compact connected Kähler manifold of dimension n, equipped with the fundamental form ω given in local coordinates by $\omega = \frac{1}{2}\sum_{\alpha,\beta} g_{\alpha\bar{\beta}} dz^\alpha \wedge d\bar{z}^\beta$, where $(g_{\alpha\bar{\beta}})$ is a positive definite Hermitian matrix and $d\omega = 0$. The smooth volume form associated with this Kähler metric is the n-th wedge product ω^n. Denote by $\mathrm{PSH}(X,\omega)$ the set of upper semi-continuous functions $u\colon X \to \mathbb{R}\cup\{-\infty\}$ such that u is integrable in X with respect to the volume form ω^n and $\omega_u := \omega + dd^c u \geq 0$ on X, where $d = \partial + \bar{\partial}$ and $d^c = i(\bar{\partial} - \partial)$. These functions are called quasi-plurisubharmonic functions (quasi-psh for short) and play an important role in the study of positive closed currents in X (see [9].) A quasi-psh function is locally the difference of a plurisubharmonic function and a smooth function. Therefore, many properties of plurisubharmonic functions hold also for quasi-psh functions. Following Bedford and Taylor [2], the complex Monge–Ampère operator $(\omega+dd^c)^n$ is locally and hence globally well defined for all bounded quasi-psh functions in X. Some important results of the complex Monge–Ampère operator for bounded quasi-psh functions have been obtained by Kolodziej [13, 14] and Blocki [4]. It is also known that the complex Monge–Ampère operator does not work well for all unbounded quasi-psh functions. Otherwise, we would lose some of the essential properties that the complex Monge-Ampère operator should have (see [1, 12]). In a bounded domain of $\mathbb{C}^n$ one usually needs certain assumptions on values of functions near the boundary of the domain to define complex Monge–Ampère measures of unbounded plurisubharmonic functions, see the Cegrell class [7, 8] where Cegrell introduced the largest subclass $\mathcal{E}(\Omega)$ of plurisubharmonic functions in a bounded hyperconvex domain Ω for which the complex Monge–Ampère operator is well defined and the monotone convergence theorem is valid. However, such a technique does not seem to work for quasi-psh functions in a compact Kähler manifold because we lose boundary. On the other hand, Bedford and Taylor already observed [3] that for each quasi-psh function

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u the complex Monge–Ampère measure $\omega_u^n := (\omega + dd^c u)^n$ is well defined on its non-polar subset $\{u > -\infty\}$. We obtained several convergence theorems for complex Monge–Ampère measures without mass on pluripolar sets [17]. In this paper we introduce a quite large subclass $\mathcal{F}(X,\omega)$ of quasi-psh functions on which images of the complex Monge–Ampère operator are well-defined positive measures and may have positive masses on pluripolar sets. We prove that the set $\mathcal{F}(X,\omega)$ is a convex cone and includes all quasi-psh functions which are in the Cegrell class. Our main result is the following convergence theorem of the complex Monge–Ampère operator in $\mathcal{F}(X,\omega)$.

Theorem 3.6 **(Convergence Theorem)** *Let $0 \le p < \infty$. Suppose that $u_0 \in \mathcal{F}(X,\omega)$ and that $g \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ is nonpositive. If $u_j,\ u \in \mathcal{F}(X,\omega)$ are such that $u_j \to u$ in Cap_ω on X and $u_j \ge u_0$, then $(-g)^p\,\omega_{u_j}^n \to (-g)^p\,\omega_u^n$ weakly in X.*

As a direct consequence we have the following

Corollary 3.7 *Let $0 \le p < \infty$ and $0 \ge g \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$. If $u_j,\ u \in \mathcal{F}(X,\omega)$ are such that $u_j \searrow u$ or $u_j \nearrow u$ in X, then $(-g)^p\,\omega_{u_j}^n \to (-g)^p\,\omega_u^n$ weakly in X.*

For bounded quasi-psh functions, Corollary 3.7 is a slightly stronger version of the well-known monotone convergence theorem due to Bedford and Taylor [2].

2 The Class $\mathcal{F}(X,\omega)$

In this section we first introduce the subclass $\mathcal{F}(X,\omega)$ of quasi-psh functions, on which images of the complex Monge–Ampère operator are finite positive measures in X. We obtain some characterizations of functions in $\mathcal{F}(X,\omega)$. Finally, we prove that $\mathcal{F}(X,\omega)$ is a star-shaped and convex set.

Recall that the Monge–Ampère capacity Cap_ω associated with the Kähler form ω is defined by

$$\mathrm{Cap}_\omega(E) = \sup\Big\{ \int_E \omega_u^n\ ;\ u \in \mathrm{PSH}(X,\omega) \text{ and } -1 \le u \le 0 \Big\}$$

for any Borel set E in X. The capacity Cap_ω was introduced by Kolodziej [13] and is comparable to the relative Monge–Ampère capacity of Bedford and Taylor [2], and hence vanishes exactly on pluripolar sets of X. Recall also that a sequence μ_j of positive Borel measures is said to be uniformly absolutely continuous with respect to Cap_ω on X, or we write that $\mu_j \ll \mathrm{Cap}_\omega$ on X uniformly for all j, if for any $\varepsilon > 0$ there exists $\delta > 0$ such that $\mu_j(E) < \varepsilon$ for all j and Borel sets $E \subset X$ with $\mathrm{Cap}_\omega(E) < \delta$. Denote by $\mathrm{PSH}^{-1}(X,\omega)$ the subset of functions u in $\mathrm{PSH}(X,\omega)$ with $\max_X u \le -1$. Given a function $u \in \mathrm{PSH}^{-1}(X,\omega)$, we define the measure $(-u)\,\omega_u^{n-1} \wedge \omega$ in X which is zero in $\{u = -\infty\}$ and

$$\int_E (-u)\,\omega_u^{n-1} \wedge \omega = \lim_{j\to\infty} \int_{E\cap\{u>-j\}} (-\max(u,-j))\,\omega_{\max(u,-j)}^{n-1} \wedge \omega$$

for all $k \ge 1$ and $E \subset \{u > -k\}$. In a completely similar way, we define the measure $\omega_u^{n-1} \wedge \omega := \chi_{\{u>-\infty\}}\,\omega_u^{n-1} \wedge \omega$, where $\chi_{\{u>-\infty\}}$ is the characteristic function of the set $\{u > -\infty\}$. It is worth pointing out that in general neither the measure $(-u)\,\omega_u^{n-1} \wedge \omega$ nor $\omega_u^{n-1} \wedge \omega$ is locally finite in X. However, we have the following result.

Proposition 2.1 *Let $u \in \mathrm{PSH}^{-1}(X, \omega)$. Suppose that*

$$-\max(u, -j)\,\omega^{n-1}_{\max(u,-j)} \wedge \omega \ll \mathrm{Cap}_\omega$$

on X uniformly for all $j = 1, 2, \ldots$. Then the following statements hold:

(i) *$(-u)\,\omega_u^{n-1} \wedge \omega$ and $\omega_u^{n-1} \wedge \omega$ are finite positive measures in X;*
(ii) *$\max(u, -j)\,\omega^{n-1}_{\max(u,-j)} \to u\,\omega_u^{n-1}$ and $\omega^{n-1}_{\max(u,-j)} \to \omega_u^{n-1}$ as currents as $j \to \infty$;*
(iii) *$(-u)\,\omega_u^{n-1} \wedge \omega \ll \mathrm{Cap}_\omega$ on X.*

Proof Since

$$\begin{aligned}\int_X (-u)\,\omega_u^{n-1} \wedge \omega &= \lim_{k\to\infty}\lim_{j\to\infty} \int_{u>-k} \bigl(-\max(u,-j)\bigr)\,\omega^{n-1}_{\max(u,-j)} \wedge \omega \\ &\le \sup_j \int_X \bigl(-\max(u,-j)\bigr)\,\omega^{n-1}_{\max(u,-j)} \wedge \omega < \infty,\end{aligned}$$

we obtain that $(-u)\,\omega_u^{n-1} \wedge \omega$ is a finite positive measure and so is $\omega_u^{n-1} \wedge \omega$. Write

$$\begin{aligned}\max(u,-j)\,\omega^{n-1}_{\max(u,-j)} = \chi_{\{u\le -j\}}\ &\max(u,-j)\,\omega^{n-1}_{\max(u,-j)} \\ &+ \chi_{\{u>-j\}}\ \max(u,-j)\,\omega^{n-1}_{\max(u,-j)},\end{aligned}$$

where the first term on the right-hand side tends to zero and the second one tends to $u\,\omega_u^{n-1}$ as $j \to \infty$. Similarly, we get that $\omega^{n-1}_{\max(u,-j)} \to \omega_u^{n-1}$ as $j \to \infty$. Moreover, for any $E \subset X$ with $\mathrm{Cap}_\omega(E) \neq 0$ we can take an open set G in X such that $E \subset G$ and $\mathrm{Cap}_\omega(G) \le 2\,\mathrm{Cap}_\omega(E)$. Then

$$\int_E (-u)\,\omega_u^{n-1} \wedge \omega \le \int_G (-u)\,\omega_u^{n-1} \wedge \omega \le \limsup_{j\to\infty} \int_G \bigl(-\max(u,-j)\bigr)\,\omega^{n-1}_{\max(u,-j)} \wedge \omega,$$

which implies that $(-u)\,\omega_u^{n-1} \wedge \omega \ll \mathrm{Cap}_\omega$ on X. ■

Let $\mathcal{F}(X, \omega)$ be the subset of functions in $\mathrm{PSH}^{-1}(X, \omega)$ which satisfy the hypotheses of Proposition 2.1. The complex Monge–Ampère measure ω_u^n of a function u in $\mathcal{F}(X, \omega)$ is defined by the sum

$$\omega_u^n := \omega \wedge \omega_u^{n-1} + dd^c(u\,\omega_u^{n-1}),$$

where the currents $u\,\omega_u^{n-1}$ and ω_u^{n-1} are the limits of two sequences

$$\max(u,-j)\,\omega^{n-1}_{\max(u,-j)} \quad \text{and} \quad \omega^{n-1}_{\max(u,-j)},$$

respectively. Locally using the inequality $\big(\omega + dd^c(\phi + u)\big)^n \geq n\,\omega_u^{n-1} \wedge \omega$, where $\omega = dd^c\phi$, we can easily see that $(-u)\,\omega_u^{n-1} \wedge \omega \ll \mathrm{Cap}_\omega$ in X for any

$$u \in \mathrm{PSH}^{-1}(X, \omega) \cap L^\infty(X),$$

where $L^\infty(X)$ denotes the set of bounded functions in X. Hence for bounded quasi-psh functions, our definition of the complex Monge–Ampère operator coincides with Bedford's and Taylor's definition [2]. Denote by $L^1(X, \mu)$ the set of integrable functions in X with respect to the positive measure μ. Now we give a characterization of functions in $\mathcal{F}(X, \omega)$.

Theorem 2.2 *Let $u \in \mathrm{PSH}^{-1}(X, \omega)$. Then $u \in \mathcal{F}(X, \omega)$ if and only if*

$$u \in L^1(X,\, \omega_u^{n-1} \wedge \omega),$$

where $\omega_u^{n-1} := \lim_{j\to\infty} \omega_{\max(u,-j)}^{n-1}$ as currents and $\omega_{\max(u,-j)}^{n-1} \wedge \omega \ll \mathrm{Cap}_\omega$ on X uniformly for $j = 1, 2, \ldots$.

Proof We prove first the "only if" part. Assume that $u \in \mathcal{F}(X, \omega)$. By Proposition 2.1 we have that $\omega_{\max(u,-j)}^{n-1} \wedge \omega \leq (-\max(u, -j))\,\omega_{\max(u,-j)}^{n-1} \wedge \omega \ll \mathrm{Cap}_\omega$ on X uniformly for all j, and $\omega_{\max(u,-j)}^{n-1} \to \omega_u^{n-1}$. Hence, by the lower semi-continuity of $-u$, we get that

$$\int_X (-\max(u, -t))\,\omega_u^{n-1} \wedge \omega \leq \limsup_{j\to\infty} \int_X (-\max(u, -j))\,\omega_{\max(u,-j)}^{n-1} \wedge \omega < \infty$$

for all $t \geq 1$. Thus, we have $u \in L^1(X,\, \omega_u^{n-1} \wedge \omega)$. Now we prove the "if" part. Observe that for any $k > 1$, by [3, Proposition 4.2] we get

$$\begin{aligned}
\chi_{\{u>-k\}}\,\omega_u^{n-1} \wedge \omega &= \lim_{j\to\infty} \chi_{\{u>-k\}} \omega_{\max(u,-j)}^{n-1} \wedge \omega \\
&= \lim_{j\to\infty} \chi_{\{\max(u,-k)>-k\}} \omega_{\max(u,-j)}^{n-1} \wedge \omega \\
&= \lim_{j\to\infty} \chi_{\{\max(u,-k)>-k\}} \omega_{\max(u,-j,-k)}^{n-1} \wedge \omega \\
&= \chi_{\{u>-k\}}\,\omega_{\max(u,-k)}^{n-1} \wedge \omega.
\end{aligned}$$

Hence, for any Borel set $E \subset X$ and $k > 1$, we have that

$$\begin{aligned}
\int_E \omega_u^{n-1} \wedge \omega &\leq \int_{u<-k+1} \omega_u^{n-1} \wedge \omega + \int_{E\cap\{u>-k\}} \omega_{\max(u,-k)}^{n-1} \wedge \omega \\
&\leq \limsup_{j\to\infty} \int_{u<-k+1} \omega_{\max(u,-j)}^{n-1} \wedge \omega + \int_E \omega_{\max(u,-k)}^{n-1} \wedge \omega,
\end{aligned}$$

where we have used that the set $\{u < -k + 1\}$ is open. Since $\omega_{\max(u,-j)}^{n-1} \wedge \omega \ll \mathrm{Cap}_\omega$ on X uniformly for j, we have $\omega_u^{n-1} \wedge \omega \ll \mathrm{Cap}_\omega$ on X. It then follows from

$u \in L^1(X, \omega_u^{n-1} \wedge \omega)$ that $(-u)\,\omega_u^{n-1} \wedge \omega \ll \mathrm{Cap}_\omega$ on X. For any $j \geq k_1 > 1$ we get

$$\begin{aligned}
&\int_{u \leq -k_1} (-\max(u,-j))\omega_{\max(u,-j)}^{n-1} \wedge \omega \\
&\quad \leq j \int_{u \leq -j} \omega_{\max(u,-j)}^{n-1} \wedge \omega + \int_{-j<u\leq -k_1} (-u)\omega_u^{n-1} \wedge \omega \\
&\quad = j \int_X \omega^n - j \int_{u>-j} \omega_{\max(u,-j)}^{n-1} \wedge \omega + \int_{-j<u\leq -k_1} (-u)\omega_u^{n-1} \wedge \omega \\
&\quad \leq j \int_X \omega^n - j \int_{u>-j} \omega_u^{n-1} \wedge \omega \\
&\qquad + \int_{u\leq -k_1} (-u)\omega_u^{n-1} \wedge \omega \leq 2 \int_{u\leq -k_1} (-u)\omega_u^{n-1} \wedge \omega.
\end{aligned}$$

Hence, for any Borel set $E_1 \subset X$ and $j \geq k_1 > 1$, we have

$$\begin{aligned}
&\int_{E_1} (-\max(u,-j))\omega_{\max(u,-j)}^{n-1} \wedge \omega \\
&\quad \leq 2 \int_{u\leq -k_1} (-u)\omega_u^{n-1} \wedge \omega + k_1 \int_{E_1 \cap \{u > -k_1\}} \omega_{\max(u,-j)}^{n-1} \wedge \omega \\
&\quad := A_{k_1} + B_{k_1,j}.
\end{aligned}$$

Given $\varepsilon > 0$, take $k_\varepsilon > 1$ such that $A_{k_\varepsilon} \leq \varepsilon$. Since $\omega_{\max(u,-j)}^{n-1} \wedge \omega \ll \mathrm{Cap}_\omega$ on X uniformly for all j, there exists $\delta > 0$ such that $k_\varepsilon \int_{E_1} \omega_{\max(u,-j)}^{n-1} \wedge \omega \leq \varepsilon$ for all j and $E_1 \subset X$ with $\mathrm{Cap}_\omega(E_1) \leq \delta$. Therefore, we have proved that

$$\int_{E_1} (-\max(u,-j))\,\omega_{\max(u,-j)}^{n-1} \wedge \omega \leq 2\,\varepsilon$$

holds for all $j \geq k_\varepsilon$ (hence for all j) and $E_1 \subset X$ with $\mathrm{Cap}_\omega(E_1) \leq \delta$. So $u \in \mathcal{F}(X,\omega)$. ■

Suppose that Ω is a hyperconvex subset in $\mathbb{C}^n$. Cegrell [8] introduced the largest subclass $\mathcal{E}(\Omega)$ of plurisubharmonic functions in Ω, for which the complex Monge–Ampère operator is well defined and the monotone convergence theorem is valid. Our next theorem says that $\mathcal{F}(X,\omega)$ includes all quasi-psh functions that are in the Cegrell class. Recall that a negative plurisubharmonic function u in Ω is said to belong to $\mathcal{E}(\Omega)$ if for each $z_0 \in \Omega$ there exists a neighborhood U_{z_0} of z_0 and a decreasing sequence u_j of bounded plurisubharmonic functions in Ω, vanishing on the boundary $\partial\Omega$, such that $u_j \searrow u$ on U_{z_0} and $\sup_j \int_\Omega (dd^c u_j)^n < \infty$. Blocki [5] proved that it is a local property to belong to $\mathcal{E}(\Omega)$, that is, if $\Omega = \bigcup_j \Omega_j$, then $u \in \mathcal{E}(\Omega)$ if and only if $u|_{\Omega_j} \in \mathcal{E}(\Omega_j)$ for each j. We call u in $\mathrm{PSH}^{-1}(X,\omega)$ for a Cegrell function in X if there exists a finite covering $\{B_s\}_1^m$ of X with hyperconvex subsets B_s such that $\phi_s + u \in \mathcal{E}(B_s)$ for all s, where ϕ_s is a local Kähler potential defined in a neighborhood of the closure of B_s, *i.e.*, $\omega = dd^c\phi_s$ on $B_s = \{\phi_s < 0\}$. Now we prove the following.

Theorem 2.3 *If u is a Cegrell function in X, then $u \in \mathcal{F}(X,\omega)$.*

Proof Take a new finite open covering $\{B'_s\}_1^m$ of X such that $B'_s \Subset B_s$ for all s. By [8] there exists a decreasing sequence u_j^s of bounded plurisubharmonic functions in B_s, vanishing on ∂B_s, such that $u_j^s \searrow \phi_s + u$ on B'_s and $\sup_j \int_{B_s} (dd^c u_j^s)^n < \infty$. Since $\operatorname{Cap}_\omega$ is comparable to the relative Monge–Ampère capacity of Bedford and Taylor, (see [2, 14], by [16, Lemma 6] we get that

$$-\max(u,-j)\,\omega^{n-1}_{\max(u,-j)} \wedge \omega \le \bigl(-\phi_s - \max(u,-j)\bigr)\,\omega^{n-1}_{\max(u,-j)} \wedge \omega \ll \operatorname{Cap}_\omega$$

uniformly for all j on each B'_s and hence on X. Therefore, $u \in \mathcal{F}(X,\omega)$. ∎

Recall that a sequence u_j of functions in X is said to be convergent to a function u in $\operatorname{Cap}_\omega$ on X if for any $\delta > 0$ we have

$$\lim_{j\to\infty} \operatorname{Cap}_\omega\bigl(\{z \in X\,;\, |u_j(z) - u(z)| > \delta\}\bigr) = 0.$$

For a uniformly bounded sequence in $\operatorname{PSH}(X,\omega)$, the convergence in capacity implies weak convergence of the complex Monge–Ampère measures [15]. Now we prove that the set $\mathcal{F}(X,\omega)$ is a convex cone. First, we need a lemma.

Lemma 2.4 *Let u, $v \in \mathcal{F}(X,\omega)$. Then*

$$\int_{u<v} (v-u)\,\omega_v^{n-1} \wedge \omega \le \int_{u<v} (v-u)\,\omega_u^{n-1} \wedge \omega.$$

If, furthermore, u and v are bounded, then for all integers $0 \le l \le n-1$ we have

$$\int_{u<v} (v-u)\,\omega_v^{l} \wedge \omega_u^{n-1-l} \wedge \omega \le \int_{u<v} (v-u)\,\omega_u^{n-1} \wedge \omega.$$

Proof We only prove the first inequality since the proof of the second one is similar. Assume first that u and v are bounded in X. By [6,9] there exist a constant $A > 1$ and two sequences u_j, $v_k \in \operatorname{PSH}(X, A\omega) \cap C^\infty(X)$ such that $u_j \searrow u$ and $v_k \searrow v$ in X. Given $\varepsilon > 0$, assume first that $\{u_j < v_k\} \ne X$. Then $\max(v_k, u_j + \varepsilon) = u_j + \varepsilon$ near the boundary of the set $\{u_j < v_k\}$. Take a smooth subset E_ε such that

$$\{u_j + \varepsilon < v_k\} \Subset E_\varepsilon \Subset \{u_j < v_k\},$$

and write $T = \sum_{l=0}^{n-2} \omega_u^l \wedge \omega_v^{n-2-l} \wedge \omega$. Using Stokes theorem we get

$$\begin{aligned}\int_{u_j<v_k} \bigl(\max(v_k, u_j+\varepsilon) - u_j - \varepsilon\bigr)\,\bigl((A\omega + dd^c u_j) - (A\omega + dd^c \max(v_k, u_j+\varepsilon))\bigr) \wedge T \\ = \int_{E_\varepsilon} d\bigl(\max(v_k, u_j+\varepsilon) - u_j\bigr) \wedge d^c\bigl(\max(v_k, u_j+\varepsilon) - u_j\bigr) \wedge T \ge 0,\end{aligned}$$

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which holds even when $\{u_j < v_k\} = X$. Hence we obtain

$$\begin{aligned}\int_{u_j<v_k} &(\max(v_k, u_j+\varepsilon) - u_j)(A\omega + dd^c u_j) \wedge T \\ &\geq \int_{u_j<v_k} (\max(v_k, u_j+\varepsilon) - u_j - \varepsilon)(A\omega + dd^c \max(v_k, u_j+\varepsilon)) \wedge T \\ &\geq \int_{u_j<v_k} (v_k - u_j)\,(A\omega + dd^c \max(v_k, u_j+\varepsilon)) \wedge T - \varepsilon A \int_X \omega^n.\end{aligned}$$

It turns out from the monotone convergence theorem [2] that

$$(v_k - u_j)\,(A\omega + dd^c \max(v_k, u_j+\varepsilon)) \wedge T \longrightarrow (v_k - u_j)\,(A\omega + dd^c v_k)) \wedge T$$

weakly in the open set $\{u_j < v_k\}$ as $\varepsilon \searrow 0$. Letting $\varepsilon \searrow 0$ and applying Lebesgue monotone convergence theorem, we obtain the inequality

$$\int_{u_j<v_k} (v_k - u_j)\,(A\omega + dd^c v_k) \wedge T \leq \int_{u_j<v_k} (v_k - u_j)\,(A\omega + dd^c u_j) \wedge T.$$

Therefore, we have $\int_{u_j<v} (v - u_j)\,(A\omega + dd^c v_k) \wedge T \leq \int_{u<v_k} (v_k - u)\,(A\omega + dd^c u_j) \wedge T$. On the other hand, we have that u_j, v_k are uniformly bounded, $u_j \to u$ in Cap_ω and $v_k \to v$ in Cap_ω on X. So for any $\delta > 0$ the inequality

$$\int_{u<v} (v - u_j)\,(A\omega + dd^c v_k) \wedge T \leq \int_{u\leq v} (v_k - u)\,(A\omega + dd^c u_j) \wedge T + \delta$$

holds for all j, k large enough. Then by the quasicontinuity of quasi-psh functions, we can assume without loss of generality that $\{u < v\}$ is open and $\{u \leq v\}$ is closed. It turns out from the proof of [15, Theorem 1] that

$$(v - u_j)\,(A\omega + dd^c v_k) \wedge T \longrightarrow (v - u_j)\,(A\omega + dd^c u) \wedge T$$

as $k \to \infty$ and $(v-u)\,(A\omega + dd^c u_j) \wedge T \longrightarrow (v-u)\,(A\omega + dd^c v) \wedge T$ as $j \to \infty$ weakly in X. Letting $k \to \infty$ and then $j \to \infty$, we obtain $\int_{u<v} (v - u)\,(A\omega + dd^c v) \wedge T \leq \int_{u\leq v} (v - u)\,(A\omega + dd^c u) \wedge T + \delta$. Applying $t\,v$ instead of v for $A > t > 1$ in the last inequality and then letting $t \searrow 1$, $\delta \searrow 0$, we get

$$\int_{u<v} (v - u)\,(A\omega + dd^c v) \wedge T \leq \int_{u<v} (v - u)\,(A\omega + dd^c u) \wedge T,$$

which yields that $\int_{u<v} (v - u)\,\omega_v^{n-1} \wedge \omega \leq \int_{u<v} (v - u)\,\omega_u^{n-1} \wedge \omega$ for all bounded quasi-psh functions u and v.

Now, for $u,\ v \in \mathcal{F}(X,\omega)$, we have

$$\int_{\max(u,-j)<\max(v,-k)} (\max(v,-k) - \max(u,-j))\omega^{n-1}_{\max(v,-k)} \wedge \omega \\ \leq \int_{\max(u,-j)<\max(v,-k)} (\max(v,-k) - \max(u,-j))\,\omega^{n-1}_{\max(u,-j)} \wedge \omega.$$

Letting $k \to \infty$, by the definition of $\omega_v^{n-1} \wedge \omega$ we get

$$\int_{\max(u,-j)<v} (v - \max(u,-j))\omega_v^{n-1} \wedge \omega \\ \leq \int_{\max(u,-j)<v} (v - \max(u,-j))\omega^{n-1}_{\max(u,-j)} \wedge \omega,$$

which by Fatou lemma implies that

$$\begin{aligned}
&\int_{u<v} (v-u)\omega_v^{n-1} \wedge \omega \\
&\quad \leq \liminf_{j\to\infty} \int_{\max(u,-j)<v} (v - \max(u,-j))\,\omega^{n-1}_{\max(u,-j)} \wedge \omega \\
&\quad \leq \liminf_{j\to\infty} \int_{u<v} (\max(v,-j) - \max(u,-j))\,\omega^{n-1}_{\max(u,-j)} \wedge \omega \\
&\quad \leq \limsup_{j\to\infty} \int_{-s<u<v} (\max(v,-j) - \max(u,-j))\,\omega^{n-1}_{\max(u,-j)} \wedge \omega \\
&\qquad + \limsup_{j\to\infty} \int_{\{u\leq -s\}\cap\{u<v\}} (\max(v,-j) - \max(u,-j))\,\omega^{n-1}_{\max(u,-j)} \wedge \omega \\
&\quad = \int_{-s<u<v} (v-u)\,\omega_u^{n-1} \wedge \omega \\
&\qquad + \limsup_{j\to\infty} \int_{\{u\leq -s\}\cap\{u<v\}} (\max(v,-j) - \max(u,-j))\,\omega^{n-1}_{\max(u,-j)} \wedge \omega
\end{aligned}$$

for all $s > 1$. Since $(-\max(v,-j))\,\omega^{n-1}_{\max(u,-j)}\wedge\omega \leq (-\max(u,-j))\,\omega^{n-1}_{\max(u,-j)}\wedge\omega \ll \mathrm{Cap}_\omega$ in the set $\{u < v\}$ uniformly for all j, letting $s \to \infty$ we get the required inequality. ■

Theorem 2.5 *Let $u_0 \in \mathcal{F}(X,\omega)$. If $u \in \mathrm{PSH}^{-1}(X,\omega)$ satisfies $u \geq u_0$ in X, then $u \in \mathcal{F}(X,\omega)$. Moreover, we have that $(-u)\,\omega_u^{n-1} \wedge \omega \ll \mathrm{Cap}_\omega$ on X uniformly for all $u \in \mathrm{PSH}^{-1}(X,\omega)$ with $u \geq u_0$ in X.*

Proof Given $k \geq 1$ and $j \geq 1$. Write $u_j = \max(u, -j)$. Then $u_j/3 \in \mathcal{F}(X, \omega)$ and by Lemma 2.4 we have

$$\begin{aligned}
\int_{u_j<-k} (-u_j)\omega_{u_j}^{n-1} \wedge \omega &\leq 2 \int_{u_j<-k} (-k/2 - u_j)\omega_{u_j}^{n-1} \wedge \omega \\
&\leq 3^{n-1}\, 2 \int_{u_j<-k/2} (-k/2 - u_j)\omega_{\frac{1}{3}u_j}^{n-1} \wedge \omega \\
&\leq 3^n \int_{u_0<u_j/3-k/3} (u_j/3 - k/3 - u_0)\, \omega_{\frac{1}{3}u_j}^{n-1} \wedge \omega \\
&\leq 3^n \int_{u_0<u_j/3-k/3} (u_j/3 - k/3 - u_0)\, \omega_{u_0}^{n-1} \wedge \omega \\
&\leq 3^n \int_{u_0<-k/3} (-u_0)\, \omega_{u_0}^{n-1} \wedge \omega.
\end{aligned}$$

Thus, by $(-u_0)\, \omega_{u_0}^{n-1} \wedge \omega \ll \mathrm{Cap}_\omega$ in X we obtain that $(-u_j)\omega_{u_j}^{n-1} \wedge \omega \ll \mathrm{Cap}_\omega$ in X uniformly for all j, which yields that $u \in \mathcal{F}(X, \omega)$. Moreover, for all $k \geq 1$, $t \geq 1$, and $u \in \mathrm{PSH}^{-1}(X, \omega)$ with $u \geq u_0$, we have

$$\begin{aligned}
\int_{\max(u,-t)<-k} (-u)\omega_u^{n-1} \wedge \omega &\leq \limsup_{j\to\infty} \int_{\max(u,-t)<-k} (-u_j)\omega_{u_j}^{n-1} \wedge \omega \\
&\leq \limsup_{j\to\infty} \int_{u_j<-k} (-u_j)\omega_{u_j}^{n-1} \wedge \omega \\
&\leq 3^n \int_{u_0<-k/3} (-u_0)\, \omega_{u_0}^{n-1} \wedge \omega.
\end{aligned}$$

Letting $t \to \infty$, we get $\int_{u<-k}(-u)\omega_u^{n-1} \wedge \omega \leq 3^n \int_{u_0<-k/3}(-u_0)\, \omega_{u_0}^{n-1} \wedge \omega$. Hence, together with $\chi_{\{u>-k-1\}}\, \omega_u^{n-1} \wedge \omega = \chi_{\{u>-k-1\}}\, \omega_{\max(u,-k-1)}^{n-1} \wedge \omega$, we obtain that $(-u)\, \omega_u^{n-1} \wedge \omega \ll \mathrm{Cap}_\omega$ on X uniformly for all $u \geq u_0$. ■

As a direct consequence of Theorem 2.5 we have the following.

Corollary 2.6 *Let* $u \in \mathcal{F}(X, \omega)$. *Then* $\max(u, v) \in \mathcal{F}(X, \omega)$ *and* $t\,u \in \mathcal{F}(X, \omega)$ *for all* $v \in \mathrm{PSH}^{-1}(X, \omega)$ *and* $0 \leq t \leq 1$.

Now we prove the following.

Theorem 2.7 *The set* $\mathcal{F}(X, \omega)$ *is convex, that is, for any* $u, v \in \mathcal{F}(X, \omega)$ *and* $0 \leq t \leq 1$ *we have that* $t\,u + (1-t)\,v \in \mathcal{F}(X, \omega)$.

Proof Given $u, v \in \mathcal{F}(X, \omega)$. Then $u/2 + v/2 \in \mathrm{PSH}^{-1}(X, \omega)$. We only need to prove that $u/2 + v/2 \in \mathcal{F}(X, \omega)$. From Corollary 2.6 it turns out that $u/2 \in \mathcal{F}(X, \omega)$

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and $v/2 \in \mathcal{F}(X,\omega)$. Then

$$\begin{aligned}\omega^{n-1}_{\max(u/2,-j)+\max(v/2,-j)} \wedge \omega &= 1/2^{n-1}\left(\omega_{\max(u,-2j)} + \omega_{\max(v,-2j)}\right)^{n-1} \wedge \omega \\ &\le n!/2^{n-1} \sum_{l=0}^{n-1} \omega^{l}_{\max(u,-2j)} \wedge \omega^{n-1-l}_{\max(v,-2j)} \wedge \omega.\end{aligned}$$

Write $u_{2j} = \max(u,-2j)$ and $v_{2j} = \max(v,-2j)$. For all $j \ge k \ge 1$ and $0 \le l \le n-1$ we have

$$\begin{aligned}\int_{u\le -k} \omega^{l}_{u_{2j}} \wedge \omega^{n-1-l}_{v_{2j}} \wedge \omega &= 1/k \int_{u\le -k}\big(-\max(u,-k)\big)\, \omega^{l}_{u_{2j}} \wedge \omega^{n-1-l}_{v_{2j}} \wedge \omega \\ &\le 1/k \int_X (-u_{2j})\, \omega^{l}_{u_{2j}} \wedge \omega^{n-1-l}_{v_{2j}} \wedge \omega \\ &\le 1/k \int_{u_{2j}\le v_{2j}} (-u_{2j})\, \omega^{l}_{u_{2j}} \wedge \omega^{n-1-l}_{v_{2j}} \wedge \omega \\ &\quad + 1/k \int_{u_{2j}> v_{2j}} (-v_{2j})\, \omega^{l}_{u_{2j}} \wedge \omega^{n-1-l}_{v_{2j}} \wedge \omega.\end{aligned}$$

From Lemma 2.4 it follows that

$$\begin{aligned}&\int_{u_{2j}\le v_{2j}} (-u_{2j})\, \omega^{l}_{u_{2j}} \wedge \omega^{n-1-l}_{v_{2j}} \wedge \omega \\ &\le 2 \int_{u_{2j}\le v_{2j}} \big(v_{2j}/2 - u_{2j}\big)\, \omega^{l}_{u_{2j}} \wedge \omega^{n-1-l}_{v_{2j}} \wedge \omega \\ &\le 2^{n-l} \int_{u_{2j}< v_{2j}/2} \big(v_{2j}/2 - u_{2j}\big)\, \omega^{l}_{u_{2j}} \wedge \omega^{n-1-l}_{v_{2j}/2} \wedge \omega \\ &\le 2^{n-l} \int_{u_{2j}< v_{2j}/2} \big(v_{2j}/2 - u_{2j}\big)\, \omega^{n-1}_{u_{2j}} \wedge \omega \le 2^{n-l} \sup_j \int_X (-u_{2j})\, \omega^{n-1}_{u_{2j}} \wedge \omega \\ &< \infty.\end{aligned}$$

Similarly, we have

$$\int_{u_{2j}>v_{2j}} (-v_{2j})\, \omega^{l}_{u_{2j}} \wedge \omega^{n-1-l}_{v_{2j}} \wedge \omega \le 2^{l+1} \sup_j \int_X (-v_{2j})\, \omega^{n-1}_{v_{2j}} \wedge \omega < \infty.$$

Hence we have proved that there exists a constant $A > 0$ such that

$$\int_{\{u\le -k\}\cup\{v\le -k\}} \omega^{n-1}_{\max(u/2,-j)+\max(v/2,-j)} \wedge \omega \le A/k$$

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for all $j \geq k \geq 1$. Thus, for $j \geq 2\,k \geq 1$ we have

$$\begin{aligned}\int_{u/2+v/2\leq -k} \omega^{n-1}_{\max(u/2+v/2,-j)} \wedge \omega &= \int_X \omega^n - \int_{u/2+v/2>-k} \omega^{n-1}_{\max(u/2+v/2,-j)} \wedge \omega \\ &= \int_X \omega^n - \int_{u/2+v/2>-k} \omega^{n-1}_{\max(u/2,-j)+\max(v/2,-j)} \wedge \omega \\ &= \int_{u/2+v/2\leq -k} \omega^{n-1}_{\max(u/2,-j)+\max(v/2,-j)} \wedge \omega \leq A/k,\end{aligned}$$

which implies that $\omega^{n-1}_{\max(u/2+v/2,-j)} \wedge \omega \ll \mathrm{Cap}_\omega$ on X uniformly for all j and hence

$$\omega^{n-1}_{u/2+v/2} \wedge \omega = \lim_{j\to\infty} \omega^{n-1}_{\max(u/2+v/2,-j)} \wedge \omega = \lim_{j\to\infty} \omega^{n-1}_{\max(u/2,-j)+\max(v/2,-j)} \wedge \omega.$$

It then follows from the lower semi-continuity of $-u/2 - v/2$ that

$$\begin{aligned}&\int_X (-u/2 - v/2)\, \omega^{n-1}_{u/2+v/2} \wedge \omega \\ &\leq \limsup_{j\to\infty} \int_X \big(-\max(u/4,-j/2) - \max(v/4,-j/2)\big)\, \omega^{n-1}_{\max(u/2,-j)+\max(v/2,-j)} \wedge \omega \\ &< \infty.\end{aligned}$$

By Theorem 2.2 we have obtained that $u/2 + v/2 \in \mathcal{F}(X,\omega)$. ■

As consequences we have the following.

Corollary 2.8 *Let $u_0,\ u_1, \dots, u_{n-1} \in \mathcal{F}(X,\omega)$. Then*

$$-u_0\, \omega_{u_1} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega \ll \mathrm{Cap}_\omega \quad \textit{on } X.$$

Proof Since

$$(u_0 + u_1 + \cdots + u_{l-1})/l = (1/l)\, u_{l-1} + (1 - 1/l)\,(u_0 + u_1 + \cdots + u_{l-2})/(l-1)$$

for $l = 2, 3 \dots, n$, using the induction principle and Theorem 2.7 we get that $f := (u_0 + u_1 + \cdots + u_{n-1})/n \in \mathcal{F}(X,\omega)$. Hence we have that

$$\begin{aligned}-u_0\, \omega_{u_1} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega &\leq -n^n f\, \omega_{u_1/n} \wedge \omega_{u_2/n} \wedge \cdots \wedge \omega_{u_{n-1}/n} \wedge \omega \\ &\leq n^n\,(-f)\, \omega^{n-1}_f \wedge \omega \ll \mathrm{Cap}_\omega\end{aligned}$$

on X. ■

Using Corollary 2.8 and following the proof of Lemma 2.4, we now get a stronger version of Lemma 2.4.

