Proceedings of the Royal Society of Edinburgh, page 1 of 23
DOI:10.1017/prm.2023.105

Topological regularity of isoperimetric sets in PI spaces having a deformation property

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(Received 19 July 2023; accepted 3 September 2023)

We prove topological regularity results for isoperimetric sets in PI spaces having a suitable deformation property, which prescribes a control on the increment of the perimeter of sets under perturbations with balls. More precisely, we prove that isoperimetric sets are open, satisfy boundary density estimates and, under a uniform lower bound on the volumes of unit balls, are bounded. Our results apply, in particular, to the class of possibly collapsed $\mathrm{RCD}(K,N)$ spaces. As a consequence, the rigidity in the isoperimetric inequality on possibly collapsed $\mathrm{RCD}(0,N)$ spaces with Euclidean volume growth holds without the additional assumption on the boundedness of isoperimetric sets. Our strategy is of interest even in the Euclidean setting, as it simplifies some classical arguments.

Keywords: isoperimetric set; PI space; deformation property; RCD space; regularity

2020 *Mathematics Subject Classification:* Primary: 53C23, 49Q20
Secondary: 26B30, 26A45, 49J40

1. Introduction

In this paper, we consider length PI spaces, i.e. metric measure spaces $(\mathrm{X},\mathsf{d},\mathfrak{m})$ where $\mathfrak{m}$ is a uniformly locally doubling Borel measure, there holds a weak local $(1,1)$-Poincaré inequality (see definition 2.4), and the distance between any two points x, y is realized as the infimum of the lengths of curves joining x and y. The well-established theory of BV functions on metric measure spaces [3, 35] allows the treatment of sets of finite perimeter in this generalized setting. Hence, it makes sense to consider the classical isoperimetric problem, defined by the following minimization:

$$\inf \left\{ P(E) \;\middle|\; E \subseteq \mathrm{X} \text{ Borel}, \, \mathfrak{m}(E) = v \right\},$$

for any assigned volume $v \in (0, \mathfrak{m}(\mathrm{X}))$, where $P(E)$ denotes the perimeter of E. A set E minimizing the previous infimum is called an *isoperimetric set*, or an *isoperimetric region*.

One of the fundamental questions about isoperimetric sets addresses their topological regularity. Namely, one aims at proving that, up to the choice of a representative, isoperimetric sets are open, bounded and enjoy density estimates at points of the topological boundary. In the Euclidean space, topological regularity follows from [25], subsequently generalized in [41]. The proof in the Euclidean setting can be further simplified, see [34, Example 21.3, Theorem 21.11]. On Riemannian manifolds the result is due to [36]. In [6] the result has been generalized to the setting of *noncollapsed* $\mathsf{RCD}(K, N)$ spaces $(\mathrm{X}, \mathsf{d}, \mathcal{H}^N)$, i.e. $N \in \mathbb{N}$ and $\mathfrak{m}$ coincides with the Hausdorff measure $\mathcal{H}^N$. We mention also [31], which addresses the case of quasi-minimal sets in PI spaces.

The purpose of this paper is to prove the topological regularity of isoperimetric sets in the general setting of length PI spaces that enjoy a so-called *deformation property*, which we are going to introduce (we refer to definition 3.3 for the precise definition). We say that a metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ has the deformation property provided the following holds: given a set $E \subseteq \mathrm{X}$ of finite perimeter and a point $x \in \mathrm{X}$, we can find $R, C > 0$ such that

$$P(E \cup B_r(y)) \leqslant C \frac{\mathfrak{m}(B_r(y) \setminus E)}{r} + P(E) \quad \text{for every } y \in B_R(x) \text{ and } r \in (0, R). \tag{1.1}$$

Classes of spaces having the deformation property are collected in remark 3.4. Notably, the class includes $\mathsf{RCD}(K, N)$ spaces $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, thanks to [6, Theorem 1.1]. We shall not introduce RCD spaces here, and we refer the reader to the survey [4] and the references therein.

We point out that being a PI space does not imply that the deformation property holds, see the examples in remarks 3.5 and 3.6. Anyway, we are not aware of any example of a PI space where the deformation property fails when tested on an isoperimetric set E, nor of an example of a PI space where the essential interior of an isoperimetric set is not topologically open.

Deformation properties for sets of locally finite perimeter are well-known in the smooth context [34, Lemma 17.21], and they represent a tool of crucial importance in several classical arguments. We refer, for instance, to [2, VI.2(3)], [24, Lemma 4.5], and [37, Lemma 3.6] in the sub-Riemannian setting, and to [21, 38] which study isoperimetric problems in a weighted setting.

In fact, it is mostly powerful to couple the topological regularity of an isoperimetric set, or of a set minimizing some variational problem, with the deformation property. For instance, knowing that such a set E has an open representative allows to apply (1.1) centred at points y in the interior, so that $\mathfrak{m}(E \cup B_r(y)) > \mathfrak{m}(E)$ only for radii r sufficiently large, and thus (1.1) implies that one can increase the volume of E controlling the perimeter of the deformed set $E \cup B_r(y)$ *linearly* with respect to the increase of mass $\mathfrak{m}(B_r(y) \setminus E)$. An analogous observation holds applying (1.1) to the complement, in case the complement of the considered set has an open representative. Observe that the previous improved deformation property with linear control follows from (1.1) only *after* topological regularity of the set has been established. This is in contrast to the Euclidean setting, where the stronger form of deformation property is always available [**34**, Lemma 17.21]. The latter result follows by deforming sets of finite perimeter by flows of vector fields, an argument out of reach in the metric setting. Hence, the simplest Euclidean proof for the topological regularity of isoperimetric sets [**34**, Example 21.3] has no hope of being performed in our framework, and we must look for an alternative argument.

We can now state our main result, which yields the topological regularity at the more general level of *volume-constrained minimizers* of the perimeter, i.e. sets which minimize the perimeter with respect to any bounded variation that locally preserves the measure, see definition 3.1. We will denote by $E^{(1)}$, $E^{(0)}$, and $\partial^e E$ the essential interior, the essential exterior, and the essential boundary, respectively, of a Borel set $E \subseteq \mathrm{X}$; see § 2.2 for their definitions.

Theorem 1.1 (Topological regularity of volume-constrained minimizers). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a length PI space having the deformation property. Let* $E \subseteq \mathrm{X}$ *be a volume-constrained minimizer of the perimeter. Then,* $E^{(1)} = \mathrm{int}(E^{(1)})$ *and* $E^{(0)} = \mathrm{int}(E^{(0)})$*. In particular, it holds that* $E^{(1)}$, $E^{(0)}$ *are open sets and* $\partial E^{(1)} = \partial E^{(0)} = \partial^e E$*.*

The previous theorem implies density estimates on the volume and on the perimeter measure of a volume-constrained minimizer at points of the essential boundary, see theorem 3.9. For an isoperimetric set, we can additionally prove its boundedness. Namely:

Theorem 1.2. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a length PI space having the deformation property. Suppose that* $\inf_{x \in \mathrm{X}} \mathfrak{m}(B_1(x)) > 0$*. Let* $E \subseteq \mathrm{X}$ *be an isoperimetric set. Then,* $E^{(1)}$ *is bounded. In particular, every isoperimetric set in* X *has a bounded representative.*

Since $\mathsf{RCD}(K, N)$ spaces with $N < \infty$ are length PI spaces (see [**39**, **40**] and [**33**]), and as recalled above they have the deformation property, putting together theorems 1.1 and 1.2 we obtain the following.

Corollary 1.3. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(K, N)$ *space with* $N < \infty$*. Let* $E \subseteq \mathrm{X}$ *be an isoperimetric set. Then, the sets* $E^{(1)}, E^{(0)}$ *are open and* $\partial^e E = \partial E^{(1)} = \partial E^{(0)}$*. Moreover, if in addition* $\inf_{x \in \mathrm{X}} \mathfrak{m}(B_1(x)) > 0$*, then* $E^{(1)}$ *is bounded.*

In the case of noncollapsed $\mathsf{RCD}(K, N)$ spaces, the above result has been previously proved in [**6**, Theorem 1.4].

As an application of corollary 1.3, we can refine the rigidity part in the sharp isoperimetric inequality on $\mathsf{RCD}(0, N)$ spaces $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ with Euclidean volume

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growth. We recall that 'Euclidean volume growth' means that the *asymptotic volume ratio*

$$\mathrm{AVR}(\mathrm{X}, \mathsf{d}, \mathfrak{m}) := \lim_{R\to\infty} \frac{\mathfrak{m}(B_R(p))}{\omega_N R^N}, \quad \text{for some } p \in \mathrm{X},$$

of the space is strictly positive. Recall that the existence of the above limit is guaranteed by the monotonicity of $(0, +\infty) \ni r \mapsto \mathfrak{m}(B_r(p))/\omega_N r^N$, which in turn follows from the Bishop–Gromov inequality (see e.g. [**40**]). Observe that the condition $\mathrm{AVR}(\mathrm{X}, \mathsf{d}, \mathfrak{m}) > 0$ implies that $\inf_{x\in\mathrm{X}} \mathfrak{m}(B_1(x)) > 0$. The sharp isoperimetric inequality on these spaces, see (1.2), was obtained at different levels of generality in [**1**, **7**, **9**, **13**, **16**, **17**]. In [**7**] the rigidity for the isoperimetric inequality was proved for noncollapsed $\mathsf{RCD}(0, N)$ spaces. Recently in [**17**, Theorem 1.5], the authors prove the rigidity for the inequality in all $\mathsf{RCD}(0, N)$ spaces with Euclidean volume growth under the additional assumption that the set achieving the equality is bounded. An application of our corollary 1.3 allows to drop the previous boundedness requirement, thus obtaining the following unconditional rigidity statement.

THEOREM 1.4 (Sharp and rigid isoperimetric inequality on $\mathsf{RCD}(0, N)$ spaces with Euclidean volume growth). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be an* $\mathsf{RCD}(0, N)$ *space with* $1 < N < \infty$ *having Euclidean volume growth. Then, for every set of finite perimeter* $E \subseteq \mathrm{X}$ *with* $\mathfrak{m}(E) < +\infty$ *it holds that*

$$P(E) \geqslant N\omega_N^{1/N} \mathrm{AVR}(\mathrm{X}, \mathsf{d}, \mathfrak{m})^{1/N} \mathfrak{m}(E)^{(N-1)/N}. \tag{1.2}$$

Moreover, the equality in (1.2) *holds for some set of finite perimeter* $E \subseteq \mathrm{X}$ *with* $\mathfrak{m}(E) \in (0, +\infty)$ *if and only if* X *is isometric to a Euclidean metric measure cone over an* $\mathsf{RCD}(N-2, N-1)$ *space and* E *is isometric, up to negligible sets, to a ball centred at one of the tips of* X.

In the previous theorem, when we say that X *is a Euclidean metric measure cone over an* $\mathsf{RCD}(N-2, N-1)$ *space* we mean that there is a compact $\mathsf{RCD}(N-2, N-1)$ metric measure space $(Z, \mathsf{d}_Z, \mathfrak{m}_Z)$ such that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is isomorphic, as a metric measure space, to the metric measure cone $(C(Z), \mathsf{d}_c, t^{N-1}\mathrm{d}t \otimes m_Z)$, where d_c is the cone metric built using d_Z. In the case of $1 < N < 2$, it is understood that in the rigidity part of the previous statement, the space X is either a weighted Euclidean half-line or a weighted Euclidean line.

We stress that theorem 1.4 is not a straightforward consequence of the results in [**7**], according to which the same result holds in the class of noncollapsed spaces. Indeed, an $\mathsf{RCD}(0, N)$ space with $1 < N < \infty$ and with Euclidean volume growth might not be noncollapsed. A simple example is given by the weighted Euclidean half-line $([0, +\infty), \mathsf{d}_{\mathrm{eu}}, t^{N-1}\mathrm{d}t)$, with $N > 1$.

We now briefly discuss our strategy for the proof of theorem 1.1. As mentioned above, the Euclidean proof [**34**, Example 21.3] cannot be adapted to our setting. As in the classical [**25**, **41**], we gain information on a volume-constrained minimizer by comparison with suitable competitors exploiting the deformation property, but our argument is different, more direct, and much shorter. The strategy of [**25**, **41**] consists in proving first that E has an interior and an exterior point, i.e. $\mathrm{int}(E^{(1)}) \neq$

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$\varnothing$ and $\mathrm{int}(E^{(0)}) \neq \varnothing$ (see [**25**, Theorem 1]), then one deduces that E is a (Λ, r_0)-perimeter minimizer, and thus finally that E is open. Instead, we prove directly that if $x \in E^{(0)}$ and $y \in E^{(1)}$ are arbitrary points, then $x \in \mathrm{int}(E^{(0)})$ and $y \in \mathrm{int}(E^{(1)})$. To do so we avoid deriving quantitative estimates on the decay of $\mathfrak{m}(B_r(y) \setminus E)$ as in [**6**, **25**, **41**], and we rather adopt a more qualitative approach. More precisely, the key point is to show (see the key lemma 3.7) that if the function $v(r) \coloneqq \mathfrak{m}(B_r(x) \cap E)$ vanishes, as $r \to 0^+$, slower than the function $w(r) \coloneqq \mathfrak{m}(B_r(y) \setminus E)$, then $x \in \mathrm{int}(E^{(0)})$ (and vice versa for $y \in \mathrm{int}(E^{(1)})$). By 'slower' we mean, roughly speaking, that $v(r) \geqslant w(r)$ for many $r > 0$ in a measure-theoretic sense (see lemma 3.7 for the precise statement). However, up to exchanging E with its complement $\mathrm{X} \setminus E$, we can always ensure that $v(r)$ vanishes slower than $w(r)$, thus deducing that $x \in \mathrm{int}(E^{(0)})$. By symmetry, we get $y \in \mathrm{int}(E^{(1)})$ as well.

We point out that the strategy of [**25**, **41**] does not seem to generalize to our setting, unless we require additional assumptions—such as Ahlfors regularity—which we do not want to make (in order to obtain a result which applies to the whole class of collapsed $\mathsf{RCD}(K, N)$ spaces). This motivated us to look for an alternative proof of the topological result, which—we believe—is of interest even in the Euclidean setting, since it brings simplifications to the classical arguments in [**25**], still (necessarily) avoiding the use of the smooth structure of the ambient.

We conclude the introduction by explicitly recording the following open problem.

Question 1.5. Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a length PI space and let $E \subseteq \mathrm{X}$ be a volume-constrained minimizer of the perimeter. Is it true that $E^{(1)}$ is open?

2. Preliminaries

Given a metric space (X, d), we denote by $\mathrm{LIP}_{\mathrm{loc}}(\mathrm{X})$ the space of all locally Lipschitz functions from X to $\mathbb{R}$, i.e. of those functions $f \colon \mathrm{X} \to \mathbb{R}$ such that for any $x \in \mathrm{X}$ there exists $r_x > 0$ for which f is Lipschitz on $B_{r_x}(x)$. The *slope* $\mathrm{lip}(f) \colon \mathrm{X} \to [0, +\infty)$ of a function $f \in \mathrm{LIP}_{\mathrm{loc}}(\mathrm{X})$ is defined as $\mathrm{lip}(f)(x) \coloneqq 0$ if $x \in \mathrm{X}$ is an isolated point and

$$\mathrm{lip}(f)(x) \coloneqq \varlimsup_{y \to x} \frac{|f(x) - f(y)|}{\mathsf{d}(x, y)} \quad \text{if } x \in \mathrm{X} \text{ is an accumulation point.}$$

The topological interior and the topological boundary of a set $E \subseteq \mathrm{X}$ are denoted by $\mathrm{int}(E)$ and ∂E, respectively. A Borel measure $\mu \geqslant 0$ on X is *locally finite* if for any $x \in \mathrm{X}$ there exists $r_x > 0$ such that $\mu(B_{r_x}(x)) < +\infty$, while we say that μ is *boundedly finite* if $\mu(B) < +\infty$ whenever $B \subseteq \mathrm{X}$ is bounded Borel. Trivially, each boundedly finite measure is locally finite, while the converse holds e.g. if (X, d) is *proper*, i.e. bounded closed subsets of X are compact. Notice that locally finite Borel measures on a complete separable metric space are σ-finite.

2.1. Sets of finite perimeter in metric measure spaces

In this paper, by a *metric measure space* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ we mean a complete separable metric space (X, d) together with a boundedly finite Borel measure $\mathfrak{m} \geqslant 0$ on X. Following [**35**], we define the *total variation* $|\mathbf{D}f|(B) \in [0, +\infty]$ of a given function

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$f \in L^1_{\rm loc}(\mathrm{X})$ in a Borel set $B \subseteq \mathrm{X}$ as

$$|\mathbf{D}f|(B) := \inf_{B\subseteq\Omega \text{ open}} \inf\left\{ \varliminf_{n\to\infty} \int_\Omega \mathrm{lip}(f_n)\,\mathrm{d}\mathfrak{m} \,\middle|\, (f_n)_{n\in\mathbb{N}} \subseteq \mathrm{LIP}_{\rm loc}(\Omega),\, f_n \to f \text{ in } L^1_{\rm loc}(\Omega) \right\}.$$

If for some open cover $(\Omega_n)_{n\in\mathbb{N}}$ of X we have that $|\mathbf{D}f|(\Omega_n) < +\infty$ holds for every $n \in \mathbb{N}$, then $|\mathbf{D}f|$ is a locally finite Borel measure on X. We say that a Borel set $E \subseteq \mathrm{X}$ is *of locally finite perimeter* if $P(E,\cdot) := |\mathbf{D}\chi_E|$ is a locally finite measure, called the *perimeter measure* of E. When $P(E) := P(E,\mathrm{X}) < +\infty$, we say that E is *of finite perimeter.*

REMARK 2.1. If $E \subseteq \mathrm{X}$ is a set of locally finite perimeter and $x \in \mathrm{X}$ is a given point, then $P(E, \partial B_r(x)) = 0$ for all but countably many radii $r > 0$. This is due to the fact that $\partial B_r(x) \cap \partial B_s(x) = \varnothing$ whenever $0 < r < s$ and to the σ-finiteness of $P(E,\cdot)$. ■

Given any $f \in \mathrm{LIP}_{\rm loc}(\mathrm{X})$, it holds that $|\mathbf{D}f|$ is a locally finite measure and $|\mathbf{D}f| \leqslant \mathrm{lip}(f)\mathfrak{m}$.

THEOREM 2.2 (Coarea formula [35, Proposition 4.2.)] *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space. Fix any* $f \in L^1_{\rm loc}(\mathrm{X})$ *such that* $|\mathbf{D}f|$ *is a locally finite measure. Fix a Borel set* $E \subseteq \mathrm{X}$. *Then,* $\mathbb{R} \ni t \mapsto P(\{f < t\}, E) \in [0, +\infty]$ *is a Borel measurable function and it holds that*

$$|\mathbf{D}f|(E) = \int_{\mathbb{R}} P(\{f < t\}, E)\,\mathrm{d}t.$$

COROLLARY 2.3. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space. Fix* $x \in \mathrm{X}$ *and a Borel set* $E \subseteq \mathrm{X}$. *Define* $f\colon (0,+\infty) \to \mathbb{R}$ *as* $f(r) := |\mathbf{D}\mathsf{d}_x|(E \cap B_r(x))$ *for every* $r > 0$, *where we denote* $\mathsf{d}_x := \mathsf{d}(\cdot, x) \in \mathrm{LIP}(\mathrm{X})$. *Then, the function* f *is locally absolutely continuous and it holds that* $f'(r) = P(B_r(x), E)$ *for* $\mathcal{L}^1$*-a.e.* $r > 0$.

Proof. By virtue of the coarea formula, we obtain that $f(r) = \int_{\mathbb{R}} P\big(\{\mathsf{d}_x < s\}, E \cap B_r(x)\big)\,\mathrm{d}s = \int_0^r P(B_s(x), E)\,\mathrm{d}s$ for every $r > 0$, whence it follows that $f(r) - f(\tilde r) = \int_{\tilde r}^r P(B_s(x), E)\,\mathrm{d}s$ for every $r > \tilde r > 0$. Hence, f is locally absolutely continuous and $f'(r) = P(B_r(x), E)$ for every Lebesgue point r of $s \mapsto P(B_s(x), E)$, thus for $\mathcal{L}^1$-a.e. $r > 0$. □

2.2. PI spaces

Even though the general theory of sets of finite perimeter is meaningful in any metric measure space, a much more refined calculus is available in the class of doubling spaces supporting a weak form of $(1,1)$-Poincaré inequality, which we refer to as *PI spaces.* Below we recall the definition of PI space we adopt in this paper, referring e.g. to [12, 29] for a thorough account of this topic. We will also recall some key features of sets of finite perimeter in PI spaces.

DEFINITION 2.4 (PI space). *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a metric measure space. Then,*

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- *We say that* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is uniformly locally doubling if there is a function* $C_D\colon(0,+\infty)\to(0,+\infty)$ *such that*

$$\mathfrak{m}\big(B_{2r}(x)\big)\leqslant C_D(R)\,\mathfrak{m}\big(B_r(x)\big)\quad\text{for every } 0<r<R \text{ and } x\in\mathrm{X}.$$

- *We say that* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *supports a weak local* $(1,1)$*-Poincaré inequality if there exist a constant* $\lambda\geqslant 1$ *and a function* $C_P\colon(0,+\infty)\to(0,+\infty)$ *such that for any function* $f\in\mathrm{LIP}_{\mathrm{loc}}(\mathrm{X})$ *it holds that*

$$\begin{aligned}&\fint_{B_r(x)}\left|f-\fint_{B_r(x)}f\,\mathrm{d}\mathfrak{m}\right|\mathrm{d}\mathfrak{m}\\&\quad\leqslant C_P(R)\,r\fint_{B_{\lambda r}(x)}\mathrm{lip}(f)\,\mathrm{d}\mathfrak{m}\quad\text{for all } 0<r<R \text{ and } x\in\mathrm{X}.\end{aligned}$$

- $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *is a PI space if it is uniformly locally doubling and it supports a weak local* $(1,1)$*-Poincaré inequality.*

We point out that if $(\mathrm{X},\mathsf{d},\mathfrak{m})$ is a uniformly locally doubling space, then (X,d) is proper, so (X,d) is locally compact, and locally finite Borel measures on (X,d) are boundedly finite.

Remark 2.5. Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a PI space such that (X,d) is a *length space*, i.e. the distance between any two points in X is the infimum of the lengths of rectifiable curves joining them. Then, the weak local $(1,1)$-Poincaré inequality is in fact *strong*, namely it holds with $\lambda=1$; see for example [**26**, Corollary 9.5 and Theorem 9.7]. Moreover, the completeness and the local compactness of (X,d) ensure that (X,d) is also geodesic. ■

Given a Borel set $E\subseteq\mathrm{X}$ in a PI space $(\mathrm{X},\mathsf{d},\mathfrak{m})$, we define its *essential interior* and *essential exterior* as

$$E^{(1)}:=\left\{x\in\mathrm{X}\;\middle|\;\lim_{r\to0}\frac{\mathfrak{m}(E\cap B_r(x))}{\mathfrak{m}(B_r(x))}=1\right\},$$
$$E^{(0)}:=\left\{x\in\mathrm{X}\;\middle|\;\lim_{r\to0}\frac{\mathfrak{m}(E\cap B_r(x))}{\mathfrak{m}(B_r(x))}=0\right\},$$

respectively. The *essential boundary* of E is defined as $\partial^e E:=\mathrm{X}\setminus(E^{(1)}\cup E^{(0)})$. Notice that $E^{(1)}$, $E^{(0)}$, and $\partial^e E$ are Borel sets and that $\partial^e E\subseteq\partial E$. It follows from the Lebesgue differentiation theorem (which holds on every uniformly locally doubling metric measure space, see e.g. [**27**, Theorem 1.8]) that $\mathfrak{m}(E^{(1)}\Delta E)=0$ and $\mathfrak{m}(E^{(0)}\Delta(\mathrm{X}\setminus E))=0$. Moreover, if E is a set of finite perimeter, then we know from [**3**, Theorem 5.3] that $P(E,\cdot)$ is concentrated on $\partial^e E$.

Proposition 2.6. *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a PI space. Let* $E,F\subseteq\mathrm{X}$ *be sets of locally finite perimeter with* $P(E,\partial^e F)=0$*. Then,*

$$P(E\cap F,\cdot)\leqslant P(E,\cdot)\llcorner F^{(1)}+P(F,\cdot)\llcorner E^{(1)}.$$

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Proof. We know from [3, Theorem 5.3] that the perimeter measure $P(G,\cdot)$ of a set $G\subseteq \mathrm{X}$ of locally finite perimeter can be written as $P(G,\cdot)=\theta_G\mathcal{H}^h|_{\partial^e G}$ for some Borel function $\theta_G\colon \mathrm{X}\to(0,+\infty)$, where $\mathcal{H}^h$ stands for the codimension-one Hausdorff measure (see [3, Section 5]). Since $P(E\cap F,\cdot)\leqslant P(E,\cdot)+P(F,\cdot)$ and $P(E,\cdot)|_{\mathrm{X}\setminus\partial^e E}=P(F,\cdot)|_{\mathrm{X}\setminus\partial^e F}=0$, we deduce that $\theta_{E\cap F}\leqslant\theta_E$ and $\theta_{E\cap F}\leqslant\theta_F$ hold $\mathcal{H}^h$-a.e. in $\partial^e E\setminus\partial^e F$ and $\partial^e F\setminus\partial^e E$, respectively. Moreover, we deduce from $\int_{\partial^e F}\theta_E\,\mathrm{d}\mathcal{H}^h|_{\partial^e E}=P(E,\partial^e F)=0$ that $\mathcal{H}^h(\partial^e E\cap\partial^e F)=0$. Given that $\partial^e(E\cap F)=(\partial^e E\cap F^{(1)})\sqcup(\partial^e F\cap E^{(1)})$ up to an $\mathcal{H}^h$-negligible set, which is shown e.g. in the proof of [6, Lemma 2.5], we conclude that

$$\begin{aligned}P(E\cap F,\cdot)&=\theta_{E\cap F}\mathcal{H}^h|_{\partial^e(E\cap F)}=\theta_{E\cap F}\mathcal{H}^h|_{\partial^e E\cap F^{(1)}}+\theta_{E\cap F}\mathcal{H}^h|_{\partial^e F\cap E^{(1)}}\\&\leqslant\theta_E\mathcal{H}^h|_{\partial^e E\cap F^{(1)}}+\theta_F\mathcal{H}^h|_{\partial^e F\cap E^{(1)}},\end{aligned}$$

which yields the statement. □

The following is a direct consequence of the study in [3], taking remark 2.5 into account.

THEOREM 2.7 (Relative isoperimetric inequality [3, Remark 4.4]). *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a length PI space. Then, there exists a function* $C_I=C_I(C_D,C_P)\colon(1,+\infty)\times(0,+\infty)\to(0,+\infty)$ *such that the following property holds: given a set* $E\subseteq\mathrm{X}$ *of finite perimeter, a radius* $R>0$*, and an exponent* $\alpha>\max\{\log_2(C_D(R)),1\}$*, we have that*

$$\begin{aligned}&\min\left\{\mathfrak{m}(B_r(x)\cap E),\mathfrak{m}(B_r(x)\setminus E)\right\}\\&\quad\leqslant C_I(\alpha,R)\left(\frac{r^\alpha}{\mathfrak{m}\big(B_r(x)\big)}\right)^{1/(\alpha-1)}P\big(E,B_r(x)\big)^{\alpha/(\alpha-1)},\end{aligned}$$

for every $x\in\mathrm{X}$ *and* $r\in(0,R)$.

In the next proposition, we recall the well-known fact that in the class of PI spaces where unit balls have measure uniformly bounded away from zero, there holds an isoperimetric inequality for sets of small volume. Such a result is essentially due to [23], after [15, 20, 30]. For a proof, we refer the reader to the argument in [19, Lemma V.2.1].

PROPOSITION 2.8 (Isoperimetric inequality for small volumes). *Let* $(\mathrm{X},\mathsf{d},\mathfrak{m})$ *be a length PI space. Then, there exist constants* $\alpha>1$, $C>0$ *such that the following holds. If* $v_0:=\inf_{x\in\mathrm{X}}\mathfrak{m}(B_1(x))>0$*, then for all Borel sets* $E\subseteq\mathrm{X}$ *with* $\mathfrak{m}(E)<v_0/2$ *it holds that*

$$P(E)\geqslant Cv_0^{1/\alpha}\mathfrak{m}(E)^{(\alpha-1)/\alpha}.$$

REMARK 2.9. Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a length PI space and $E\subseteq\mathrm{X}$ a set of finite perimeter such that $\mathfrak{m}(E),\mathfrak{m}(\mathrm{X}\setminus E)>0$. Then, the relative isoperimetric inequality ensures that $P(E)\neq 0$. In order to prove it, fix any $x\in\mathrm{X}$ and notice that we have $\mathfrak{m}(B_R(x)\cap E),\mathfrak{m}(B_R(x)\setminus E)>0$ for some $R>0$ sufficiently large, thus $P(E)\geqslant P(E,B_R(x))>0$. ■

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LEMMA 2.10. *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a length PI space. Then, there exists* $c = c(\inf C_D, \inf C_P) \in (0,1)$ *such that*

$$c\,\mathfrak{m} \leqslant |\mathbf{D}\mathsf{d}_x| \leqslant \mathfrak{m} \quad \text{for every } x \in \mathrm{X}, \tag{2.1}$$

where we denote $\mathsf{d}_x \coloneqq \mathsf{d}(\cdot, x) \in \mathrm{LIP}_{\mathrm{loc}}(\mathrm{X})$. *In particular, it holds that*

$$\mathfrak{m}\big(\partial B_r(x)\big) = 0 \quad \text{for every } x \in \mathrm{X} \text{ and } r > 0. \tag{2.2}$$

Proof. Recall that $|\mathbf{D}\mathsf{d}_x| \leqslant \mathrm{lip}(\mathsf{d}_x)\mathfrak{m}$. Moreover, we deduce from [5, Equation (4.5)] that there exists a constant $c = c(\inf C_D, \inf C_P) \in (0,1)$ such that $c\,\mathrm{lip}(\mathsf{d}_x)\mathfrak{m} \leqslant |\mathbf{D}\mathsf{d}_x|$. To obtain (2.1), observe that $\mathrm{lip}(\mathsf{d}_x) \equiv 1$: the inequality $\mathrm{lip}(\mathsf{d}_x) \leqslant 1$ holds in any metric space, while the converse inequality readily follows from the fact that (X, d) is geodesic. Finally, (2.2) can be proved by combining (2.1) with the coarea formula: we can estimate

$$\mathfrak{m}\big(\partial B_r(x)\big) \leqslant \frac{1}{c}|\mathbf{D}\mathsf{d}_x|\big(\partial B_r(x)\big) = \frac{1}{c}\int_{\mathbb{R}} P\big(B_s(x), \partial B_r(x)\big)\,\mathrm{d}s = 0,$$

where the last identity follows from the fact that $P(B_s(x), \cdot)$ is concentrated on $\partial^e B_s(x) \subseteq \partial B_s(x)$. □

3. Topological regularity

Let us begin with the definition of a volume-constrained minimizer of the perimeter.

DEFINITION 3.1 (Volume-constrained minimizer). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space. Then, a set* $E \subseteq \mathrm{X}$ *of locally finite perimeter is said to be a volume-constrained minimizer of the perimeter if the following property is verified: given a Borel set* $F \subseteq \mathrm{X}$ *and a compact set* $K \subseteq \mathrm{X}$ *satisfying* $\mathfrak{m}((E \Delta F) \setminus K) = 0$ *and* $\mathfrak{m}(E \cap K) = \mathfrak{m}(F \cap K)$, *it holds* $P(E, K) \leqslant P(F, K)$.

Observe that E is a volume-constrained minimizer if and only if $\mathrm{X} \setminus E$ is a volume-constrained minimizer.

REMARK 3.2. An *isoperimetric set*, i.e. a set $E \subseteq \mathrm{X}$ of finite perimeter with $0 < \mathfrak{m}(E) < +\infty$ such that $P(E) \leqslant P(F)$ for any Borel set $F \subseteq \mathrm{X}$ with $\mathfrak{m}(F) = \mathfrak{m}(E)$, is a volume-constrained minimizer of the perimeter. ■

Next, we introduce our definition of a metric measure space having the *deformation property*, which will be our standing assumption throughout the rest of the paper.

DEFINITION 3.3 (Deformation property). *Let* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *be a metric measure space with* (X, d) *proper. Then, we say that* $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ *has the deformation property if the following property holds: for every set of locally finite perimeter* $E \subseteq \mathrm{X}$ *and any* *point $x \in \mathrm{X}$, there exist constants $R \in (0,1]$ and $C \geqslant 0$ such that*

$$P(E \setminus B_r(y), B_{2R}(x)) \leqslant C\frac{\mathfrak{m}(B_r(y) \cap E)}{r} + P(E, B_{2R}(x)) \quad \forall y \in B_R(x), r \in (0,R), \tag{3.1a}$$

$$P(E \cup B_r(y), B_{2R}(x)) \leqslant C\frac{\mathfrak{m}(B_r(y) \setminus E)}{r} + P(E, B_{2R}(x)) \quad \forall y \in B_R(x), r \in (0,R), \tag{3.1b}$$

For convenience, we define from now on $R_x(E) \in (0,1]$ to be the maximal $R \in (0,1]$ such that the above holds for some $C \geqslant 0$ and we define $C_x(E) \geqslant 0$ to be the minimal constant such that (3.1a) and (3.1b) hold with $R = R_x(E)$. Note that, by symmetry, we have that $R_x(E) = R_x(\mathrm{X} \setminus E)$ and $C_x(E) = C_x(\mathrm{X} \setminus E)$; this is the reason why in definition 3.3 we require the validity of both (3.1a) and (3.1b) with the same constants C and R. We also observe that if $E \subseteq \mathrm{X}$ is a given set of finite perimeter (resp. of locally finite perimeter), then (3.1a) is equivalent to asking that $P(E \setminus B_r(y), S) \leqslant C(\mathfrak{m}(B_r(y) \cap E)/r) + P(E, S)$ holds for every $(y, r) \in B_R(x) \times (0, R)$ and every Borel set (resp. bounded Borel set) $S \subseteq \mathrm{X}$ with $B_{2R}(x) \subseteq S$. Similarly for (3.1b). We will often make use of this observation without further notice. Also,

$$\inf_{x \in B} R_x(E) > 0 \qquad \text{for every bounded set } B \subseteq \mathrm{X}. \tag{3.2}$$

Indeed, the compactness of the closure of B ensures that $B \subseteq \bigcup_{i=1}^n B_{R_{x_i}(E)/2}(x_i)$ for some $x_1, \ldots, x_n \in B$, which gives $R_x(E) \geqslant \delta \coloneqq \min\left\{R_{x_i}(E)/2 \,:\, i = 1, \ldots, n\right\} > 0$ for every $x \in B$. The same argument shows also that (3.1a) and (3.1b) hold for every $x \in B$ for some R and C that depend only on B and E, e.g. by taking $R \coloneqq \delta$ and $C \coloneqq \max\left\{C_{x_i}(E) \,:\, i = 1, \ldots, n\right\}$.

Remark 3.4 (Spaces having the deformation property). These are some spaces with the deformation property:

(i) Euclidean spaces (see e.g. [**25**] and the references therein).

(ii) Riemannian manifolds (this can be proved e.g. by following the proof of [**6**, Theorem 1.1] and using the fact that the Ricci curvature is locally bounded from below).

(iii) $\mathsf{RCD}(K, N)$ spaces with $K \in \mathbb{R}$ and $N \in [1, \infty)$ (proved in [**6**, Theorem 1.1] building upon the Gauss–Green formula in [**14**, Theorem 2.4]).

We point out that in the above cases a stronger version of the deformation property holds, since, given an arbitrary $R > 0$, the constants $C_x(E)$ for which the deformation property holds at every point $x \in \mathrm{X}$ and for every $0 < r < R$, can be chosen to be independent of E, x, and to be dependent only on K, N, R.

It would be interesting to study whether there are other distinguished examples of PI spaces having the deformation property. One natural class to investigate is the one of sub-Riemannian manifolds, or, more specifically, the one of Carnot groups. For example, in the first Heisenberg group one has a sub-Laplacian comparison

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theorem. Being r the Carnot–Carathéodory distance from the origin, we have that $\Delta_{\mathrm{H}} r \leqslant 4/r$ holds in the distributional sense, where Δ_{H} is the horizontal Laplacian. See [**11**] for the study of sub-Laplacian comparison theorems in more general sub-Riemannian structures, and [**18**, Corollary 4.19] for the Laplacian comparison theorem in arbitrary essentially non-branching MCP spaces. Then, coupling this with the Gauss–Green formulae for Carnot groups in [**22**], one could argue following the lines of [**6**, Theorems 2.32 and 1.1] to obtain that at least $\mathbb{H}^1$, and more in general all the groups that are essentially non-branching $\mathsf{MCP}(K,N)$ spaces, with $K \in \mathbb{R}$ and $N \in (1,\infty)$ (cf. [**10**], [**8**]), have the deformation property. Since this is out of the scope of the present paper, and since there are also some regularity issues of the distance function to deal with, we do not treat these examples here, but we leave it to possible future investigations.

We mention that, on the contrary, the topological regularity of isoperimetric sets is already proved in [**32**] in the setting of Carnot groups and in [**24**] on a certain class of sub-Riemannian manifolds. ∎

Remark 3.5. There exist PI spaces where the deformation property fails. For example, fix a sequence of pairwise well-separated non-empty balls $B_n := B_{r_n}(x_n)$ in $\mathbb{R}^2$ such that $x_n \to 0$ and $\sum_n r_n < +\infty$. Now, consider the density function $\rho\colon \mathbb{R}^2 \to [1,2]$ given by $\rho := \chi_E + 2\chi_{\mathbb{R}^2\setminus E}$, where $E := \bigcup_n B_n$. Letting $\mathfrak{m} := \rho\mathcal{L}^2$ we have $\mathcal{L}^2 \leqslant \rho\mathcal{L}^2 \leqslant 2\mathcal{L}^2$, so that $(\mathbb{R}^2, |\cdot|, \mathfrak{m})$ is an Ahlfors regular geodesic PI space. We claim that the deformation property is not valid for the set of finite perimeter E at the origin 0. To check it, notice that for any $n \in \mathbb{N}$ it holds $P(B_n) = 2\pi r_n$, while $P(B_{r_n+\varepsilon}(x_n)) = 4\pi(r_n+\varepsilon)$ and $\mathfrak{m}(B_{r_n+\varepsilon}(x_n)\setminus B_n) = 2\pi(2r_n\varepsilon + \varepsilon^2)$ for any $\varepsilon \in (0,\varepsilon_n)$ for some $\varepsilon_n > 0$ sufficiently small. Therefore,

$$\frac{P(E\cup B_{r_n+\varepsilon}(x_n)) - P(E)}{\mathfrak{m}(B_{r_n+\varepsilon}(x_n)\setminus E)/(r_n+\varepsilon)} = \frac{(2\pi r_n + 4\pi\varepsilon)(r_n+\varepsilon)}{2\pi(2r_n\varepsilon+\varepsilon^2)} \to +\infty \quad \text{as } \varepsilon \searrow 0,$$

which shows that the deformation property fails at the origin. However, we are not aware of any example of a PI space where the deformation property fails when tested on an isoperimetric set, nor of an example of a PI space where the essential interior of some isoperimetric set is not topologically open. ∎

Remark 3.6. The validity of the deformation property on a metric measure space $(\mathrm{X},\mathsf{d},\mathfrak{m})$ entails a growth condition: given $x \in \mathrm{X}$, there exist $C_x, r_x > 0$ such that

$$P(B_r(y)) \leqslant C_x \frac{\mathfrak{m}(B_r(y))}{r} \quad \text{for every } y \in B_{r_x}(x) \text{ and } r \in (0,r_x). \tag{3.3}$$

Equation (3.3) follows just by taking $E := \varnothing$ in the deformation property. We have that (3.3) is not equivalent to the deformation property (e.g. in the example in remark 3.5 property (3.3) is satisfied). However, there are examples of PI spaces where also (3.3) fails. The example we are going to describe has been pointed out to the authors by Panu Lahti. Consider the measure $\mathfrak{m} := |x|^{-1/2}\mathrm{d}x$ in $\mathbb{R}$. Since the function $|x|^{-1/2}$ is an A_1-Muckenhoupt weight, we know that $(\mathbb{R}, |\cdot|, \mathfrak{m})$ is a PI space (see e.g. [**28**]). Using that $\mathfrak{m}(B_r(0))/r = 2 \fint_0^r (1/\sqrt{x})\,\mathrm{d}x = 4/\sqrt{r} \to +\infty$ as $r \searrow 0$, one can easily check that the codimension-one Hausdorff measure of the singleton $\{0\}$ diverges, i.e. $\mathcal{H}^h(\{0\}) = +\infty$. It follows from [**3**, Theorem 5.3] that

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$B_{|y|}(y)$ is not a set of locally finite perimeter when $y \in (0, +\infty)$. Hence, (3.3) fails for $x = 0$. ■

Given a metric measure space $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$, a point $x \in \mathrm{X}$, and a Borel set $E \subseteq \mathrm{X}$, we introduce the notation

$$v_{E,x}(r) := \mathfrak{m}(B_r(x) \cap E), \quad w_{E,x}(r) := \mathfrak{m}(B_r(x) \setminus E) \quad \text{for every } r > 0. \tag{3.4}$$

The core of the proof of our main theorem 1.1 is contained in the following technical result.

