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AN EXTENSION OF KESTEN'S GENERALISED LAW OF THE ITERATED LOGARITHM

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Let X_i be independent and identically distributed random variables with $S_n = X_1 + X_2 + \ldots + X_n$. We extend a classic result of Kesten, by showing that if X_i are in the domain of partial attraction of the normal distribution, there are sequences α_n and $B(n)$ for which

$$-1 = \liminf_{n\to+\infty} \left(S_n-\alpha_n\right)/B(n) < \limsup_{n\to+\infty} \left(S_n-\alpha_n\right)/B(n) = 1$$

almost surely, and the almost sure limit points of $\left(S_n-\alpha_n\right)/B(n)$ coincide with the interval $[-1, 1]$. The norming sequence $B(n)$ is slightly different to that used by Kesten, and has properties that are less desirable. The converse to the above result is known to be true by results of Heyde and Rogozin.

Let X_i be independent and identically distributed random variables with distribution F, and let $S_n = X_1 + X_2 + \ldots + X_n$. In 1968 Heyde [2] and Rogozin [8] showed that if F is not in the domain of partial attraction of the normal distribution (*cf*. Lévy [5], p. 113) then necessarily $\limsup \left|S_n-\delta_n\right|/\gamma(n) = +\infty$ almost surely, or $\left(S_n-\delta_n\right)/\gamma(n) \to 0$ almost surely, where δ_n and $\gamma(n)$ are constants. Kesten [4] in 1972 proved the converse to this, thus giving the following elegant

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generalisation of the classical law of the iterated logarithm: F is in the domain of partial attraction of the normal distribution if and only if there is a positive sequence $\gamma(n)$ and constants δ_n satisfying $P(S_n \geq \delta_n) \geq \pi$, $P(S_n \leq \delta_n) \leq \pi$ for some $\pi \in (0, 1)$ such that

$$-\infty < \liminf_{n\to+\infty} (S_n-\delta_n)/\gamma(n) < \limsup_{n\to+\infty} (S_n-\delta_n)/\gamma(n) < +\infty$$

almost surely.

The purpose of the present paper is to give the following extended version of the above result, which partially answers the problem on page 717 of [4].

THEOREM 1. *F is in the domain of partial attraction of the normal distribution if and only if there is a positive sequence $B(n) \uparrow +\infty$ such that*

$$-1 = \liminf_{n\to\infty} (S_n-\alpha_n)/B(n) < \limsup_{n\to\infty} (S_n-\alpha_n)/B(n) = 1$$

almost surely, where $\alpha_n = n \int_{|u|\leq B(n)} u\,dF(u)$. *Furthermore the almost sure limit points of* $(S_n-\alpha_n)/B(n)$ *are precisely the interval* $[-1, 1]$.

Our proof of Theorem 1 is based on the methods of Kesten but differs in detail. The norming sequence $B(n)$ we use, though derived from Kesten's $\gamma(n)$, is defined differently and, unlike $\gamma(n)$, fails to have the property that $n^{-\frac{1}{2}+\varepsilon}B(n)$ is nondecreasing for $0 < \varepsilon < \frac{1}{2}$. We discuss this point further following the proof of Theorem 1.

Our method does not depend on symmetrisation of the random variables in an essential way. It can be shown as in [4] that the centering sequence α_n may be replaced by any sequence δ_n for which $P(S_n \geq \delta_n) \geq \pi$ and $P(S_n \leq \delta_n) \leq \pi$ for some $\pi \in (0, 1)$. Thus α_n may be replaced by the median of S_n.

We derive the recurrence part of Theorem 1 from Lemma 1 below, which is a modified version of the criterion of Binmore and Katz (Theorem 2 of [3]) for the recurrence of S_n. Our result is given in a slightly more general form than is required for the proof of Theorem 1, since it is hoped that it may have other applications.

We now prove Theorem 1. The statements and proofs of the lemma just mentioned, and of a second lemma we use (which is a version of Lévy's inequality [6, p. 259]), follow this proof.

Proof of Theorem 1. We need only give the sufficiency part of the proof, so suppose F is in the domain of partial attraction of the normal distribution. Thus there is a sequence $x_k \uparrow +\infty$ for which

$$\zeta_k = x_k^2 P(|X| > x_k)/V(x_k) \to 0 ,$$

where

$$V(x) = \int_{-x}^{x} u^2 dF(u) - \left[\int_{-x}^{x} udF(u)\right]^2 .$$

Fixing ε with $0 < \varepsilon < \frac{1}{4}$ we can assume $\zeta_k \le k^{-2/\varepsilon}$.

Define a sequence $r_k \uparrow +\infty$ by $r_k = \left[\log_2\left\{\zeta_k^{-3/4} x_k^2/V(x_k)\right\}\right]$ ($[x]$ denotes the integer part of x and $\log_2$ the logarithm to base 2), so that

$$2^{r_k} \le \zeta_k^{-3/4}\left\{x_k^2/V(x_k)\right\} = \zeta_k^{\frac{1}{4}}/P(|X| > x_k) < 2.2^{r_k} ,$$

and define $B(n)$ by

$$B(n) = 2^{r_k/2}\sqrt{2 \log kV(x_i)} \quad \text{when} \quad 2^{r_{k-1}} < n \le 2^{r_k} .$$

Since V is ultimately nondecreasing, it is no restriction to assume that $B(n)$ is nondecreasing for $n \ge 1$. Now we truncate at x_k : let $X_i^k = X_i$ if $|X_i| \le x_k$, 0 otherwise, let

$$\beta_k = EX_i^k = \int_{-x_k}^{x_k} udF(u) ,$$

and note that X_i^k has variance

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$$E\left(X_i^k\right)^2 - \beta_k^2 = \int_{-x_k}^{x_k} u^2 dF(u) - \beta_k^2 = V\left(x_k\right) .$$

Hence if $S_n^k = X_1^k + X_2^k + \ldots + X_n^k$, we have by the Berry-Esseen theorem (Feller [1, p. 542]), if $r \geq 1$, $k \geq 1$, and $\Phi(x) = (2\pi)^{-\frac{1}{2}} \int_{-\infty}^{x} e^{-\frac{1}{2}u^2} du$, that

$$(1) \quad \sup_{-\infty<x<+\infty} \left|P\left\{S_{2^r}^k - 2^r \beta_k < x\sqrt{2^r V\left(x_k\right)}\right\} - \Phi(x)\right|$$
$$\leq 3E\left|X_i^k - \beta_k\right|^3 / 2^{r/2} V^{3/2}\left(x_k\right) = L_r^k ,$$

say.

Suppose $r_k - \varepsilon^{-1}\log_2 k < r \leq r_k$. Then since for some constants c_0, $c > 0$,

$$E\left|X_i^k - \beta_i\right|^3 \leq c_0 E\left|X_i^k\right|^3 \leq c_0 x_k \int_{-x_k}^{x_k} u^2 dF(u) \leq c^2 x_k V\left(x_k\right) ,$$

we have

$$L_r^k \leq c\left\{x_k^2 / 2^r V\left(x_k\right)\right\}^{\frac{1}{2}} = c2^{\left(r_k - r\right)/2}\left\{x_k^2 / 2^{r_k} V\left(x_k\right)\right\}^{\frac{1}{2}} \leq c\left\{2^{r_k - r + 1} \zeta_k^{3/4}\right\}^{\frac{1}{2}}$$
$$\leq c\left\{2k^{1/\varepsilon} \zeta_k^{3/4}\right\}^{\frac{1}{2}} \leq c2^{\frac{1}{2}} k^{-1/4\varepsilon} ,$$

so that, because $\varepsilon < \frac{1}{4}$,

$$\sum_{k \geq 1} \sum_{r_k - \varepsilon^{-1}\log_2 k < r \leq r_k} L_r^k \leq c2^{\frac{1}{2}}\varepsilon^{-1} \sum_{k \geq 1} k^{-1/4\varepsilon} \log_2 k < +\infty .$$