Corollary 2.9 *Let $u,\ v \in \mathcal{F}(X,\omega)$ and $0 \le l \le n-1$. Then*

$$\int_{u<v} (v-u)\,\omega_v^l \wedge \omega_u^{n-1-l} \wedge \omega \le \int_{u<v} (v-u)\,\omega_u^{n-1} \wedge \omega.$$

Corollary 2.10 *Let $u_0 \in \mathcal{F}(X,\omega)$. Then*

$$-u_1\,\omega_{u_2} \wedge \omega_{u_3} \wedge \cdots \wedge \omega_{u_n} \wedge \omega \ll \mathrm{Cap}_\omega \quad \text{on } X$$

uniformly for all $u_l \in \mathrm{PSH}^{-1}(X,\omega)$ with $u_l \ge u_0$ and $l = 1,2,\ldots,n$.

Proof Since $f := (u_1 + u_2 + \cdots + u_n)/n \ge u_0$ and $f \in \mathcal{F}(X,\omega)$, by Theorem 2.5 we get that $-u_1\,\omega_{u_2} \wedge \omega_{u_3} \wedge \cdots \wedge \omega_{u_n} \wedge \omega \le n^n\,(-f)\,\omega_f^{n-1} \wedge \omega \ll \mathrm{Cap}_\omega$ on X uniformly for all such functions u_l. ■

Remark. Corollary 2.10 implies that a function $u \in \mathrm{PSH}^{-1}(X,\omega)$ belongs to $\mathcal{F}(X,\omega)$ if and only if $\big(-\max(u,-j)\big)\,\omega^l_{\max(u,-j)} \wedge \omega^{n-l} \ll \mathrm{Cap}_\omega$ on X uniformly for all $j \ge 1$ and $0 \le l \le n-1$. The ω_u^n concentrating on $\{u > -\infty\}$ were studied by Guedj and Zeriahi [10].

3 A Convergence Theorem of the Complex Monge–Ampère Operator

In this section we prove a convergence theorem of the complex Monge–Ampère operator in $\mathcal{F}(X,\omega)$. We divide its proof into several lemmas.

Given $u_1, u_2, \ldots, u_{n-1} \in \mathcal{F}(X,\omega)$, by Corollary 2.8 the current $\omega_{u_1} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}}$ is well defined. Now for any $g \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$, we define the wedge product $\omega_{u_1} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega_g$ in a natural way:

$$\omega_{u_1} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega_g := \omega \wedge \omega_{u_1} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} + dd^c(g\,\omega_{u_1} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}}).$$

Then we have the following.

Lemma 3.1 *Let $u_0,\ u_1, \ldots, u_{n-1} \in \mathcal{F}(X,\omega)$ and $f,\ g \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$. Then the following equalities hold.*

(i) $\int_X (-g)\,dd^c f \wedge \omega_{u_1} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} = \int_X (-f)\,dd^c g \wedge \omega_{u_1} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}}$.
(ii) $\int_X (-g)\,dd^c u_0 \wedge \omega_{u_1} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} = \int_X (-u_0)\,dd^c g \wedge \omega_{u_1} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}}$.

Proof It is no restriction to assume that $f, g \le -2$ in X. Write $T = \omega_{u_1} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}}$. Take two sequences $f_j,\ g_k \in \mathrm{PSH}^{-1}(X, A\omega) \cap C^\infty(X)$ for some $A \ge 1$ such that $f_j \searrow f$ and $g_k \searrow g$ in X, (see [6, 9]. It follows from Dini's theorem and quasicontinuity of quasi-psh functions that $f_j \to f$ in Cap_ω on X. So, using $T \wedge \omega \ll \mathrm{Cap}_\omega$, we get $f_j\,T \to f\,T$ and hence $dd^c f_j \wedge T \to dd^c f \wedge T$ weakly in X.

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Similarly, $dd^c g_k \wedge T \to dd^c g \wedge T$ weakly in X. Thus we have

$$\begin{aligned}\int_X (-f_j)\, dd^c g \wedge T &= \lim_{k\to\infty} \int_X (-f_j)\, dd^c g_k \wedge T \\ &= \lim_{k\to\infty} \int_X (-g_k)\, dd^c f_j \wedge T \\ &= \lim_{k\to\infty} \int_X (-g_k)\, (A\omega + dd^c f_j) \wedge T - \lim_{k\to\infty} \int_X (-g_k)\, (A\omega) \wedge T \\ &= \int_X (-g)\, dd^c f_j \wedge T,\end{aligned}$$

where the last equality follows from the Lebesgue monotone convergence theorem. Then, by lower semi-continuity of $-g$, we get

$$\begin{aligned}\int_X (-f)\, dd^c g \wedge T &= \lim_{j\to\infty} \int_X (-f_j)\, dd^c g \wedge T \\ &= \lim_{j\to\infty} \int_X (-g) dd^c f_j \wedge T \\ &= \lim_{j\to\infty} \int_X (-g)(A\omega + dd^c f_j) \wedge T - \int_X (-g)(A\omega) \wedge T \\ &\geq \int_X (-g)\, dd^c f \wedge T.\end{aligned}$$

By symmetry we have abtained equality (i). Let $u_l = \max(u_0, -l)$. By (i) we have $\int_X (-g)\, dd^c u_l \wedge T = \int_X (-u_l)\, dd^c g \wedge T$. It follows from Corollary 2.8 that $u_0 T$ is a well-defined current and $u_l T \to u_0 T$ as currents in X. Hence we get

$$\begin{aligned}\int_X (-g)\, dd^c u_0 \wedge T \leq \lim_{l\to\infty} \int_X (-g)\, dd^c u_l \wedge T &= \lim_{l\to\infty} \int_X (-u_l)\, dd^c g \wedge T \\ &= \int_X (-u_0)\, dd^c g \wedge T.\end{aligned}$$

On the other hand,

$$\begin{aligned}\int_X (-u_0)\, dd^c g_k \wedge T = \lim_{l\to\infty} \int_X (-u_l)\, dd^c g_k \wedge T &= \lim_{l\to\infty} \int_X (-g_k)\, dd^c u_l \wedge T \\ &= \int_X (-g_k)\, dd^c u_0 \wedge T.\end{aligned}$$

Letting $k \to \infty$ we get $\int_X (-u_0)\, dd^c g \wedge T \leq \int_X (-g)\, dd^c u_0 \wedge T$. Hence we have proved equality (ii). ■

Lemma 3.2 *Let $u \in \mathcal{F}(X, \omega)$ and $g \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$. Then the following statements hold.*

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(i) $\omega^{n-1}_{\max(u,-j)} \wedge \omega_g \ll \mathrm{Cap}_\omega$ *on X uniformly for all j;*
(ii) *for each* $f \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$*, we have that* $f\,\omega^{n-1}_{\max(u,-j)} \wedge \omega_g \longrightarrow f\,\omega^{n-1}_u \wedge \omega_g$ *weakly in X as* $j \to \infty$*;*
(iii) $(-u)\,\omega^{n-1}_u \wedge \omega_g \ll \mathrm{Cap}_\omega$ *on X.*

Proof It is no restriction to assume that $g \le -2$ in X. Given $j \ge k \ge 1$. By Lemma 3.1 we have

$$\begin{aligned}\int_{u\le -k} \omega^{n-1}_{\max(u,-j)} \wedge \omega_g &\le 1/k \int_X \big(-\max(u,-k)\big)\, \omega^{n-1}_{\max(u,-j)} \wedge \omega_g \\ &= 1/k \int_X \big(-\max(u,-k)\big)\, \omega^{n-1}_{\max(u,-j)} \wedge \omega \\ &\quad + 1/k \int_X (-g)\, \omega^{n-1}_{\max(u,-j)} \wedge dd^c \max(u,-k) \\ &\le 1/k \int_X \big(-\max(u,-j)\big)\, \omega^{n-1}_{\max(u,-j)} \wedge \omega \\ &\quad + 1/k \int_X (-g)\, \omega^{n-1}_{\max(u,-j)} \wedge \omega_{\max(u,-k)} \\ &\le 1/k \sup_j \int_X \big(-\max(u,-j)\big)\, \omega^{n-1}_{\max(u,-j)} \wedge \omega \\ &\quad + 1/k \sup_X |g| \int_X \omega^n.\end{aligned}$$

Given a Borel set $E \subset X$. By [3, Proposition 4.2] for bounded quasi-psh functions, we get that $\int_E \omega^{n-1}_{\max(u,-j)} \wedge \omega_g \le \int_{u\le k} \omega^{n-1}_{\max(u,-j)} \wedge \omega_g + \int_E \omega^{n-1}_{\max(u,-k)} \wedge \omega_g$ for all $j \ge k \ge 1$, which implies (i).

To prove (ii), we prove first that $\omega^{n-1}_{\max(u,-j)} \wedge \omega_g \longrightarrow \omega^{n-1}_u \wedge \omega_g$ weakly in X as $j \to \infty$. Given a smooth function ψ, multiplying a small positive constant if necessary, we can assume $\psi \in \mathrm{PSH}(X,\omega) \cap C^\infty(X)$. Then we have

$$\begin{aligned}&\int_X \psi\, \omega^{n-1}_{\max(u,-j)} \wedge \omega_g - \int_X \psi\, \omega^{n-1}_u \wedge \omega_g \\ &\quad = \int_X \psi \left(\omega^{n-1}_{\max(u,-j)} \wedge \omega - \omega^{n-1}_u \wedge \omega\right) + \int_X g \left(\omega^{n-1}_{\max(u,-j)} - \omega^{n-1}_u\right) \wedge dd^c\psi,\end{aligned}$$

where by Proposition 2.1 the first term on the right-hand side tends to zero as $j \to \infty$. Take a sequence $g_k \in \mathrm{PSH}^{-1}(X, A\omega) \cap C^\infty(X)$ for some $A \ge 1$ such that $g_k \searrow g$ in X, (see [6, 9]). Write the second term as

$$\int_X g_k \left(\omega^{n-1}_{\max(u,-j)} - \omega^{n-1}_u\right) \wedge dd^c\psi + \int_X (g-g_k) \left(\omega^{n-1}_{\max(u,-j)} - \omega^{n-1}_u\right) \wedge dd^c\psi := B_{k,j} + C_{k,j}.$$

By the smoothness of ψ we have that $(\omega^{n-1}_{\max(u,-j)} + \omega^{n-1}_u) \wedge \omega_\psi \ll \mathrm{Cap}_\omega$ on X uniformly for all j. Since $g_k \to g$ in Cap_ω on X, we get that $C_{k,j} \to 0$ as $k \to \infty$ uniformly for all j. Then for each fixed k, $B_{k,j} \to 0$ as $j \to \infty$. Hence we have proved that

$$\omega^{n-1}_{\max(u,-j)} \wedge \omega_g \longrightarrow \omega^{n-1}_u \wedge \omega_g$$

weakly in X as $j \to \infty$. Together with (i), we get $\omega^{n-1}_u \wedge \omega_g \ll \mathrm{Cap}_\omega$ on X, (see the proof of Proposition 2.1). Now for $f \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$, we take a sequence $f_k \in \mathrm{PSH}(X, A\omega) \cap C^\infty(X)$ for some $A \ge 1$ such that $f_k \searrow f$ in X. Write

$$\begin{aligned} f\,\omega^{n-1}_{\max(u,-j)} \wedge \omega_g - f\,\omega^{n-1}_u \wedge \omega_g &= (f - f_k)\left(\omega^{n-1}_{\max(u,-j)} \wedge \omega_g - \omega^{n-1}_u \wedge \omega_g\right) \\ &\quad + f_k\left(\omega^{n-1}_{\max(u,-j)} \wedge \omega_g - \omega^{n-1}_u \wedge \omega_g\right), \end{aligned}$$

where for each fixed k the second term on the right-hand side tends to zero weakly as $j \to \infty$. Using (i) and $\omega^{n-1}_u \wedge \omega_g \ll \mathrm{Cap}_\omega$, we get that the first term converges weakly to zero uniformly for all j as $k \to \infty$. Thus we have obtained (ii).

Finally, by the lower semi-continuity of $-u$, for any $k \ge 1$ we obtain

$$\begin{aligned} &\int_X (-\max(u,-k))\omega^{n-1}_u \wedge \omega_g \\ &\quad \le \limsup_{j\to\infty} \int_X (-\max(u,-k))\,\omega^{n-1}_{\max(u,-j)} \wedge \omega_g \\ &\quad \le \sup_j \int_X (-\max(u,-j))\,\omega^{n-1}_{\max(u,-j)} \wedge \omega + \sup_X |g| \int_X \omega^n < \infty, \end{aligned}$$

which yields $u \in L^1(X, \omega^{n-1}_u \wedge \omega_g)$. Thus we have that $(-u)\,\omega^{n-1}_u \wedge \omega_g \ll \omega^{n-1}_u \wedge \omega_g \ll \mathrm{Cap}_\omega$ on X. ■

Lemma 3.3 *Let $u_0, u_1, \dots, u_{n-1} \in \mathcal{F}(X,\omega)$ and $g \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$. Suppose that a sequence $u_{1j} \in \mathrm{PSH}^{-1}(X,\omega)$ decreases to u_1 in X. Then the following statements hold:*

(i) $(-u_0)\,\omega_{u_1} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega_g \ll \mathrm{Cap}_\omega$ *on X;*

(ii) *for each $f \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$, we have that*

$$f\,\omega_{u_{1j}} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega_g \longrightarrow f\,\omega_{u_1} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega_g$$

weakly in X as $j \to \infty$;

(iii) $\omega_{u_{1j}} \wedge \omega_{u_2} \wedge \omega_{u_3} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega_g \ll \mathrm{Cap}_\omega$ *on X uniformly for all j.*

Proof Since $(u_0 + u_1 + \cdots + u_{n-1})/n \in \mathcal{F}(X,\omega)$, assertion (i) follows directly from (iii) of Lemma 3.2. Now we prove (ii). Given a smooth function ψ in X, we assume without loss of generality that $0 \le f, \psi \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$. Observe that $\varepsilon h^2 \in \mathrm{PSH}(X,\omega)$ if h is a bounded positive quasi-psh function in X and the constant ε satisfies $\max_X h \le 1/(2\varepsilon)$. Hence, applying the equality $\frac{\psi f}{2} = (\frac{\psi+f}{2})^2 - (\frac{\psi}{2})^2 - (\frac{f}{2})^2$, we can assume that $h := \psi f$ or $-h$ is a bounded quasi-psh function in X. By Lemma3.1, for each $k \geq 1$ we get

$$\begin{aligned}
&\Big| \int_X \psi f \,\omega_{u_{1j}} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega_g - \int_X \psi f \,\omega_{u_1} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega_g \Big| \\
&= \Big| \int_X (u_{1j} - u_1)\, dd^c h \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega_g \Big| \\
&\leq \int_X |u_{1j} - u_1|\, (\omega_h + \omega) \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega_g \\
&\leq \int_{u_1 < -k} |u_1|\, (\omega_h + \omega) \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega_g \\
&\qquad + \int_X |\max(u_{1j}, -k) - \max(u_1, -k)|\, (\omega_h + \omega) \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega_g,
\end{aligned}$$

where by (i) the first term on the right-hand side tends to zero as $k \to \infty$. For each fixed k, since $\max(u_{1j}, -k) \to \max(u_1, -k)$ in Cap_ω on X as $j \to \infty$, we get that the second term converges to zero as $j \to \infty$. Hence we have obtained (ii).

By (i) and [3, Theorem 3.2], assertion (iii) follows from the property that for any hyperconvex subset $\Omega \Subset X$ with $dd^c\phi = \omega$ and $\phi = 0$ on $\partial\Omega$ and any $h \in \mathrm{PSH}(\Omega) \cap L^\infty(\Omega)$, we have that $h\,\omega_{u_{1j}} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega_g \to h\,\omega_{u_1} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega_g$ weakly in Ω as $j \to \infty$. To prove this property, for each $\psi \in C_0^\infty(\Omega)$, we take a constant $\varepsilon > 0$ such that $\varepsilon\,(h - \sup_\Omega h - 1) > \phi$ on $\operatorname{supp}\psi$, and $\varepsilon\,(h - \sup_\Omega h - 1) < \phi$ near $\partial\Omega$. Set

$$f = \begin{cases} \max\big(\varepsilon\,(h - \sup_\Omega h - 1), \phi\big) - \phi & \text{in } \Omega, \\ 0 & \text{in } X \setminus \Omega. \end{cases}$$

Then $f \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$ and $\psi\, h = \varepsilon^{-1}\psi\phi + \varepsilon^{-1}\psi f + \psi \sup_\Omega h + \psi$. Hence, by the smoothness of ϕ and (ii), we get that

$$h\,\omega_{u_{1j}} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega_g \longrightarrow h\,\omega_{u_1} \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega_g$$

weakly in Ω as $j \to \infty$. Therefore, we have proved (iii). ■

Lemma 3.4 *Let $u_0, u_1, u_2, \ldots, u_{n-1} \in \mathcal{F}(X, \omega)$ and $g \in \mathrm{PSH}(X, \omega) \cap L^\infty(X)$. Then for almost all constants $1 \leq k < \infty$,*

$$\int_{u_1 < -k} (-k - u_1)\, dd^c u_0 \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega_g \leq \int_{u_1 < -k} (-u_0)\, dd^c u_1 \wedge \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega_g.$$

Proof Write $T = \omega_{u_2} \wedge \cdots \wedge \omega_{u_{n-1}} \wedge \omega_g$. Assume first that $0 \geq u_0, u_1 \in \mathrm{PSH}(X, A\omega) \cap C^\infty(X)$ with $A \geq 1$. Given $\varepsilon > 0$ and $k \geq 1$. Since $\max(u_1 + \varepsilon, -k) = u_1 + \varepsilon$ near

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$\partial\{u_1 < -k\}$ if it is not empty, we have that

$$\begin{aligned}\int_{u_1<-k} &(-k-u_1)\, dd^c u_0 \wedge T \\ &= \lim_{\varepsilon \searrow 0} \int_{u_1<-k} (\max(u_1+\varepsilon, -k) - u_1 - \varepsilon)\, dd^c u_0 \wedge T \\ &= \lim_{\varepsilon \searrow 0} \int_{u_1<-k} u_0\, dd^c\big(\max(u_1+\varepsilon,-k) - u_1 - \varepsilon\big) \wedge T \\ &= \int_{u_1<-k} (-u_0)\, dd^c u_1 \wedge T + \lim_{\varepsilon \searrow 0} \int_{u_1<-k} u_0\, dd^c \max(u_1+\varepsilon,-k) \wedge T.\end{aligned}$$

Since $\max(u_1+\varepsilon,-k)\, T \to \max(u_1,-k)\, T$ weakly in X as $\varepsilon \searrow 0$, we have

$$(A\omega + dd^c \max(u_1+\varepsilon,-k)) \wedge T \longrightarrow (A\omega + dd^c \max(u_1,-k)) \wedge T$$

weakly as $\varepsilon \searrow 0$. From the upper semi-continuity of $u_0 \leq 0$ in the open set $\{u_1 < -k\}$, it turns out that

$$\begin{aligned}\lim_{\varepsilon \searrow 0} \int_{u_1<-k} &u_0\, dd^c \max(u_1+\varepsilon,-k) \wedge T \\ &= \lim_{\varepsilon \searrow 0} \int_{u_1<-k} u_0 \left[\big(A\omega + dd^c \max(u_1+\varepsilon,-k)\big) - A\omega\right] \wedge T \\ &\leq \int_{u_1<-k} u_0\, dd^c \max(u_1,-k) \wedge T = 0.\end{aligned}$$

Hence we get $\int_{u_1<-k}(-k-u_1)\, dd^c u_0 \wedge T \leq \int_{u_1<-k}(-u_0)\, dd^c u_1 \wedge T$ for all $k \geq 1$ in the case of $0 \geq u_0,\ u_1 \in \mathrm{PSH}(X, A\omega) \cap C^\infty(X)$.