Lemma 3.7. *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a length PI space having the deformation property. Let $E \subseteq \mathrm{X}$ be a volume-constrained minimizer of the perimeter. Fix any $x \in E^{(0)}$ and $y \in E^{(1)}$. Define the functions $v_{E,x}, w_{E,y} \colon (0, +\infty) \to [0, +\infty)$ as in (3.4). Fix a sequence $(r_n)_n \subseteq (0, 1)$ such that $r_n \to 0$. For any $n \in \mathbb{N}$, we define the Borel set $A^{x,y}_{E,r_n} \subseteq (0, r_n)$ as*

$$A^{x,y}_{E,r_n} := \big\{ r \in (0, r_n) \,\big|\, v_{E,x}(r) \geqslant w_{E,y}(r) \big\}. \tag{3.5}$$

Suppose the following conditions are verified:

(i) *There exists $\delta \in (0, R_y(E))$ such that $\bar{B}_\delta(x) \cap \bar{B}_\delta(y) = \varnothing$, $v_{E,x}(\delta) > 0$, and $w_{E,y}(\delta) > 0$.*

(ii) *The inequality $\mathcal{L}^1(A^{x,y}_{E,r_n}) \geqslant r_n/2$ holds for infinitely many $n \in \mathbb{N}$.*

Then, it holds that $x \in \mathrm{int}(E^{(0)})$.

Proof. We argue by contradiction: suppose that $x \notin \mathrm{int}(E^{(0)})$. Recalling that $\mathfrak{m}(B_\delta(y) \setminus E) = w_{E,y}(\delta) > 0$ and noticing that $\mathfrak{m}(B_r(x) \cap E) \to 0$ as $r \to 0$, we can extract a (not relabelled) subsequence of $(r_n)_n$ for which

$$r_n < \delta, \quad \mathfrak{m}(B_{r_n}(x) \cap E) < \mathfrak{m}(B_\delta(y) \setminus E), \quad \mathcal{L}^1(A^{x,y}_{E,r_n}) \geqslant \frac{r_n}{2}, \tag{3.6}$$

for every $n \in \mathbb{N}$. Now, let $n \in \mathbb{N}$ be fixed. We claim that for any $r \in A_n := A^{x,y}_{E,r_n}$ there exists $s(r) \in [r, \delta)$ such that

$$v_{E,x}(r) = \mathfrak{m}(B_r(x) \cap E) = \mathfrak{m}(B_{s(r)}(y) \setminus E) = w_{E,y}(s(r)).$$

Indeed, if $w_{E,y}(r) = v_{E,x}(r)$, then we can take $s(r) := r$. If $w_{E,y}(r) \neq v_{E,x}(r)$, then $v_{E,x}(r) > w_{E,y}(r)$ by definition of A_n, thus the continuity of $w_{E,y}$ (which follows from (2.2)) ensures that $w_{E,y}(s(r)) = v_{E,x}(r)$ for some $s(r) > 0$. Since $w_{E,y}$ is non-decreasing, we infer that $s(r) \geqslant r$. Moreover, the second inequality in (3.6) implies that $s(r) < \delta$.

Given any $r \in A_n$, we define the Borel set $E_r \subseteq \mathrm{X}$ as $E_r := (E \setminus B_r(x)) \cup B_{s(r)}(y)$. The first inequality in (3.6) ensures that $\bar{B}_r(x) \cap \bar{B}_{s(r)}(y) = \varnothing$, whence it follows that $\mathfrak{m}\big(E_r \cap (\bar{B}_r(x) \cup \bar{B}_{s(r)}(y))\big) = \mathfrak{m}\big(E \cap (\bar{B}_r(x) \cup \bar{B}_{s(r)}(y))\big)$. Denote $K := \bar{B}_{2\delta}(x) \cup \bar{B}_{2\delta}(y)$ for brevity. The assumption that E is a volume-constrained minimizer of the perimeter then implies that $P(E, K) \leqslant P(E_r, K)$. For ease of notation from now on we will denote $C_y(E)$ simply by C_y. Thanks to proposition 2.6, remark 2.1, the deformation property, and $s(r) \geqslant r$, we deduce that for $\mathcal{L}^1$-a.e. $r \in A_n$ one has

$$\begin{aligned}
P(E,K) &\leqslant P(E_r,K) \\
&\leqslant P\big(E\cup B_{s(r)}(y), B_r(x)^{(0)}\cap K\big) + P\big(B_r(x), (E\cup B_{s(r)}(y))^{(1)}\cap K\big) \\
&= P(E\cup B_{s(r)}(y), K) - P\big(E\cup B_{s(r)}(y), \partial^e B_r(x)\cup B_r(x)^{(1)}\big) \\
&\quad + P(B_r(x), E^{(1)}) \\
&\leqslant P(E\cup B_{s(r)}(y), K) - P(E, B_r(x)) + P(B_r(x), E^{(1)}) \\
&\leqslant P(E,K) + C_y\frac{\mathfrak{m}(B_{s(r)}(y)\setminus E)}{s(r)} - P(E,B_r(x)) + P(B_r(x),E^{(1)}) \\
&= P(E,K) + C_y\frac{\mathfrak{m}(B_r(x)\cap E)}{s(r)} - P(E,B_r(x)) + P(B_r(x),E^{(1)}) \\
&\leqslant P(E,K) + C_y\frac{\mathfrak{m}(B_r(x)\cap E)}{r} - P(E,B_r(x)) + P(B_r(x),E^{(1)}).
\end{aligned}$$

Notice that the constant C_y depends on y and E, but neither on n nor on r. Therefore, we have shown that

$$\begin{aligned}
&P(E,B_r(x)) \\
&\quad\leqslant C_y\frac{\mathfrak{m}(B_r(x)\cap E)}{r} + P(B_r(x),E^{(1)}) \quad \text{for all } n\in\mathbb{N} \text{ and } \mathcal{L}^1\text{-a.e. } r\in A_n.
\end{aligned} \tag{3.7}$$

Now, fix any $\alpha > \max\{\log_2(C_D(\delta)),1\}$. We know from the relative isoperimetric inequality, i.e. theorem 2.7, that

$$\begin{aligned}
&P(E,B_r(x)) \\
&\quad\geqslant 2\tilde{C}\min\{v_{E,x}(r),w_{E,x}(r)\}^{1-1/\alpha}\frac{\mathfrak{m}(B_r(x))^{1/\alpha}}{r} \quad \text{for every } n\in\mathbb{N} \text{ and } r\in A_n,
\end{aligned} \tag{3.8}$$

where we define $\tilde{C} := 1/2C_I(\alpha,\delta)^{(\alpha-1)/\alpha}$. Exploiting the fact that $x\in E^{(0)}$, we can find $\bar{n}\in\mathbb{N}$ such that

$$v_{E,x}(r) < w_{E,x}(r), \qquad C_y\left(\frac{\mathfrak{m}(B_r(x)\cap E)}{\mathfrak{m}(B_r(x))}\right)^{1/\alpha} \leqslant \tilde{C} \quad \text{for every } n\geqslant\bar{n} \text{ and } r\in A_n. \tag{3.9}$$

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By combining (3.7), (3.8), and (3.9), we deduce that for every $n \geqslant \bar{n}$ and $\mathcal{L}^1$-a.e. $r \in A_n$ it holds that

$$\begin{aligned}
&2\tilde{C}\mathfrak{m}(B_r(x)\cap E)^{1-1/\alpha}\frac{\mathfrak{m}(B_r(x))^{1/\alpha}}{r}\\
&\quad\leqslant P(B_r(x),E^{(1)}) + \mathfrak{m}(B_r(x)\cap E)^{1-1/\alpha}C_y\left(\frac{\mathfrak{m}(B_r(x)\cap E)}{\mathfrak{m}(B_r(x))}\right)^{1/\alpha}\frac{\mathfrak{m}(B_r(x))^{1/\alpha}}{r}\\
&\quad\leqslant P(B_r(x),E^{(1)}) + \tilde{C}\mathfrak{m}(B_r(x)\cap E)^{1-1/\alpha}\frac{\mathfrak{m}(B_r(x))^{1/\alpha}}{r}.
\end{aligned}$$

Rearranging the terms, we infer that

$$\begin{aligned}
&\tilde{C}\frac{\mathfrak{m}(B_r(x))^{1/\alpha}}{r}\mathfrak{m}(B_r(x)\cap E)^{1-1/\alpha}\\
&\quad\leqslant P(B_r(x),E^{(1)}) \quad \text{for every } n \geqslant \bar{n} \text{ and } \mathcal{L}^1\text{-a.e. } r \in A_n.
\end{aligned} \tag{3.10}$$

Now, define the function $f\colon (0,+\infty) \to \mathbb{R}$ as $f(r) := |\mathbf{D}\mathsf{d}_x|(B_r(x)\cap E^{(1)})$ for every $r > 0$. Corollary 2.3 tells that f is locally absolutely continuous and $f'(r) = P(B_r(x),E^{(1)})$ for $\mathcal{L}^1$-a.e. $r > 0$. Moreover, lemma 2.10 gives $f(r) \leqslant \mathfrak{m}(B_r(x)\cap E)$ for every $r > 0$. Consequently, it follows from (3.10) that

$$\tilde{C}\frac{\mathfrak{m}(B_r(x))^{1/\alpha}}{r}f(r)^{1-1/\alpha} \leqslant f'(r) \quad \text{for every } n \geqslant \bar{n} \text{ and } \mathcal{L}^1\text{-a.e. } r \in A_n. \tag{3.11}$$

Using that $x \notin \mathrm{int}(E^{(0)})$, which is the contradiction assumption, and lemma 2.10 we see that $f(r) \geqslant c\,\mathfrak{m}(B_r(x)\cap E) > 0$ for every $r > 0$, thus we can divide both sides of (3.11) by $\alpha f(r)^{1-1/\alpha}$, obtaining that

$$\frac{\tilde{C}}{\alpha}\frac{\mathfrak{m}(B_r(x))^{1/\alpha}}{r} \leqslant \frac{f'(r)}{\alpha f(r)^{1-1/\alpha}} = (f^{1/\alpha})'(r) \quad \text{for every } n \geqslant \bar{n} \text{ and } \mathcal{L}^1\text{-a.e. } r \in A_n. \tag{3.12}$$

The third inequality in (3.6) implies that $\mathcal{L}^1([r_n/4,r_n]\cap A_n) \geqslant r_n/4$ for every $n \in \mathbb{N}$, thus integrating (3.12) (and taking into account that $(f^{1/\alpha})'(r) \geqslant 0$ holds for $\mathcal{L}^1$-a.e. $r > 0$) we get that

$$\begin{aligned}
\frac{\tilde{C}}{4\alpha}\mathfrak{m}(B_{r_n/4}(x))^{1/\alpha} &\leqslant \frac{\tilde{C}}{\alpha}\frac{\mathfrak{m}(B_{r_n/4}(x))^{1/\alpha}}{r_n}\mathcal{L}^1([r_n/4,r_n]\cap A_n)\\
&\leqslant \frac{\tilde{C}}{\alpha}\int_{[r_n/4,r_n]\cap A_n}\frac{\mathfrak{m}(B_r(x))^{1/\alpha}}{r}\,\mathrm{d}r\\
&\leqslant \int_{[r_n/4,r_n]\cap A_n}(f^{1/\alpha})'(r)\,\mathrm{d}r \leqslant \int_0^{r_n}(f^{1/\alpha})'(r)\,\mathrm{d}r\\
&= f(r_n)^{1/\alpha} \leqslant \mathfrak{m}(B_{r_n}(x)\cap E)^{1/\alpha}
\end{aligned}$$

for every $n \geqslant \bar{n}$. Letting $C := (1/C_D(\delta)^2)(\tilde{C}/4\alpha)^\alpha$, we can conclude that $\mathfrak{m}(B_{r_n}(x)\cap E) \geqslant C\mathfrak{m}(B_{r_n}(x))$ for every $n \geqslant \bar{n}$. This leads to a contradiction with the fact that $x \in E^{(0)}$. Therefore, the proof of the statement is achieved. □

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Having lemma 3.7 at our disposal, we can now easily prove theorem 1.1.

Proof of theorem 1.1. Since $E^{(1)} = (\mathrm{X} \setminus E)^{(0)}$, it is sufficient to check that $E^{(0)} = \mathrm{int}(E^{(0)})$. To prove it, we argue by contradiction: suppose there exists a point $x \in E^{(0)} \setminus \mathrm{int}(E^{(0)})$. This implies that both $\mathfrak{m}(E) > 0$ (otherwise $E^{(0)} = \mathrm{X} = \mathrm{int}(E^{(0)})$) and $\mathfrak{m}(\mathrm{X} \setminus E) > 0$ (otherwise $E^{(0)} = \varnothing$), thus we know from remark 2.9 that $P(E) \neq 0$. Since $P(E, \cdot)$ is concentrated on $\partial^e E$, we can find a point $z \in \partial^e E$. Notice that $\mathfrak{m}(B_r(z) \cap E) > 0$ and $\mathfrak{m}(B_r(z) \setminus E) > 0$ for all $r > 0$. Since $z \neq x$, we can fix some radius $\delta \in (0, R_x(E)) \cap (0, 2R_z(E)/3) \cap (0, \mathsf{d}(x, z)/3)$. Thanks to the fact that $\mathfrak{m}(B_{\delta/2}(z) \cap E) > 0$, we can find a point $y \in E^{(1)} \cap B_{\delta/2}(z)$. Notice that $B_{\delta/2}(z) \setminus E \subseteq B_\delta(y) \setminus E$, so that $\mathfrak{m}(B_\delta(y) \setminus E) \geqslant \mathfrak{m}(B_{\delta/2}(z) \setminus E) > 0$. The fact that $x \notin \mathrm{int}(E^{(0)})$ implies that also $\mathfrak{m}(B_\delta(x) \cap E) > 0$. Hence, letting $v_{E,x}$, $w_{E,y}$ be defined in (3.4), we have proved that $v_{E,x}(\delta) > 0$ and $w_{E,y}(\delta) > 0$. By our construction and by the definition of $R_z(E), R_y(E)$ it holds $2R_z(E)/3 \leqslant R_y(E)$, hence we have $\delta \in (0, R_y(E))$. Moreover, the inequality $\delta < \mathsf{d}(x, z)/3$ implies that $\mathsf{d}(x, y) > 2\mathsf{d}(x, z)/3 > 2\delta$, which means that $\bar{B}_\delta(x) \cap \bar{B}_\delta(y) = \varnothing$. All in all, we showed that item (i) of lemma 3.7 holds. Hence, fixed any sequence $(r_n)_n \subseteq (0, 1)$ with $r_n \to 0$, we deduce from the assumption $x \notin \mathrm{int}(E^{(0)})$ that item (ii) of lemma 3.7 fails. Letting $A^{x,y}_{E,r_n}$ be as in (3.5), we get that

$$\mathcal{L}^1(A^{x,y}_{E,r_n}) \geqslant \frac{r_n}{2} \quad \text{holds only for finitely many } n \in \mathbb{N}. \tag{3.13}$$

Since $A^{x,y}_{E,r_n} \cup A^{y,x}_{\mathrm{X}\setminus E,r_n} = (0, r_n)$ for every $n \in \mathbb{N}$, we infer that $\mathcal{L}^1(A^{y,x}_{\mathrm{X}\setminus E,r_n}) \geqslant r_n/2$ for infinitely many $n \in \mathbb{N}$. Given that $v_{\mathrm{X}\setminus E,y}(\delta) = w_{E,y}(\delta) > 0$ and $w_{\mathrm{X}\setminus E,x}(\delta) = v_{E,x}(\delta) > 0$, we are in a position to apply lemma 3.7 again, obtaining that $y \in \mathrm{int}((\mathrm{X} \setminus E)^{(0)}) = \mathrm{int}(E^{(1)})$. This gives some $\bar{r} > 0$ satisfying $w_{E,y}(r) = 0$ for every $r \in (0, \bar{r})$. On the other hand, we know from $x \notin \mathrm{int}(E^{(0)})$ that $v_{E,x}(r) > 0$ for all $r \in (0, \bar{r})$. Choosing $\bar{n} \in \mathbb{N}$ so that $r_n < \bar{r}$ for all $n \geqslant \bar{n}$, we conclude that $A^{x,y}_{E,r_n} = (0, r_n)$ for every $n \geqslant \bar{n}$, in contradiction with (3.13). This proves that $E^{(0)} = \mathrm{int}(E^{(0)})$. □

Remark 3.8 (Some generalizations of theorem 1.1). To keep the presentation of theorem 1.1 as clear as possible, we decided not to prove it in its utmost generality. However, below we discuss some generalizations of our result that can be obtained by slightly adapting our arguments. The standing assumption is that $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ is a length PI space.

(i) By inspecting the proof of lemma 3.7, one can see that assuming the validity of a weaker notion of deformation property is sufficient. Namely, one can allow for the constant C appearing in (3.1a), (3.1b) to depend on y and it is sufficient to require the deformation property only for volume-constrained minimizers E of the perimeter.

(ii) A localized version of theorem 1.1 holds as well: let $E \subseteq \mathrm{X}$ be a volume-constrained minimizer of the perimeter in some open set $\Omega \subseteq \mathrm{X}$ (i.e. as in definition 3.1 but requiring that $K \subseteq \Omega$ and with $P(\cdot)$ replaced by $P(\cdot, \Omega)$) satisfying $P(E, \Omega) > 0$. Then, $E^{(1)} \cap \Omega$, $E^{(0)} \cap \Omega$ are open sets and $\partial E^{(1)} \cap \Omega = \partial E^{(0)} \cap \Omega = \partial^e E \cap \Omega$.

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(iii) Theorem 1.1 can be generalized to volume-constrained minimizers of a suitable class of *quasi-perimeters*. Fix an open set $\Omega \subseteq \mathrm{X}$ and a functional $G\colon \mathscr{B}(\Omega) \to \mathbb{R} \cup \{+\infty\}$ with $G(\varnothing) < +\infty$ having the following property: for any $U \Subset \Omega$ open, there exist constants $C = C(U) > 0$ and $\sigma = \sigma(U) \in \big(1 - (1/\max\{1, \log_2(\inf C_D)\}), 1\big]$ such that

$$G(E) \leqslant G(F) + C\mathfrak{m}(E\Delta F)^\sigma \quad \text{whenever } E, F \in \mathscr{B}(\Omega) \text{ satisfy } E\Delta F \subseteq U.$$

We then define the quasi-perimeter $\mathscr{P}_G$ restricted to Ω as $\mathscr{P}_G(E, \Omega) := P(E, \Omega) + G(E \cap \Omega)$ for every $E \in \mathscr{B}(\Omega)$. Then, an adaption of the previous arguments yields the validity of the following statement: if $E \subseteq \mathrm{X}$ is a volume-constrained minimizer of the quasi-perimeter $\mathscr{P}_G$ in Ω (i.e. as in definition 3.1 but requiring that $K \subseteq \Omega$, and with $P(\cdot)$ replaced by $\mathscr{P}_G(\cdot, \Omega)$) satisfying $P(E, \Omega) > 0$, then $E^{(1)} \cap \Omega$ and $E^{(0)} \cap \Omega$ are open sets, and it holds that $\partial E^{(1)} \cap \Omega = \partial E^{(0)} \cap \Omega = \partial^e E \cap \Omega$. ■

Once we know that volume-constrained minimizers of the perimeter have an open representative, we can obtain the following expected boundary density estimates by suitably adapting the arguments in the proof of lemma 3.7.

THEOREM 3.9 (Boundary density estimates). *Let $(\mathrm{X}, \mathsf{d}, \mathfrak{m})$ be a length PI space having the deformation property. Let $E \subseteq \mathrm{X}$ be a volume-constrained minimizer of the perimeter. Let $B \subseteq \mathrm{X}$ be a given bounded set. Then, there exist constants $\bar{r} = \bar{r}(E, B, C_D, C_I) > 0$ and $C = C(E, B, C_D, C_I) > 1$ such that*

$$\frac{1}{C} \leqslant \frac{\mathfrak{m}(B_r(x) \cap E)}{\mathfrak{m}(B_r(x))} \leqslant 1 - \frac{1}{C}, \qquad \frac{1}{C} \leqslant \frac{rP(E, B_r(x))}{\mathfrak{m}(B_r(x))} \leqslant C, \tag{3.14}$$

for every $x \in \partial^e E \cap B$ and $r \in (0, \bar{r})$.

In particular, there exists a constant $\tilde{C} = \tilde{C}(C, C_D(\bar{r}/2)) \geqslant 1$ such that

$$P(E, B_{2r}(x)) \leqslant \tilde{C}\, P(E, B_r(x)) \quad \text{for every } x \in \partial^e E \cap B \text{ and } r \in (0, \bar{r}/2). \tag{3.15}$$

Proof. If $\partial^e E$ contains only one point, the first one in (3.14) follows by the definition $\partial^e E$, while the second follows from [**3**, Theorem 5.4]. Thus, we can assume that $\partial^e E$ contains at least two distinct points z and $\tilde{z}$, otherwise there is nothing to prove. In particular, letting $\rho := \min\big\{R_z(E), R_{\tilde{z}}(E), (1/5)\mathsf{d}(z, \tilde{z})\big\}$, we can find two points $y \in B_{\rho/2}(z) \cap E^{(1)}$ and $\tilde{y} \in B_{\rho/2}(\tilde{z}) \cap E^{(1)}$. In fact, theorem 1.1 ensures that $y, \tilde{y} \in \mathrm{int}(E^{(1)})$, so that there exists $r_0 \in (0, \rho)$ such that

$$\mathfrak{m}(B_{r_0}(y) \setminus E) = \mathfrak{m}(B_{r_0}(\tilde{y}) \setminus E) = 0. \tag{3.16}$$

Notice that $\mathfrak{m}(B_\rho(y) \setminus E) \geqslant \mathfrak{m}(B_{\rho/2}(z) \setminus E) > 0$ and similarly $\mathfrak{m}(B_\rho(\tilde{y}) \setminus E) > 0$. The doubling assumption ensures that the closure K of B is compact, thus an

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application of Dini's theorem yields the existence of $r_1 > 0$ such that

$$\mathfrak{m}(B_r(x) \cap E) < \min\{\mathfrak{m}(B_\rho(y) \setminus E), \mathfrak{m}(B_\rho(\tilde{y}) \setminus E)\} \quad \text{for every } x \in K \text{ and } r \in (0, r_1). \tag{3.17}$$

Thanks to (3.2), we can also find $r_2 > 0$ such that $r_2 < R_y(E)$, $r_2 < R_{\tilde{y}}(E)$, and $r_2 < R_x(\varnothing)$ hold for every $x \in K$. Now, define $\bar{r}_0 := \min\{r_0, r_1, r_2\} > 0$. Let $x \in \partial^e E \cap B$ be fixed. Our choice of ρ ensures that $\bar{B}_\rho(x)$ is disjoint from at least one between $\bar{B}_\rho(y)$ and $\bar{B}_\rho(\tilde{y})$. Up to relabelling y and $\tilde{y}$, say that $\bar{B}_\rho(x) \cap \bar{B}_\rho(y) = \varnothing$. Given any $r \in (0, \bar{r}_0)$, we deduce from (3.16), (3.17), and the continuity of $s \mapsto \mathfrak{m}(B_s(y) \setminus E)$ that there exists $s(r) \in (\bar{r}_0, \rho)$ such that $\mathfrak{m}(B_r(x) \cap E) = \mathfrak{m}(B_{s(r)}(y) \setminus E)$. Define the Borel set $E_r \subseteq \mathrm{X}$ as $E_r := (E \setminus B_r(x)) \cup B_{s(r)}(y)$. By the minimality assumption on E, arguing as we did in the proof of lemma 3.7 we obtain

$$P(E, B_r(x)) \leqslant \max\{C_z, C_{\tilde{z}}\} \frac{\mathfrak{m}(B_r(x) \cap E)}{\bar{r}_0} + P(B_r(x), E^{(1)}), \tag{3.18}$$

for any $x \in \partial^e E \cap B$ and $\mathcal{L}^1$-a.e. $r \in (0, \bar{r}_0)$. For any $x \in \partial^e E \cap B$, define $A_x(E) := \{r > 0 : |\mathbf{D}\mathsf{d}_x|(B_r(x) \cap E) \leqslant |\mathbf{D}\mathsf{d}_x|(B_r(x) \setminus E)\}$. Fix $\alpha > \max\{\log_2(C_D(\rho)), 1\}$. Applying the relative isoperimetric inequality to the left-hand side of (3.18) and using lemma 2.10, we deduce that

$$\begin{aligned} &2C_0 \frac{\mathfrak{m}(B_r(x))^{1/\alpha}}{r} |\mathbf{D}\mathsf{d}_x|(B_r(x) \cap E)^{1-1/\alpha} \\ &= 2C_0 \frac{\mathfrak{m}(B_r(x))^{1/\alpha}}{r} \min\{|\mathbf{D}\mathsf{d}_x|(B_r(x) \cap E), |\mathbf{D}\mathsf{d}_x|(B_r(x) \setminus E)\}^{1-1/\alpha} \\ &\leqslant 2C_0 \frac{\mathfrak{m}(B_r(x))^{1/\alpha}}{r} \min\{\mathfrak{m}(B_r(x) \cap E), \mathfrak{m}(B_r(x) \setminus E)\}^{1-1/\alpha} \\ &\leqslant P(B_r(x), E^{(1)}) + \frac{\max\{C_z, C_{\tilde{z}}\}}{c^{1-1/\alpha}} \frac{r}{\bar{r}_0} \frac{\mathfrak{m}(B_r(x))^{1/\alpha}}{r} |\mathbf{D}\mathsf{d}_x|(B_r(x) \cap E)^{1-1/\alpha} \end{aligned} \tag{3.19}$$

holds for $\mathcal{L}^1$-a.e. $r \in (0, \bar{r}_0) \cap A_x(E)$, where we set $C_0 := 1/(2\, C_I(\alpha, \rho)^{(\alpha-1)/\alpha})$ for brevity. Therefore, if we let

$$\bar{r} := \min\left\{\frac{c^{1-1/\alpha} C_0 \bar{r}_0}{\max\{C_z, C_{\tilde{z}}\}}, \bar{r}_0\right\} \in (0, \bar{r}_0],$$

then we infer from (3.19) that

$$C_0 \frac{\mathfrak{m}(B_r(x))^{1/\alpha}}{r} |\mathbf{D}\mathsf{d}_x|(B_r(x) \cap E)^{1-1/\alpha} \leqslant P(B_r(x), E^{(1)}) \quad \text{for } \mathcal{L}^1\text{-a.e. } r \in (0, \bar{r}) \cap A_x(E). \tag{3.20}$$

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This also proves (by considering $\mathrm{X} \setminus E$ instead of E) that, up to shrinking $\bar{r} > 0$, it holds that

$$C_0 \frac{\mathfrak{m}(B_r(x))^{1/\alpha}}{r} |\mathbf{D}\mathsf{d}_x|(B_r(x) \setminus E)^{1-1/\alpha} \leqslant P(B_r(x), E^{(0)}) \quad \text{for } \mathcal{L}^1\text{-a.e. } r \in (0, \bar{r}) \cap A_x(\mathrm{X} \setminus E). \tag{3.21}$$

Let us now define the function $f_x \colon (0, +\infty) \to \mathbb{R}$ as

$$f_x(r) \coloneqq \min\left\{|\mathbf{D}\mathsf{d}_x|(B_r(x) \cap E^{(1)}), |\mathbf{D}\mathsf{d}_x|(B_r(x) \cap E^{(0)})\right\} \quad \text{for every } r > 0.$$

Corollary 2.3 ensures that f_x is locally absolutely continuous and

$$f_x'(r) = \begin{cases} P(B_r(x), E^{(1)}) & \text{for } \mathcal{L}^1\text{-a.e. } r \in A_x(E), \\ P(B_r(x), E^{(0)}) & \text{for } \mathcal{L}^1\text{-a.e. } r \in A_x(\mathrm{X} \setminus E). \end{cases}$$

Observe that $A_x(E) \cup A_x(\mathrm{X} \setminus E) = (0, +\infty)$. Arguing as in lemma 3.7, we deduce from (3.20) and (3.21) that

$$\frac{C_0}{\alpha} \frac{\mathfrak{m}(B_r(x))^{1/\alpha}}{r} \leqslant (f_x^{1/\alpha})'(r) \quad \text{for } \mathcal{L}^1\text{-a.e. } r \in (0, \bar{r}). \tag{3.22}$$

Given any $r \in (0, \bar{r})$, we can integrate the inequality in (3.22) over the interval $[r/2, r]$, thus obtaining that

$$\begin{aligned} \frac{C_0}{2\alpha\,(C_D(\bar{r}/2))^{1/\alpha}} \mathfrak{m}(B_r(x))^{1/\alpha} &\leqslant \frac{C_0}{\alpha} \frac{\mathfrak{m}(B_{r/2}(x))^{1/\alpha}}{r} \frac{r}{2} \leqslant \frac{C_0}{\alpha} \int_{r/2}^{r} \frac{\mathfrak{m}(B_s(x))^{1/\alpha}}{s} \,\mathrm{d}s \\ &\leqslant \int_0^r (f_x^{1/\alpha})'(s) \,\mathrm{d}s = f_x(r)^{1/\alpha} \leqslant \min\left\{\mathfrak{m}(B_r(x) \cap E), \mathfrak{m}(B_r(x) \setminus E)\right\}^{1/\alpha}. \end{aligned} \tag{3.23}$$

It follows that $\mathfrak{m}(B_r(x)) \leqslant C_1 \mathfrak{m}(B_r(x) \cap E)$ for every $x \in \partial^e E \cap B$ and $r \in (0, \bar{r})$, where we define $C_1 \coloneqq C_D(\bar{r}/2)\big(2\alpha/C_0\big)^{\alpha}$.

