Acknowledgement

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KARL BUSHNELL
Ribston Hall High School,
Stroud Road,
Gloucester
GL1 5LE
email: *kbu@ribstonhall.gloucs.sch.uk*

108.45 The golden section from three congruent semicircles

Let R be a positive real number and let A_1B_1 be a line segment with length $2R$. Two rays ℓ, ℓ' with origins at A_1, B_1, respectively, are perpendicular to A_1B_1. We show how to obtain the following configuration where $A_2B_2 = A_3B_3 = 2R$, points A_3, B_3 are on ℓ, B_2 is on ℓ', and the semicircles ω_1, ω_2, ω_3 with respective diameters A_1B_1, A_2B_2, A_3B_3 satisfy:

- A_2B_2 is tangent to ω_1 at A_2
- ω_2 is tangent to ω_3.

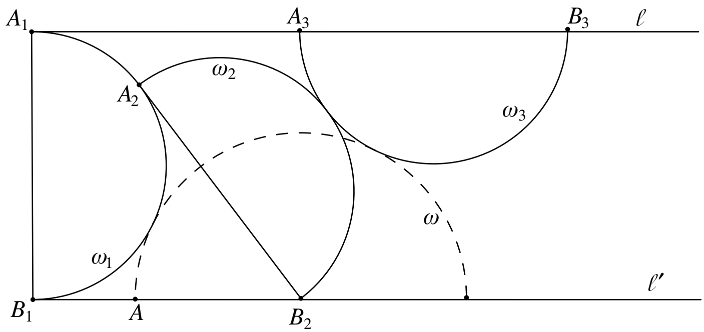

FIGURE 1

As a by-product, the construction will provide the following proposition:

Proposition 1: The semicircle ω with centre B_2 externally tangent to ω_1 is also tangent to ω_3. In addition, if it intersects the line segment B_1B_2 in A, then $\frac{AB_2}{AB_1} = \phi$, the golden ratio $\left(\phi = \frac{1}{2}(\sqrt{5} + 1)\right)$.

Constructing Figure 1

The construction of ω_2 is easy: since the tangents to ω_1 from B_2 are of equal length, we must have $B_2B_1 = B_2A_2 = 2R$. Thus, we first locate B_2 on ℓ such that $B_1B_2 = 2R$, then draw the tangent B_2A_2 to ω_1 and ω_2 follows.

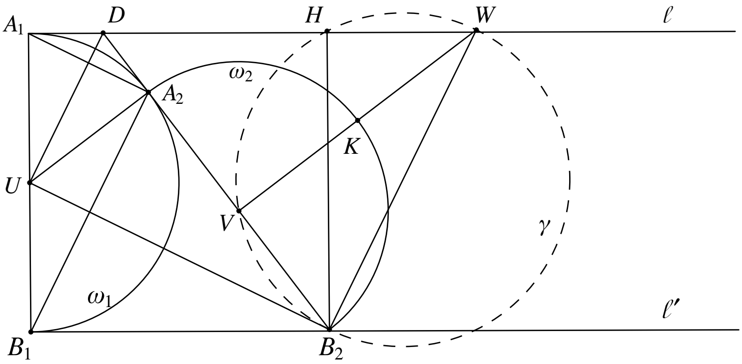

FIGURE 2

To conclude the first part of the construction, we introduce the centre U of ω_1 and the point D of intersection of the line B_2A_2 and ℓ (Figure 2). Since A_1A_2 is perpendicular to A_2B_1, the line segment DU is perpendicular to B_2U. It follows that UA_2 is the altitude on the hypotenuse of the right triangle DUB_2. As a result, we have $A_2D \times A_2B_2 = A_2U^2$ and so $DA_2 = \frac{1}{2}R$.

To proceed, we consider the orthogonal projection H of B_2 onto ℓ and the perpendicular bisector of A_2B_2, which intersects ω_2 at K, ℓ at W, and A_2B_2 at its midpoint V (Figure 2). We show that $A_3 = H$ and that W is the centre of the desired semicircle ω_3.

The points V and H, which are on the circle γ with diameter B_2W (since $\angle B_2HW = \angle B_2VW = 90°$), satisfy $DV = DA_2 + A_2V = \frac{1}{2}R + R = \frac{3}{2}R$ and $DH^2 = DB_2^2 - B_2H^2 = \left(\frac{5}{2}R\right)^2 - 4R^2 = \frac{9}{4}R^2$, hence $HD = VD$. Since the power of D with respect to γ is $DH \times DW$ as well as $DV \times DB_2$, it follows that $DW = DB_2 = \frac{5}{2}R$ and so $HW = R$. We deduce that $VW^2 = B_2W^2 - VB_2^2 = B_2H^2 + HW^2 - R^2 = 4R^2 + R^2 - R^2 = 4R^2$, that is, $VW = 2R$. Since $VK = R$, we obtain $WK = R$ and the semicircle with centre W and radius R passes through H and is tangent to ω_2 at K. This semicircle is ω_3.

Proof of Proposition 1

We observe that $B_2U^2 = B_2A_2^2 + A_2U^2 = 5R^2 = B_2H^2 + HW^2 = B_2W^2$ so that the radius r of ω is $B_2U - R = R(\sqrt{5} - 1)$. Since $B_2W - R = R(\sqrt{5} - 1)$ as well, the semicircle ω is also tangent to ω_3. Note the relation $\frac{2R}{r} = \phi$.

There just remains to calculate

$$\frac{AB_2}{AB_1} = \frac{r}{2R - r} = \frac{1}{\phi - 1} = \phi.$$

In passing, we can extract the following quick construction of G dividing a given line segment XY in the golden ratio.

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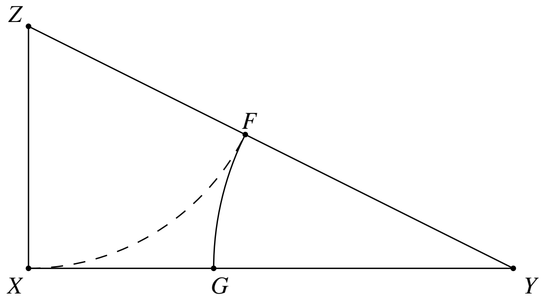

FIGURE 3

First locate Z with ZX perpendicular to XY and $XZ = \frac{1}{2}XY$ and then F on the line segment ZY with $ZF = ZX$. Lastly, let G on XY be such that $YG = YF$. This point G satisfies $\frac{GY}{GX} = \phi$.

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TRAN QUANG HUNG
High School for Gifted Students,
Hanoi University of Science,
Hanoi National University,
Hanoi, Vietnam
e-mail: *analgeomatica@gmail.com*
MICHEL BATAILLE
6 square des Boulots,
76520 Franqueville-Saint-Pierre, France
e-mail: *michelbataille@wanadoo.fr*

108.46 A generalisation of Fuss' theorem

Introduction

Fuss' theorem for bicentric quadrilaterals is a classic theorem of plane geometry that appeared in the 18th century in the works of Nikolai Fuss, an assistant of the great Leonhard Euler, see [1, 2, 3]. In [3], Juan Carlos Salazar gave a very simple and elegant solution to this theorem using only classical tools. This is an interesting idea, and we have exploited this idea to give a generalisation of Fuss' theorem. Here we shall propose a 'weaker' condition that only the inscribed quadrilateral is enough. The theorem is as follows: