COCOMMUTATIVE HOPF ALGEBRAS

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1. Introduction. A *coalgebra* over the field F is a vector space A over F, with maps $\delta: A \to A \otimes A$ and $\epsilon: A \to F$ such that

$$(1) \qquad (1 \otimes \delta)\delta = (\delta \otimes 1)\delta$$

and

$$(2) \qquad (1 \otimes \epsilon)\delta = (\epsilon \otimes 1)\delta = 1.$$

The notion of coalgebra is dual to the notion of algebra with unit, with δ as coproduct (equation (1) says that δ is associative) and ϵ as the unit map (equation (2) is just the statement that ϵ is a unit for the coproduct δ). If A is also an algebra with unit and δ and ϵ are algebra homomorphisms, A is a *Hopf algebra*.

An example of Hopf algebra is the group algebra $\Gamma(G, F)$ of a semigroup G with unit. In this case δ and ϵ are defined by $\delta(g) = g \otimes g$ and $\epsilon(g) = 1$ for $g \in G$. Another example is the universal enveloping algebra $U(L)$ of a Lie algebra L. Here δ and ϵ are defined by $\delta(x) = 1 \otimes x + x \otimes 1$ and $\epsilon(x) = 0$ for $x \in L$. Both of these examples are cocommutative, that is, they satisfy $\delta = T\delta$ where $T: A \otimes A \to A \otimes A$ is defined by $T(a \otimes b) = b \otimes a$.

Note that if A is a coalgebra, the dual vector space A^* has a natural algebra structure. In this paper we characterize the types of Hopf algebras described in the examples given above in terms of the structure of the dual algebra. Specifically, if F is algebraically closed, a cocommutative Hopf algebra is the group algebra of a semigroup with unit if and only if its dual algebra is semi-simple. If F has characteristic 0, a cocommutative Hopf algebra is the universal enveloping algebra of a Lie algebra if and only if its dual algebra is local. Then we prove that a cocommutative Hopf algebra over an algebraically closed field can be written as the direct sum of sub-coalgebras with local dual algebras. This enables us to give a proof of a theorem discovered by B. Kostant which gives conditions for a cocommutative Hopf algebra over an algebraically closed field to be the product (in the sense of Definition 3.1) of a Hopf algebra with a local dual algebra by a group algebra. Such a Hopf algebra is called invertible. Finally we give conditions for a cocommutative Hopf algebra over

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an algebraically closed field to be embeddable in an invertible cocommutative Hopf algebra.

If A is an algebra with unit, it will be convenient to denote by $\mu: A \otimes A \to A$ the product in A and by $\eta: F \to A$ the map sending $\alpha \in F$ into $\alpha 1 \in A$.

2. Coalgebras and topological algebras. In this section we show that the dual of a coalgebra is a topological algebra satisfying certain conditions. This allows us to prove some useful facts about coalgebras in later sections.

If L is a vector space over the field F, denote by L^* the space of all linear transformations from L to F. If M is a subset of L, denote by M^{perp} the set of all elements of L^* which vanish on M. We shall identify L with its image under the natural injection $L \to L^{**}$.

DEFINITION 2.1. *Let L be a vector space over the field F. The finite topology for L^* is the topology such that $\{x + S^{\text{perp}}\}$, where S ranges over the finite subsets of L, is a base for the neighbourhood system of the point x in L^*.*

If the field F is given the discrete topology, this definition makes L^* into a complete Hausdorff topological vector space.

The following two lemmas are immediate consequences of (3, Propositions IV. 6.1, 2).

LEMMA 2.2. *Let M be a linear subspace of L^*. Then*

$$\mathrm{Cl}(M) = (M^{\text{perp}} \cap L)^{\text{perp}}.$$

LEMMA 2.3. *Let M be a closed linear subspace of L^* of finite codimension. If N is a linear subspace of L^* containing M, then N is closed.*

By a *topological algebra* we mean an algebra over the topological field F whose underlying vector space is a Hausdorff topological vector space, and whose multiplication is continuous. Throughout this paper we shall assume that the field F is given the discrete topology.

The following lemma is a list of trivial but useful facts.

LEMMA 2.4. *If A is a coalgebra over the field F, A^* with the finite topology is a complete topological algebra. If B is a sub-coalgebra of A, B^{perp} is a closed ideal in A^*. If I is an ideal in A^*, $I^{\text{perp}} \cap A$ is a sub-coalgebra of A.*

PROPOSITION 2.5. *Let A be a coalgebra over the field F. If V is a finite-dimensional subspace of A, there exists a finite-dimensional sub-coalgebra B containing V.*

Proof. Since the span of a set of sub-coalgebras is a sub-coalgebra, it is enough to show that any element $a \in A$ is contained in a finite-dimensional sub-coalgebra.

Let B be the minimal sub-coalgebra of A containing a. Let $\{b_i | i \in I\}$ be a basis of B and let b'_i be the element of B^* defined by $(b_i, b'_i) = 1$ and $(b_j, b'_i) = 0$ if $j \neq i$. If $(a, B^* b'_i B^*) = 0$, then by Lemma 2.4

$$(B^* b'_i B^*)^{\text{perp}} \cap B$$

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is a *proper* sub-coalgebra of B containing a, contradicting the minimality of B. Therefore for each $i \in I$ there exist x_i and y_i in B^* such that $(a, x_i b_i' y_i) \neq 0$. Let

$$(\delta \otimes 1)\delta(a) = \sum \alpha_{\text{kmn}} b_k \otimes b_m \otimes b_n.$$

Since

$$\sum \alpha_{\text{kin}} (b_k, x_i)(b_n, y_i) = (a, x_i b'_i y_i) \neq 0$$

for each $i \in I$, there are $k, n \in I$ such that $\alpha_{\text{kin}} \neq 0$. Therefore, since only finitely many of the $\alpha_{\text{kmn}} \neq 0$, B is finite dimensional. This completes the proof of the proposition.

The following proposition is an immediate consequence of Lemma 2.4 and Proposition 2.5.

PROPOSITION 2.6. *Let A be a cocommutative coalgebra. Then A^* with the finite topology is a complete commutative topological algebra with a base for the neighbourhood system of* 0 *consisting of closed ideals of finite codimension.*

PROPOSITION 2.7. *Let R be a complete commutative topological algebra with a base for the neighbourhood system of* 0 *consisting of closed ideals of finite codimension. Then the radical $J(R)$ of R is the intersection of the closed maximal ideals of R.*

Proof. Denote the intersection of the closed maximal ideals of R by T. Then $J(R) \subseteq T$, since $J(R)$ is the intersection of *all* maximal ideals of R.

We wish to show that if $t \in T$, then 1 lies in the ideal $\text{Cl}(R(1 + t))$. If not, there exists a closed ideal of U of finite codimension which is a neighbourhood of 0 such that $(1 + U) \cap \text{Cl}(R(1 + t)) = \emptyset$. Therefore,

$$1 \notin U + \text{Cl}(R(1 + t)).$$

Let K' be a maximal ideal in R/U containing $(\text{Cl}(R(1 + t)) + U)/U$ and K the complete inverse image of K' in R. By Lemma 2.3, K is a closed maximal ideal containing $\text{Cl}(R(1 + t))$. But $t \in T \subseteq K$, so $1 \in K$, which is a contradiction.

To complete the proof of the proposition, we show that every element $t \in T$ is quasi-regular, which implies that $T \subseteq J(R)$. By the above discussion $\text{Cl}(R(1 + t)) = R$. Therefore there exists a net $\{r_n | n \in D\}$ in R such that $\lim r_n(1 + t) = 1$. A straightforward calculation shows that $\{r_n | n \in D\}$ is a Cauchy net. By the completeness of R there exists $r \in R$ such that $\lim r_n = r$. It follows that $r = (1 + t)^{-1}$. Therefore, t is quasi-regular. This completes the proof of the proposition.