Secondly, assume that $u_0,\ u_1 \in \mathcal{F}(X,\omega) \cap L^\infty(X)$. By [6, 9] there exist negative functions $u_{0t},\ u_{1s} \in \mathrm{PSH}(X, A\omega) \cap C^\infty(X)$ with some $A \geq 1$ such that $u_{0t} \searrow u_0$ and $u_{1s} \searrow u_1$ in X. Since $\int_{u_1 \leq -k}\big(\omega_{u_1} + \omega\big) \wedge T$ is a decreasing function of k and hence continuous almost everywhere with respect to the Lebesgue measure, we have that $\int_{u_1=-k}\big(\omega_{u_1} + \omega\big) \wedge T = 0$ holds for almost all k in $[1,\infty)$. Given such a constant k, by the Fatou lemma and the lower semi-continuity of $-u_{1s}$, we get that

$$\begin{aligned}\int_{u_1<-k} &(-k-u_1)\, dd^c u_0 \wedge T \\ &= \int_{u_1<-k} (-k-u_1)\, (A\omega + dd^c u_0) \wedge T - A \int_{u_1<-k} (-k-u_1)\, \omega \wedge T \\ &\leq \liminf_{s\to\infty} \int_{u_{1s}<-k} (-k-u_{1s})\, (A\omega + dd^c u_0) \wedge T - A \int_{u_1<-k} (-k-u_1)\, \omega \wedge T\end{aligned}$$

$$\le \liminf_{s\to\infty} \limsup_{t\to\infty} \int_{u_{1s}<-k} (-k-u_{1s})\,(A\omega + dd^c u_{0t}) \wedge T$$
$$- \liminf_{s\to\infty} A \int_{u_1<-k} (-k-u_{1s})\,\omega \wedge T$$
$$= \liminf_{s\to\infty} \limsup_{t\to\infty} \int_{u_{1s}<-k} (-k-u_{1s})\, dd^c u_{0t} \wedge T$$
$$- A \liminf_{s\to\infty} \int_{u_{1s}\ge -k > u_1} (-k-u_{1s})\,\omega \wedge T.$$

Given $\delta > 0$, we have that

$$\Big| \int_{u_{1s}\ge -k>u_1} (-k-u_{1s})\,\omega\wedge T \Big| \le \delta \int_X \omega\wedge T + \int_{u_{1s}-u_1\ge\delta} (-u_1)\,\omega\wedge T \longrightarrow \delta \int_X \omega\wedge T$$

as $s\to\infty$, since $u_{1s}\to u_1$ in Cap_ω and $(-u_1)\,\omega\wedge T \ll \mathrm{Cap}_\omega$ on X. Hence we have

$$\int_{u_1<-k} (-k-u_1)\, dd^c u_0 \wedge T$$
$$\le \liminf_{s\to\infty} \limsup_{t\to\infty} \int_{u_{1s}<-k} (-k-u_{1s})\, dd^c u_{0t} \wedge T$$
$$\le \liminf_{s\to\infty} \limsup_{t\to\infty} \int_{u_{1s}<-k} (-u_{0t})\, dd^c u_{1s} \wedge T$$
$$= \liminf_{s\to\infty} \int_{u_{1s}<-k} (-u_0)\, dd^c u_{1s} \wedge T$$
$$\le \liminf_{s\to\infty} \int_{u_1\le -k} (-u_0)\,(A\omega + dd^c u_{1s}) \wedge T - A \liminf_{s\to\infty} \int_{u_{1s}<-k} (-u_0)\,\omega\wedge T$$
$$= \liminf_{s\to\infty} \int_{u_1\le -k} (-u_0)\,(A\omega + dd^c u_{1s}) \wedge T - A \int_{u_1\le -k} (-u_0)\,\omega\wedge T.$$

By Lemma 3.3 and quasicontinuity of quasi-psh functions, it is no restriction to assume that $\{u_1 \le -k\}$ is a closed set and hence the last limit inferior does not exceed $\int_{u_1\le -k} (-u_0)\,(A\omega + dd^c u_1) \wedge T$. So we have obtained

$$\int_{u_1<-k} (-k-u_1)\, dd^c u_0 \wedge T \le \int_{u_1<-k} (-u_0)\, dd^c u_1 \wedge T$$

for all $u_0,\ u_1 \in \mathcal{F}(X,\omega)\cap L^\infty(X)$ and almost all k in $[1,\infty)$.

Finally, let $u_0,\ u_1 \in \mathcal{F}(X,\omega)$. For almost all constants k in $[1,\infty)$ we have that $\int_{u_1=-k} (\omega_{u_1}+\omega)\wedge T = 0$ and

$$\int_{\max(u_1,-s)<-k} (-k-\max(u_1,-s))\, dd^c \max(u_0,-t) \wedge T$$
$$\le \int_{\max(u_1,-s)<-k} (-\max(u_0,-t))\, dd^c \max(u_1,-s) \wedge T$$

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for all integers $s, t \geq 1$. Letting $s \to \infty$ and applying the same proof as above, we have $\int_{u_1<-k}(-k-u_1)\, dd^c \max(u_0,-t) \wedge T \leq \int_{u_1<-k}\big(-\max(u_0,-t)\big)\, dd^c u_1 \wedge T$, and then letting $t \to \infty$ we get the required inequality. ■

Lemma 3.5 *Let $u_0 \in \mathcal{F}(X,\omega)$ and $g \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$. Then*

$$\int_{u<-k} (-u)\, \omega_u^{n-1} \wedge \omega_g \longrightarrow 0, \quad \text{as} \quad k \to \infty,$$

uniformly for all $u \in \mathrm{PSH}^{-1}(X,\omega)$ with $u \geq u_0$ in X.

Proof Given $u \in \mathrm{PSH}^{-1}(X,\omega)$ with $u \geq u_0$. Take a sequence $1 \leq k_1 \leq k_2 \leq \cdots \leq k_j \to \infty$ such that Lemma 3.4 holds for the functions u and u_0 when $k = k_j/2^i$, where $i = 1,\dots,n-1$ and $j = 1,2,\dots$. Hence we have

$$\begin{aligned}
\int_{u<-k_j} (-u)\, \omega_u^{n-1} \wedge \omega_g &\leq \int_{u_0<-k_j} (-u_0)\, \omega_u^{n-1} \wedge \omega_g \\
&\leq 2 \int_{u_0<-k_j} (-k_j/2 - u_0)\, \omega_u^{n-1} \wedge \omega_g \\
&\leq 2 \int_{u_0<-k_j/2} (-k_j/2 - u_0)\, \omega \wedge \omega_u^{n-2} \wedge \omega_g \\
&\quad + 2 \int_{u_0<-k_j/2} (-k_j/2 - u_0)\, dd^c u \wedge \omega_u^{n-2} \wedge \omega_g \\
&\leq 2 \int_{u_0<-k_j/2} (-k_j/2 - u_0)\, \omega \wedge \omega_u^{n-2} \wedge \omega_g \\
&\quad + 2 \int_{u_0<-k_j/2} (-u)\, dd^c u_0 \wedge \omega_u^{n-2} \wedge \omega_g \\
&\leq 2 \int_{u_0<-k_j/2} (-u_0)\, \omega \wedge \omega_u^{n-2} \wedge \omega_g + 2 \int_{u_0<-k_j/2} (-u_0)\, \omega_{u_0} \wedge \omega_u^{n-2} \wedge \omega_g \\
&= 2 \int_{u_0<-k_j/2} (-u_0)\, (\omega + \omega_{u_0}) \wedge \omega_u^{n-2} \wedge \omega_g \\
&\leq 2^2 \int_{u_0<-k_j/2^2} (-u_0)\, (\omega + \omega_{u_0})^2 \wedge \omega_u^{n-3} \wedge \omega_g \leq \dots \\
&\leq 2^{n-1} \int_{u_0<-k_j/2^{n-1}} (-u_0)\, (\omega + \omega_{u_0})^{n-1} \wedge \omega_g,
\end{aligned}$$

which, by Lemma 3.3 and the equality $(\omega+\omega_{u_0})^{n-1} = \sum_{l=0}^{n-1} \binom{n-1}{l}\, \omega^l \wedge \omega_{u_0}^{n-1-l}$, tends to zero as $j \to \infty$. ■

We are now in a position to prove the convergence theorem.

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***Theorem 3.6* (Convergence Theorem)** *Let $0 \le p < \infty$. Suppose that $0 \ge g \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$ and $u_0 \in \mathcal{F}(X,\omega)$. If $u_j,\ u \in \mathrm{PSH}^{-1}(X,\omega)$ are such that $u_j \to u$ in Cap_ω on X and $u_j \ge u_0$, then $(-g)^p\,\omega_{u_j}^n \to (-g)^p\,\omega_u^n$ weakly in X.*

Proof Given $k \ge 1$, write

$$(-g)^p\,\omega_{u_j}^n - (-g)^p\,\omega_u^n = (-g)^p(\omega_{u_j}^n - \omega_{\max(u_j,-k)}^n) + (-g)^p\,(\omega_{\max(u_j,-k)}^n - \omega_{\max(u,-k)}^n) + (-g)^p(\omega_{\max(u,-k)}^n - \omega_u^n) := A_{k,j} + B_{k,j} + C_k.$$

For each fixed k, by [17, Theorem 1] we have that $B_{k,j} \to 0$ weakly in X as $j \to \infty$. Given a smooth function ψ in X, and following the proof of [17, Theorem 1], we can assume that $\psi(-g)^p$ is the sum of finite terms of form $\pm f$, where f are bounded quasi-psh functions in X. For such a function f, by Lemma 3.1 we get

$$\begin{aligned}\Big|\int_X f\left(\omega_{u_j}^n - \omega_{\max(u_j,-k)}^n\right)\Big| &= \Big|\int_X (u_j - \max(u_j,-k))\, dd^c f \wedge \sum_{l=0}^{n-1} \omega_{u_j}^l \wedge \omega_{\max(u_j,-k)}^{n-1-l}\Big| \\ &= \Big|\int_{u_j<-k} (u_j + k)\, dd^c f \wedge \sum_{l=0}^{n-1} \omega_{u_j}^l \wedge \omega_{\max(u_j,-k)}^{n-1-l}\Big| \\ &\le \int_{u_j<-k} (-u_j)\,(\omega_f + \omega) \wedge \omega_{u_j}^{n-1},\end{aligned}$$

which by Lemma 3.5 tends to zero uniformly for all j as $k \to \infty$. Hence, $A_{k,j} \to 0$ uniformly for all j as $k \to \infty$. Similarly, we have that $C_k \to 0$ weakly as $k \to \infty$. Therefore, we have obtained that $(-g)^p\,\omega_{u_j}^n \to (-g)^p\,\omega_u^n$ weakly. ■

Applying Dini's theorem and quasicontinuity of quasi-psh functions, we get the following consequence.

Corollary 3.7 *Let $0 \le p < \infty$ and $0 \ge g \in \mathrm{PSH}(X,\omega) \cap L^\infty(X)$. If $u_j,\ u \in \mathcal{F}(X,\omega)$ are such that $u_j \searrow u$ or $u_j \nearrow u$ in X, then $(-g)^p\,\omega_{u_j}^n \to (-g)^p\,\omega_u^n$ weakly in X.*

Corollary 3.8 *Let $u,\ v \in \mathcal{F}(X,\omega)$. Then $\chi_{\{u>v\}}\,\omega_{\max(u,v)}^n = \chi_{\{u>v\}}\,\omega_u^n$.*

Proof This proof is similar to the proof of [11, Theorem 4.1]. Given a constant $k \ge 0$, Write $u_j = \max(u, -j)$. By [3, Proposition 4.2] we have that

$$\max(u_j + k, 0)\,\omega_{\max(u_j,-k)}^n = \max(u_j + k, 0)\,\omega_{u_j}^n$$

for all j. Using $\max(u_j + k, 0) \ge \max(u + k, 0) \ge 0$, we get

$$\max(u + k, 0)\,\omega_{\max(u_j,-k)}^n = \max(u + k, 0)\,\omega_{u_j}^n.$$

Letting $j \to \infty$ and applying Theorem 3.6, we get

$$\max(u + k, 0)\,\omega_{\max(u,-k)}^n = \max(u + k, 0)\,\omega_u^n.$$

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Hence we have obtained that $\chi_{\{u>-k\}}\,\omega^n_{\max(u,-k)} = \chi_{\{u>-k\}}\,\omega^n_u$ holds for any $u \in \mathcal{F}(X,\omega)$ and $k \geq 0$. Therefore, $\omega^n_{\max(u,v)} = \omega^n_{\max(u,v,-k)}$ and $\omega^n_u = \omega^n_{\max(u,-k)}$ on each set $\{u > -k > v\}$ with a rational number $k \geq 0$. But $\omega^n_{\max(u,v,-k)} = \omega^n_{\max(u,-k)}$ on the open set $\{-k > v\}$ and hence $\chi_{\{u>-k>v\}}\,\omega^n_{\max(u,v)} = \chi_{\{u>-k>v\}}\,\omega^n_u$, which implies the required equality. ■

Corollary 3.9 *Let u, $v \in \mathcal{F}(X,\omega)$. Then*

$$\omega^n_{\max(u,v)} \geq \chi_{\{u\geq v \text{ and } u\neq-\infty\}}\,\omega^n_u + \chi_{\{u<v\}}\,\omega^n_v.$$

Proof Given $\varepsilon > 0$, by Corollary 3.8 we have

$$\omega^n_{\max(u,v-\varepsilon)} \geq \chi_{\{u>v-\varepsilon\}}\,\omega^n_u + \chi_{\{u<v-\varepsilon\}}\,\omega^n_v \geq \chi_{\{u\geq v \text{ and } u\neq-\infty\}}\,\omega^n_u + \chi_{\{u<v-\varepsilon\}}\,\omega^n_v.$$

Letting $\varepsilon \searrow 0$ and using Theorem 3.6, we obtain the required inequality. ■

Corollary 3.10 *Let u, $v \in \mathcal{F}(X,\omega)$. Then*

$$\int_{u<v} \omega^n_v \leq \int_{u<v} \omega^n_u + \int_{u=v=-\infty} \omega^n_u.$$

Proof By Corollary 3.8 we have

$$\begin{aligned}\int_{u<v} \omega^n_v &= \int_{u<v} \omega^n_{\max(u,v)} = \int_X \omega^n - \int_{u\geq v} \omega^n_{\max(u,v)}\\ &\leq \int_X \omega^n - \int_{u>v} \omega^n_{\max(u,v)} = \int_X \omega^n - \int_{u>v} \omega^n_u = \int_{u\leq v} \omega^n_u.\end{aligned}$$

Using $\delta\, v$ instead of v and letting $\delta \nearrow 1$, we get the required inequality. ■

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References

[1] E. Bedford, *Survey of pluri-potential theory*. In: Several Complex Variables. Math. Notes 38, Princeton University Press, Princeton, NJ, 1993, pp. 48–97.

[2] E. Bedford and B. A. Taylor, *A new capacity for plurisubharmonic functions*. Acta Math. **149**(1982), no. 1-2, 1-40. doi:10.1007/BF02392348

[3] ———, *Fine topology, Šilov boundary and $(dd^c)^n$*. J. Funct. Anal. **72**(1987), no. 2, 225–251. doi:10.1016/0022-1236(87)90087-5

[4] Z. Błocki, *Uniqueness and stability of the complex Monge–Ampère operator on compact Kähler manifolds*. Indina Univ. Math. J. **52**(2003), no. 6, 1697–1701. doi:10.1512/iumj.2003.52.2346

[5] ———, *The domain of definition for the complex Monge–Ampère operator*. Amer. J. Math. **128**(2006), no. 2, 519–530. doi:10.1353/ajm.2006.0010

[6] Z. Błocki and S. Kołodziej, *On regularization of plurisubharmonic functions on manifolds*. Proc. Amer. Math. Soc. **135**(2007), no. 7, 2089–2093. doi:10.1090/S0002-9939-07-08858-2

[7] U. Cegrell, *Pluricomplex energy*. Acta Math. **180**(1998), no. 2, 187–217. doi:10.1007/BF02392899

[8] ———, *The general definition of the complex Monge–Ampère operator*. Ann. Inst. Fourier (Grenoble) **54**(2004), no. 1, 159–179.

[9] J.-P.Demailly, *Regularization of closed positive currents and intersection theory*. J. Algebraic Geom. **1**(1992), 361-409.

[10] V. Guedj and A. Zeriahi, *The weighted Monge–Ampère energy of quasiplurisubharmonic functions.* J. Funct. Anal. **250**(2007), no. 2, 442–482. doi:10.1016/j.jfa.2007.04.018

[11] N. V. Khue and P. H. Hiep, *Some properties of the complex Monge–Ampère operator in Cegrell's classes and applications.* www.arXiv.org:0704.0359.

[12] C. O. Kiselman, *Sur la définition de l'opérateur de Monge–Ampère complexe.* In: Analyse Complexe. Lecture Notes in Math. 1094. Springer, Berlin, 1984, pp. 139–150.

[13] S. Kołodziej, *The complex Monge–Ampère equation and pluripotential theory.* Mem. Amer. Math. Soc. **178**(2005), no. 840.

[14] ———, *The set of measures given by bounded solutions of the complex Monge–Ampère equation on compact Kähler manifolds.* J. London Math. Soc. **72**(2005), no. 1, 225–238. doi:10.1112/S0024610705006460

[15] Y. Xing, *Continuity of the complex Monge–Ampère operator.* Proc. Amer. Math. Soc. **124**(1996), no. 2, 457–467. doi:10.1090/S0002-9939-96-03316-3

[16] ———, *Convergence in capacity.* Ann. Inst. Fourier (Grenoble) **58**(2008), no. 5, 1839–1861.

[17] ———, *Continuity of the complex Monge-Ampère operator on compact Kähler manifolds.* Math. Z. **263**(2009), 331–344.

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