Let $x \in \partial^e E \cap B$ and $r \in (0, \bar{r})$ be fixed. Since $\mathfrak{m}(B_r(x)) \leqslant C_1 \min\big\{\mathfrak{m}(B_r(x) \cap E), \mathfrak{m}(B_r(x) \setminus E)\big\}$ by (3.23), by using the relative isoperimetric inequality, and recalling that $2C_0 = 1/C_I(\alpha, \rho)^{(\alpha-1)/\alpha}$, we get that

$$\begin{aligned} \frac{2C_0}{C_1^{1-1/\alpha}} \frac{\mathfrak{m}(B_r(x))}{r} &\leqslant 2C_0 \frac{\mathfrak{m}(B_r(x))^{1/\alpha}}{r} \min\left\{\mathfrak{m}(B_r(x) \cap E), \mathfrak{m}(B_r(x) \setminus E)\right\}^{1-1/\alpha} \leqslant P(E, B_r(x)). \end{aligned}$$

On the contrary, up to shrinking $\bar{r}$ (depending only on B), we can find a constant $C_2 > 0$ (depending only on B) such that $P(B_{\tilde{r}}(x)) \leqslant C_2 \mathfrak{m}(B_{\tilde{r}}(x))/\tilde{r}$ for every $\tilde{r} \in$

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$(0,\bar{r})$; recall the discussion after (3.2). Then,

$$\begin{aligned}
P(E,B_r(x)) &\leqslant P(E,B_{\tilde r}(x)) \leqslant \max\{C_z,C_{\tilde z}\}\frac{\mathfrak{m}(B_{\tilde r}(x)\cap E)}{\tilde r} + P(B_{\tilde r}(x),E^{(1)})\\
&\leqslant \max\{C_z,C_{\tilde z}\}\frac{\mathfrak{m}(B_{\tilde r}(x))}{\tilde r} + P(B_{\tilde r}(x))\\
&\leqslant \big(\max\{C_z,C_{\tilde z}\}+C_2\big)\frac{\mathfrak{m}(B_{\tilde r}(x))}{\tilde r},
\end{aligned}$$

for $\mathcal{L}^1$-a.e. $\tilde r\in(r,\bar r)$, thanks to (3.18) and to the deformation property. Hence, $rP(E,B_r(x))/\mathfrak{m}(B_r(x))\leqslant \max\{C_z,C_{\tilde z}\}+C_2$ for all $x\in\partial^e E\cap B$ and $r\in(0,\bar r)$. Taking

$$C \coloneqq \max\big\{C_1, C_1^{(\alpha-1)/\alpha}/(2C_0), \max\{C_z,C_{\tilde z}\}+C_2\big\},$$

we conclude that (3.14) holds. Finally, applying (3.14) we conclude that for every $x\in\partial^e E\cap B$ and $r\in(0,\bar r/2)$ it holds that

$$\frac{P(E,B_{2r}(x))}{P(E,B_r(x))}\leqslant \frac{C\mathfrak{m}(B_{2r}(x))}{2r}\frac{Cr}{\mathfrak{m}(B_r(x))}\leqslant \frac{C^2C_D(\bar r/2)}{2},$$

which proves the validity of (3.15). Consequently, the statement is achieved. □

We conclude with a final comment on further minimality properties satisfied by volume-constrained minimizers. Such properties can be derived by reproducing well-known arguments, see, e.g. [**6**, Remark 3.23, Theorem 3.24], exploiting theorem 1.1 and the deformation property.

Remark 3.10. Let $(\mathrm{X},\mathsf{d},\mathfrak{m})$ be a length PI space having the deformation property. Let $E\subseteq \mathrm{X}$ be a volume-constrained minimizer of the perimeter. Using theorem 1.1 and with arguments similar to those in the proof of theorem 3.9, it is possible to prove that for any compact set $K\subseteq\mathrm{X}$ there exist $\Lambda,r_0>0$ such that E is a (Λ,r_0)-perimeter minimizer on K, i.e. whenever $F\Delta E\subseteq B_r(x)$ for some $x\in K$ and $r<r_0$ it holds $P(E,B_r(x))\leqslant P(F,B_r(x))+\Lambda\,\mathfrak{m}(E\Delta F)$.

Moreover, for any given compact set $K\subseteq\mathrm{X}$ there exist constants $L,r_0>0$ such that E is (L,r_0)-quasi minimal on K, i.e. whenever $F\Delta E\subseteq B_r(x)$ for some $x\in K$ and $r<r_0$ it holds that $P(E,B_r(x))\leqslant L\,P(F,B_r(x))$. The class of quasi-minimal sets has been studied e.g. in [**31**].

It is worth pointing out that, once we know that volume-constrained minimizers of the perimeter are (L,r_0)-quasi minimal sets, theorem 3.9 follows directly from [**31**, Theorem 4.2 and Lemma 5.1]. Nevertheless, we opted for a self-contained proof of theorem 3.9, which takes advantage of the openness of volume-constrained minimizers. ■

4. Boundedness of isoperimetric sets

In this final section, we prove the boundedness of isoperimetric sets in length PI spaces satisfying the deformation property and with a uniform lower bound on the

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volume of unit balls (theorem 1.2). The argument makes use of the topological regularity given by our main result theorem 1.1.

Proof of theorem 1.2. Suppose by contradiction that E has no bounded representatives, i.e. $\mathfrak{m}(E \setminus B_R(x)) > 0$ for all $R > 0$ and $x \in \mathrm{X}$. In particular X is unbounded and, since

$$v_0 := \inf_{x \in \mathrm{X}} \mathfrak{m}(B_1(x)) > 0,$$

we have $\mathfrak{m}(\mathrm{X}) = \infty$ and $\mathfrak{m}(\mathrm{X} \setminus E) > 0$. Then, $P(E) > 0$ and, arguing as in the proof of theorem 1.1, we can find $y \in E^{(1)}$ and $\rho \in (0, R_y(E))$ such that $\delta := \mathfrak{m}(B_\rho(y) \setminus E) > 0$. By theorem 1.1 it holds that $y \in \mathrm{int}(E^{(1)})$, i.e. there exists $r_0 > 0$ such that $\mathfrak{m}(B_{r_0}(y) \setminus E) = 0$. We consider the function $f : (0, +\infty) \to \mathbb{R}$ defined by

$$f(R) := |\mathbf{D}\mathsf{d}_y|(E^{(1)} \setminus B_R(y)) = |\mathbf{D}\mathsf{d}_y|(E^{(1)}) - |\mathbf{D}\mathsf{d}_y|(E^{(1)} \cap B_R(y)),$$

and observe that $|\mathbf{D}\mathsf{d}_y|(E^{(1)}) \leqslant \mathrm{lip}(\mathsf{d}_y)\mathfrak{m}(E^{(1)}) = \mathfrak{m}(E) < +\infty$. By corollary 2.3 the function f is locally absolutely continuous and satisfies $f'(r) = -P(B_R(x), E^{(1)})$ for $\mathcal{L}^1$-a.e. $R > 0$. Thanks to lemma 2.10 and since $\mathfrak{m}(E^{(1)} \Delta E) = 0$, there also exists a constant $c > 0$ such that

$$0 < c\,\mathfrak{m}(E \setminus B_R(y)) \leqslant f(R) \leqslant \mathfrak{m}(E \setminus B_R(y)), \quad \forall\, R > 0. \tag{4.1}$$

Observe that, since $\mathfrak{m}(E) < +\infty$, it holds $\mathfrak{m}(E \setminus B_R(y)) \to 0$ as $R \to +\infty$. Hence, $f(R) \to 0$ as $R \to +\infty$ and so we can find $R_0 > \rho$ such that $f(R) < \min\{\delta, v_0/2\}$ for all $R \geqslant R_0$. By continuity, for every $R \geqslant R_0$ there exists $r(R) \in (0, \rho)$ such that

$$\mathfrak{m}(B_{r(R)}(y) \setminus E) = \mathfrak{m}(E \setminus B_R(y)). \tag{4.2}$$

For every $R \geqslant R_0$ we define the set $F_R := (E \cup B_{r(R)}(y)) \cap B_R(y)$, which satisfies $\mathfrak{m}(F_R) = \mathfrak{m}(E)$ thanks to (4.2) and $r(R) < R$. Hence, by minimality, $P(E) \leqslant P(F_R)$ for every $R \geqslant R_0$. Moreover, using proposition 2.6 and the deformation property, for $\mathcal{L}^1$-a.e. $R \geqslant R_0$ we have

$$\begin{aligned}
P(E) &\leqslant P(F_R) \\
&= P((E \cup B_{r(R)}(y)) \cap B_R(y)) \\
&\leqslant P(E \cup B_{r(R)}(y), B_R(y)^{(1)}) + P(B_R(y), (E \cup B_{r(R)}(y))^{(1)}) \\
&\leqslant P(E \cup B_{r(R)}(y)) - P(E \cup B_{r(R)}(y), B_R(y)^{(0)}) + P(B_R(y), E^{(1)}) \\
&\leqslant P(E) + C_y(E)\frac{\mathfrak{m}(B_{r(R)}(y) \setminus E)}{r_0} - P(E, B_R(y)^{(0)}) + P(B_R(y), E^{(1)}) \\
&\leqslant P(E) + C_y(E)\frac{\mathfrak{m}(B_{r(R)}(y) \setminus E)}{r_0} - P(E \setminus B_R(y)) + 2P(B_R(y), E^{(1)}) \\
&\leqslant P(E) + C_y(E)\frac{\mathfrak{m}(E \setminus B_R(y))}{r_0} - Cv_0^{1/\alpha}\mathfrak{m}(E \setminus B_R(y))^{(\alpha-1)/\alpha} \\
&\quad + 2P(B_R(y), E^{(1)}),
\end{aligned}$$

with $C > 0, \alpha > 1$ constants independent of R, where in the fifth line we used again proposition 2.6 and in the last line we used the isoperimetric inequality for small

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volumes in proposition 2.8 (recall that $\mathfrak{m}(E \setminus B_R(y)) < v_0/2$). This combined with (4.1) shows that

$$\begin{aligned} 2f'(R) &\leqslant C_y(E)c^{-1}r_0^{-1}f(R) - Cv_0^{1/\alpha}f(R)^{(\alpha-1)/\alpha} \\ &\leqslant -C_1 f(R)^{(\alpha-1)/\alpha}, \quad \text{for a.e. } R \geqslant R_1, \end{aligned}$$

for some constant $R_1 \geqslant R_0$ big enough and where $C_1 > 0$ is a constant independent of R. Note that in the last inequality we used that $f(R) \to 0$ as $R \to +\infty$ and $\alpha > 1$. Since $f(R) > 0$ for all $R > 0$, this shows that

$$(f^{1/\alpha})'(R) \leqslant -\frac{C_1}{2\alpha}, \quad \text{for a.e. } R \geqslant R_1,$$

which contradicts the fact that $f(R)$ is strictly positive for any $R > 0$. □

Acknowledgements

Part of this research has been carried out at the Fields Institute (Toronto) in November 2022, during the Thematic Program on Nonsmooth Riemannian and Lorentzian Geometry. The authors gratefully acknowledge the warm hospitality and the stimulating atmosphere. The authors thank Panu Lahti for pointing out the example in remark 3.6. The authors also thank Camillo Brena, Vesa Julin, Tapio Rajala, and Daniele Semola for fruitful discussions on the topic of the paper. The authors also thank the reviewer for the careful reading and for pointing out an inaccuracy in a preliminary version of the paper.

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