If X is a set, denote by F^X the algebra of all functions from X to F with the topology of pointwise convergence. If $x \in X$, denote by e_x the characteristic function of the set $\{x\}$. If R is an algebra, denote by p_R the natural projection $R \rightarrow R/J(R)$.

PROPOSITION 2.8. *Let R be a complete commutative topological algebra with a base for the neighbourhood system of 0 consisting of closed ideals of finite codimension. Assume there is a set X and a continuous algebra isomorphism $\phi: F^X \to R/J(R)$. Then there exists $\{f_x \in R | x \in X\}$ such that $p_R(f_x) = \phi(e_x)$, $f_x^2 = f_x$, $f_x f_y = 0$ if $x \neq y$, and*

$$\lim (f_{x1} + \ldots + f_{xn}) = 1$$

where $\{x_1, \ldots, x_n\}$ ranges over all finite subsets of X.

Proof. It is easily seen that if $z \in J(R)$, $\lim z^n = 0$. This implies that any power series in z with coefficients in F converges to an element of R. Therefore, the proof of (**3**, Proposition III.8.3) shows that there exist $f_x \in R$ such that $p_R(f_x) = \phi(e_x)$ and $f_x^2 = f_x$. Since $f_x f_y$ is an idempotent in $J(R)$, $f_x f_y = 0$. To show that

$$\lim (f_{x1} + \ldots + f_{xn}) = 1,$$

it is enough to show that for every closed ideal U of finite codimension which is a neighbourhood of 0, there exists a finite set $Y = \{x_1, \ldots, x_n\} \subseteq X$ such that

$$1 - (f_{x1} + \ldots + f_{xn}) \in U$$

and $f_z \in U$ for $z \notin Y$. We claim that

$$Y = \{x \in X | e_x \notin \phi^{-1}(p_R(U))\}$$

is the desired set. It is finite because it is linearly independent modulo $\phi^{-1}(p_R(U))$. It is easily verified that

$$\phi^{-1}(p_R(U)) = \{f \in F^X | f(Y) = 0\}.$$

But this implies that

$$1 - (f_{x1} + \ldots + f_{xn})$$

and f_z, $z \notin Y$, are idempotents in $U + J(R)$. It follows that they are in U. This completes the proof of the proposition.

3. The radical of the dual algebra.

Given a cocommutative Hopf algebra A, the radical of the dual algebra $J(A^*)$ plays a central role in determining the structure of A. If F is algebraically closed, $J(A^*) = 0$ if and only if A is the group algebra of a semigroup. If F has characteristic 0, $J(A^*)$ is a maximal ideal in A^* if and only if A is the universal enveloping algebra of a Lie algebra.

LEMMA 3.1. *Let A be a Hopf algebra over the field F. Then*

$$G(A) = \{a \in A | a \neq 0 \text{ and } \delta(a) = a \otimes a\}$$

is linearly independent, is closed under multiplication, and contains 1.

We may characterize $G(A)$ as follows: if A is a Hopf algebra, $\Gamma(G(A), F)$ is the maximal sub-Hopf algebra of A which is the group algebra of some semigroup.

The following theorem was first proved (unpublished) by D. K. Harrison for finite-dimensional Hopf algebras.

THEOREM 3.2. *Let A be a cocommutative Hopf algebra over the algebraically closed field F. Then A^* is a semisimple algebra if and only if A is the group algebra of a semigroup with identity.*

Proof. The following lemma implies that if A^* is semisimple, $A = \Gamma(G(A), F)$. The converse of the theorem is trivial.

LEMMA 3.3. *$J(A^*)$ is closed in the finite topology and*

$$\Gamma(G(A), F) = J(A^*)^{\text{perp}} \cap A.$$

Proof. $J(A^*)$ is closed since it is the intersection of closed ideals by Propositions 2.6 and 2.7. Each $g \in G(A)$ induces a homomorphism of A^* onto F with a closed kernel. Since $J(A^*)$ is contained in the kernel of this homomorphism, $g \in J(A^*)^{\text{perp}}$. Therefore,

$$\Gamma(G(A), F) \subseteq J(A^*)^{\text{perp}} \cap A.$$

To show that

$$\Gamma(G(A), F) \supseteq J(A^*)^{\text{perp}} \cap A,$$

it is enough to show that $K^{\text{perp}} \cap A \subseteq \Gamma(G(A), F)$ for every closed maximal ideal K in A^*. Since K is closed and maximal, $K^{\text{perp}} \cap A$ is a minimal subcoalgebra of A. Therefore $K^{\text{perp}} \cap A$ is finite dimensional by Proposition 2.5. Since $\dim A^*/K = \dim K^{\text{perp}} \cap A$ is finite and F is algebraically closed, $\dim A^*/K = 1$. Let $g \in K^{\text{perp}} \cap A$ be such that $\epsilon(g) = 1$. It is easily checked that $\delta(g) = g \otimes g$, so

$$K^{\text{perp}} \cap A = Fg \subseteq \Gamma(G(A), F).$$

This completes the proofs of the lemma and the theorem.

DEFINITION 3.4. *The Hopf algebra over the field F is colocal if $J(A^*)$ is a maximal ideal in A^*.*

THEOREM 3.5. *Let A be a cocommutative Hopf algebra over the field F of characteristic* 0. *Then A is colocal if and only if A is the universal enveloping algebra of a Lie algebra.*

Proof. It is trivial to show that if A is the universal enveloping algebra of a Lie algebra, A is colocal. To show the converse, by (**5**, 5.18) it is enough to show that if A is colocal, A is generated as an algebra by

$$P(A) = \{a \in A \mid \delta(a) = 1 \otimes a + a \otimes 1\}.$$

Define the length of $a \in A$ as follows:

$$l(a) = \min\{n > 0 \mid (a, J(A^*)^{n+1}) = 0\}.$$

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Note that $l(a) = 1$ if and only if $a \in P(A)$. We show by induction on length that every element of A can be written as a linear combination of products of elements of $P(A)$. Assume that every element whose length is less than n can be written as such a linear combination, and that $l(a) = n$. Let B be the minimal sub-coalgebra of A containing a. Let $\{a_1, \ldots, a_k\}$ be a basis of $P(B)$, and $\{x_1, \ldots, x_k\}$ be elements of B^* such that $(a_i, x_j) = \delta_{ij}$. Let

$$b = a - \sum (e_1! \ldots e_k!)^{-1}(a, x_1^{e_1} \ldots x_k^{e_k})a_1^{e_1} \ldots a_k^{e_k}$$

where the sum is taken over all k-tuples of non-negative integers $(e_1, \ldots, e_k)$ such that $e_1 + \ldots + e_k = n$. We claim $l(b) < n$. Since $l(b) \leqslant n$, it is enough to show that $(b, z_1 \ldots z_n) = 0$ for $z_i \in J(A^*)$. Writing $\beta_{ij} = (a_j, z_i)$, we have

$$(3) \qquad (a, z_1 \ldots z_n) = \sum_{j_i=1}^{k} \beta_{1j_1} \ldots \beta_{nj_n}(a, x_{j_1} \ldots x_{j_n}),$$

and

$$(4) \qquad (\sum (e_1! \ldots e_k!)^{-1}(a, x_1^{e_1} \ldots x_k^{e_k})a_1^{e_1} \ldots a_k^{e_k}, z_1 \ldots z_n) = \sum (a, x_1^{e_1} \ldots x_k^{e_k}) \sum \beta_{1j_1} \ldots \beta_{nj_n}$$

where the sum on the left-hand side and the first sum on the right-hand side are taken over all k-tuples $(e_1, \ldots, e_k)$ with $e_1 + \ldots + e_k = n$, and for each k-tuple $(e_1, \ldots, e_k)$, the second sum on the right-hand side is taken over all *distinct* orderings $(j_1, \ldots, j_n)$ of 1 taken e_1 times, ..., k taken e_k times. Since the right-hand sides of equations (3) and (4) are equal, $(b, z_1 \ldots z_n) = 0$. By induction b is a linear combination of products of elements of $P(A)$. But a is the sum of b and a linear combination of products of elements of $P(B)$. This completes the proof of the theorem.

Remarks. If the characteristic of F is $p \neq 0$, a slight modification of this proof shows that A is the u-algebra of a restricted Lie algebra (**2**, §V.7) if and only if $(1, x) = 0$ implies $x^p = 0$ for every $x \in A^*$.

The Referee has pointed out that the functor G defined in Lemma 3.1 is coadjoint (**4**, §8) to the functor Γ from the category of semigroups with identity to the category of cocommutative Hopf algebras over F, and similarly (if the characteristic of F is 0) the functor P defined in the proof of Theorem 3.5 is coadjoint to the functor U.

4. Cocommutative Hopf algebras. We now prove a theorem which is the basis for most of the structure theory developed in later sections of this paper.

THEOREM 4.1. *Let A be a cocommutative Hopf algebra over the algebraically closed field F. Then there exists a set of colocal sub-coalgebras $\{A_g | g \in G(A)\}$ such that $g \in A_g$, $A = \Sigma \oplus A_g$, and $A_g A_h \subseteq A_{gh}$.*

Proof. Since

$$J(A^*) = \mathrm{Cl}(J(A^*)) = \Gamma(G(A), F)^{\mathrm{perp}}$$

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by Lemma 3.3, we have

$$A^*/J(A^*) = \Gamma(G(A), F)^* = F^{G(A)}.$$

It is easily checked that the isomorphism $F^{G(A)} \to A^*/J(A^*)$ is continuous. Therefore by Proposition 2.8 there exists a family of orthogonal idempotents $\{f_g \in A^* | g \in G(A)\}$ such that $f_g + J(A^*) = e_g$ and

$$\lim (f_{g_1} + \ldots + f_{g_n}) = 1.$$

Define

$$A_g = \mathrm{Cl}(\textstyle\sum_{h \neq g} f_h A^*)^{\mathrm{perp}} \cap A.$$

It is not very hard to see that $g \in A_g$. A_g is a colocal coalgebra because $A_g^* = f_g A^*$.

We wish to show that $\sum A_g = A$. If not, there exists $x \neq 0$ such that $(\sum A_g, x) = 0$. But then

$$x \in A_g^{\mathrm{perp}} = \textstyle\sum_{h \neq g} f_h A^*$$

for every g, so $f_g x = 0$ for every g. This implies that $x = 0$, which is a contradiction.

We now show that the sum $\sum A_g$ is direct. Suppose $\sum a_g = 0$, with $a_g \in A_g$. If $a_h \neq 0$, there exists $x_h = f_h x_h$ such that $(a_h, x_h) = 1$. But

$$0 = (\textstyle\sum a_g, x_h) = (a_h, x_h).$$

We have proved that $A = \Sigma \oplus A_g$.

If $a \in A_g$, define

$$l_g(a) = \min\{n \geqslant 0 | (a, J(A_g^*)^{n+1}) = 0\}.$$

Note that $l_g(a) = 0$ if and only if $a = \alpha g$ for some $\alpha \in F$. Suppose $a \in A_g$ and $l_g(a) = n$. Then it is easily shown that we can write $\delta(a) = \sum a_i \otimes a'_i$, where $l_g(a_i) < n$ or $l_g(a'_i) < n$ for each i.

We now show that if $a \in A_g$ and $b \in A_h$, then $ab \in A_{gh}$. The proof is by induction on $l_g(a) + l_h(b)$. If $l_g(a) + l_h(b) = 0$, then $a = \alpha g$ and $b = \beta_h$ where $\alpha, \beta \in F$. Therefore $ab = \alpha\beta gh \in A_{gh}$. Let $n > 0$, and assume that $c \in A_g$, $d \in A_h$, $l_g(c) + l_h(d) < n$ implies $cd \in A_{gh}$. Suppose $l_g(a) + l_h(b) = n$. If $ab \notin A_{gh}$, then $\delta(ab) \notin A \otimes A_{gh} + A_{gh} \otimes A$. But by the above discussion, we can write $\delta(a) = \sum a_i \otimes a'_i$ and $\delta(b) = \sum b_j \otimes b'_j$ with $l_g(a_i) + l_h(b_j) < n$ or $l_g(a'_i) + l_h(b'_j) < n$ for each pair (i, j). By induction

$$\delta(ab) = \textstyle\sum a_i b_j \otimes a'_i b'_j \in A \otimes A_{gh} + A_{gh} \otimes A.$$

Therefore, $ab \in A_{gh}$. This completes the proof of the theorem.

Remark. It can be shown that the coalgebras A_g are filtered. That is, there exists a filtration

$$Fg = F^0 A_g \subseteq F^1 A_g \subseteq \ldots$$

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in A_g, with

$$\bigcup F^n A_g = A_g, \qquad \delta(F^n A_g) \subseteq \sum_{k+l=n} F^k A_g \otimes F^l A_g,$$

and

$$(F^k A_g)(F^l A_h) \subseteq F^{k+l} A_{gh}.$$

5. Invertible Hopf algebras. If $G(A)$ is a group, we can reduce the problem of finding the structure of A to finding the structure of $G(A)$, finding the structure of the colocal Hopf algebra A_1, and finding the way the inner automorphisms induced by elements of $G(A)$ act on A_1.

DEFINITION 5.1. *Let G be a semigroup with identity, B a Hopf algebra, and $\phi: G \to \mathrm{Aut}(B)$ a homomorphism of semigroups with identity. Then*

$$\Gamma(G, F)_\phi \otimes B$$

is the Hopf algebra with underlying vector space $\Gamma(G, F) \otimes B$ and maps

$$\mu(g_1 \otimes b_1 \otimes g_2 \otimes b_2) = g_1 g_2 \otimes b_1{}^{\phi(g_2)} b_2,$$

$$\delta(g \otimes b) = (1 \otimes T \otimes 1)(g \otimes g \otimes \delta(b)),$$

$$\eta(\alpha) = \alpha 1 \otimes 1, \qquad \epsilon(g \otimes b) = \epsilon(b).$$

Note that the inner automorphism of $\Gamma(G, F)_\phi \otimes B$ induced by $g \otimes 1$ restricted to $1 \otimes B$ is given by $1 \otimes \phi(g)$.

DEFINITION 5.2. *Let A be a Hopf algebra over the field F. A is invertible if for every $g \in G(A)$ there exists $a \in A$ such that $ga = ag = 1$.*

THEOREM 5.3. *Let A be a cocommutative Hopf algebra over the algebraically closed field F. If A is invertible, then $A = \Gamma(G(A), F)_\phi \otimes A_1$, where $\phi(g)$ is the inner automorphism of A induced by g restricted to the sub-Hopf algebra A_1. Conversely, if $A = \Gamma(G, F)_\phi \otimes R$ for a group G, a cocommutative colocal Hopf algebra R, and a group homomorphism $\phi: G \to \mathrm{Aut}(R)$, then A is invertible.*

Proof. Let $g \in G(A)$, and suppose $ag = ga = 1$. It is easily checked that $a \in G(A)$. Therefore, $G(A)$ is a group. Map $\Gamma(G(A), F)_\phi \otimes A_1 \to A$ by $g \otimes b \to gb$, where $g \in G(A)$, $b \in A_1$. It is immediate from Definition 5.1 and the fact that $G(A)$ is a group that this map is a Hopf algebra homomorphism. Since

$$gA_1 \subseteq A_g = g(g^{-1}A_g) \subseteq gA_1,$$

we have $A = \Sigma \oplus A_g = \Sigma \oplus gA_1$. Therefore, the map is an isomorphism. The converse is trivial.

It is possible to define a semigroup with identity (or a group) in terms of maps in **Ens** (the category of sets). The maps in the definition are the product $m: S \times S \to S$ and the identity $h: \{\emptyset\} \to S$ (and the inverse $c: S \to S$ for the definition of a group). The only constructions needed in the definition are the product and $\{\emptyset\}$ (a terminal object in **Ens**). Therefore, we could define a semigroup with identity (or a group) in any category with products and a terminal object. The category **C** of cocommutative coalgebras over the field F is such a category, with $\otimes$ as product and F as a terminal object.

A semigroup with identity in **C** is just a cocommutative Hopf algebra. A group in **C** is a cocommutative Hopf algebra A with a map $\gamma: A \rightarrow A$ which is a coalgebra homomorphism satisfying

$$\mu(1 \otimes \gamma)\delta = \mu(\gamma \otimes 1)\delta = \eta\epsilon.$$

It can be shown that this implies that γ is an algebra anti-automorphism of period 2.

DEFINITION 5.4. *Let A be a cocommutative Hopf algebra over the field F. A conjugation in A is a map $\gamma: A \rightarrow A$ which is a coalgebra automorphism and an algebra anti-automorphism of period 2 satisfying*

$$\mu(1 \otimes \gamma)\delta = \mu(\gamma \otimes 1)\delta = \eta\epsilon.$$

The preceding discussion implies that the Hopf algebra A is a group in **C** if and only if it has a conjugation. The following theorem says that A has a conjugation if and only if the sub-Hopf algebra $\Gamma(G(A), F)$ has one.

THEOREM 5.5. *Let A be a cocommutative Hopf algebra over the algebraically closed field F. Then A is invertible if and only if A has a conjugation.*

Proof. Assume A is invertible. By Theorem 5.3,

$$A = \Gamma(G(A), F)_\phi \otimes A_1.$$

Define

$$F^n A_1 = (I(A_1^*)^{n+1})^{\text{perp}} \cap A_1.$$

It is easily checked that $F^n A_1$ is a coalgebra filtration. The argument in **(5**, §8**)** with grading replaced by filtration shows that we can define a conjugation γ_1 in A_1. It is easily verified (using the fact that γ_1 commutes with all automorphisms of A_1) that the map $\gamma: A \rightarrow A$ defined by

$$\gamma(g \otimes r) = g^{-1} \otimes \gamma_1(r^{\phi(g^{-1})})$$

for $g \in G(A)$ and $r \in A_1$ is a conjugation in A.

The converse is trivial. This completes the proof of the theorem.

Remark. Theorem 5.3 was first proved (unpublished) by B. Kostant, who showed that a cocommutative Hopf algebra A over the algebraically closed field F has a conjugation if and only if it is the product of a cocommutative filtered Hopf algebra by a group algebra.

6. An embedding theorem. In the last section a cocommutative invertible Hopf algebra (or, equivalently, a cocommutative Hopf algebra with a conjugation) over an algebraically closed field was characterized as the product of a colocal cocommutative Hopf algebra by a group algebra. In this section we prove that a cocommutative Hopf algebra satisfying conditions less restrictive than invertibility can be embedded in the product of a colocal cocommutative Hopf algebra by the group algebra of a semi-group. A generalization of Ore's theorem to Hopf algebras follows from this.

DEFINITION 6.1. *Let A be a Hopf algebra. A is G-cancellative if for every $a \in A$ and every $g \in G(A)$, $ga = 0$ or $ag = 0$ implies $a = 0$. A is G-right reversible if for every $a \in A$ and every $g \in G(A)$, there exist $b \in A$ and $h \in G(A)$ such that $ha = bg$.*

THEOREM 6.2. *Let A be a cocommutative Hopf algebra over the algebraically closed field F. A is G-cancellative and G-right reversible if and only if $G(A)$ is a cancellative right reversible semigroup and there exist a colocal cocommutative Hopf algebra R and a semigroup homomorphism $\phi: G(A) \to \mathrm{Aut}\,(R)$ such that*

$$A \subseteq \Gamma(G(A), F)_\phi \otimes R$$

and the following conditions are satisfied:

(a) *if $\sum g \otimes r_g \in A$ with $g \in G(A)$ and $r_g \in R$, then $g \otimes r_g \in A$;*

(b) *if $r \in R$ there exists $g \in G(A)$ such that $g \otimes r \in A$.*

Proof. It is immediate from the fact that A is G-cancellative that $G(A)$ is cancellative. Given $g, h \in G(A)$, there exist $a \in A$ and $k \in G(A)$ such that $ag = kh$. Since A is G-cancellative, $a \in G(A)$. This proves that $G(A)$ is right reversible.

Let L be the subspace of A spanned by all elements of the form $a - ga$, where $a \in A$ and $g \in G(A)$. Define $R = A/L$. A simple computation shows that R is a quotient coalgebra of A. Denote by p_g the restriction of the projection $A \to R$ to the sub-coalgebra A_g. The maps p_g are homomorphisms of augmented coalgebras.

To derive some properties of the maps p_g, we now make $G(A)$ into a category. Let $\hom(g, h)$ have exactly one element if there exists (a necessarily unique) $l \in G(A)$ with $lg = h$, and be empty otherwise. We have a functor from $G(A)$ to the category of vector spaces taking the object g into A_g, and taking the map in $\hom(g, h)$ into the map sending $a \in A_g$ into $la \in A_h$, where $h = lg$. R is the direct limit of this functor in the sense of (**4**, §8). Since the category $G(A)$ has the property that for any g, h objects in $G(A)$ there exists an object k such that $\hom(g, k)$ and $\hom(h, k)$ are non-empty (the semigroup $G(A)$ is right reversible) and the functor carries maps in $G(A)$ into monomorphisms (A is G-cancellative), the classical argument for direct limit on a directed set shows that the maps $p_g: A_g \to R$ are injections and that $R = \bigcup \mathrm{Im}\, p_g$. This last fact implies that R is a colocal coalgebra.

Now we define a multiplication on the colocal cocommutative coalgebra R to make it a Hopf algebra. Let $r, s \in R$, and let $a \in A_g$, $b \in A_h$ be such that $r = p_g(a)$ and $s = p_h(b)$. There exist $c \in A_k$ and $l \in G(A)$ such that $ch = la$.

Define $rs = p_{kh}(cb)$. A series of straightforward arguments shows that this product is well defined, associative, has an identity, and that $\delta: R \to R \otimes R$ is an algebra homomorphism.

If $h \in G(A)$ and $p_g(a) = r \in R$, let $r^{\phi(h)} = p_{gh}(ah)$. Another series of straightforward arguments shows that $\phi(h)$ is a well-defined endomorphism of the Hopf algebra R. It is surjective because A is G-right reversible, and injective because the maps $p_g: A_g \to R$ are injective and A is G-cancellative. We thus have a semigroup homomorphism $\phi: G \to \mathrm{Aut}(R)$.

Define a map $i: A \to \Gamma(G(A), F)_\phi \otimes R$ as follows: if $a \in A_g$, let $i(a) = g \otimes p_g(a)$, and extend i to all of A by linearity. It is immediate that i is an isomorphism of A into the Hopf algebra $\Gamma(G(A), F)_\phi \otimes R$. Condition (a) in the statement of the theorem is obvious, and condition (b) is satisfied because $R = \bigcup \mathrm{Im}\, p_g$.

The converse is trivial. This completes the proof of the theorem.

COROLLARY 6.3. *Let A be a cocommutative Hopf algebra over the algebraically closed field F. A is G-cancellative and G-right reversible if and only if there exists a cocommutative invertible Hopf algebra B containing A such that for every $b \in B$ there exist $a \in A$ and $g \in G(A)$ with $b = g^{-1}a$.*

Proof. By Ore's Theorem (**1**, Theorem 1.25) there is a group G containing $G(A)$ with $G = (G(A))^{-1}G(A)$. The homomorphism $\phi: G(A) \to \mathrm{Aut}(R)$ extends to a homomorphism $G \to \mathrm{Aut}(R)$ (which we also call ϕ). Let $B = \Gamma(G, F)_\phi \otimes R$. A simple calculation shows that every $b \in B$ is of the form $g^{-1}a$ for some $g \in G(A)$ and $a \in A$. The converse is immediate. This completes the proof of the corollary.

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