Also if $r_k - \varepsilon^{-1}\log_2 k < r \leq r_k$,

$$B\left(2^r\right) = 2^{r_k/2}\sqrt{2 \log k V\left(x_k\right)} \geq \sqrt{2^r 2 \log k V\left(x_k\right)} ,$$

so it follows from (1) with $x = a\sqrt{2 \log k}$, $a > 1$, that

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$$\sum_{k\geq 1} \sum_{r_k-\varepsilon^{-1}\log_2 k<r\leq r_k} P\left\{S^k_{2^r}-2^r\beta_k > aB\left(2^r\right)\right\}$$

$$\leq \sum_{k\geq 1} \sum_{r_k-\varepsilon^{-1}\log_2 k<r\leq r_k} P\left\{S^k_{2^r}-2^r\beta_k > a\sqrt{2^r 2 \log kV\left(x_k\right)}\right\}$$

$$\leq \sum_{k\geq 1} \sum_{r_k-\varepsilon^{-1}\log_2 k<r\leq r_k} \left\{L^k_r+1-\Phi(a\sqrt{2\log k})\right\}$$

$$< +\infty + 2\varepsilon^{-1} \sum_{k\geq 1} k^{-a^2}\log_2 k < +\infty ,$$

using the approximation $1-\Phi(x) \leq 2e^{-\frac{1}{2}x^2}$, valid when $x \geq 2$. It is easy to check that

$$P\left\{S_{2^r}-2^r\beta_k > aB\left(2^r\right)\right\} \leq P\left\{S^k_{2^r}-2^r\beta_k > aB\left(2^r\right)\right\} + 2^r P\left(|X| > x_k\right) ,$$

while since

$$\sum_{k\geq 1} \sum_{r_{k-1}<r\leq r_k} 2^r P\left(|X| > x_k\right) \leq 2 \sum_{k\geq 1} 2^{r_k} P\left(|X| > x_k\right)$$

$$\leq \sum_{k\geq 1} \zeta_k^{\frac{1}{4}} \leq \sum_{k\geq 1} k^{-1/2\varepsilon} < +\infty$$

we have

(2) $$\sum_{k\geq 1} \sum_{r_k-\varepsilon^{-1}\log_2 k<r\leq r_k} P\left\{S_{2^r}-2^r\beta_k > aB\left(2^r\right)\right\} < +\infty .$$

Suppose $r_{k-1} < r \leq r_k - \varepsilon^{-1}\log_2 k$. By Chebychev's inequality, if $a > 0$,

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$$\sum_{k\geq 1} \sum_{r_{k-1}<r\leq r_k-\varepsilon^{-1}\log_2 k} P\left\{S^k_{2^r}-2^r\beta_k > aB(2^r)\right\}$$

$$\leq a^{-2} \sum_{k\geq 1} \sum_{r_{k-1}<r\leq r_k-\varepsilon^{-1}\log_2 k} B^{-2}(2^r)2^r V(x_k)$$

$$= a^{-2} \sum_{k\geq 1} \sum_{r_{k-1}<r\leq r_k-\varepsilon^{-1}\log_2 k} 2^{r-r_k}(\log k)^{-1}$$

$$\leq 2a^{-2} \sum_{k\geq 1} (\log k)^{-1}k^{1/\varepsilon} < +\infty$$

because $B^2(2^r) = 2^{r_k}(2\log kV(x_k))$ when $r_{k-1} < r \leq r_k$. Again since $\sum_{k\geq 1} \sum_{r_{k-1}<r\leq r_k} 2^r P(|X| > x_k) < +\infty$, we can ignore the truncation, and together with (2) the inequality just derived gives

(3) $$\sum_{k\geq 1} \sum_{r_{k-1}<r\leq r_k} P\left\{S_{2^r}-2^r\beta_k > aB(2^r)\right\} < +\infty \quad \text{when} \quad a > 1 .$$

Now we need the following argument: replacing X_i by $-X_i$ we see from (3) that

$$\sum_{k\geq 1} \sum_{r_{k-1}<r\leq r_k} P\left\{S_{2^r}-2^r\beta_k < -aB(2^r)\right\} < +\infty \quad \text{when} \quad a > 1 ,$$

so

$$\sum_{k\geq 1} \sum_{r_{k-1}<r\leq r_k} P\left\{\left|S_{2^r}-2^r\beta_k\right| > aB(2^r)\right\} < +\infty \quad \text{when} \quad a > 1 .$$

Letting S^s_n be a symmetrisation of S_n in the usual way, this means $\sum_{r\geq 1} P\left\{\left|S^s_{2^r}\right| > 2aB(2^r)\right\} < +\infty$, and hence by Lemma 2 of Kesten [4], $\limsup_{n\to+\infty} \left|S^s_n\right|/B(n) \leq 4$ almost surely. Applying Lemma 4 of [4] now gives $S^s_n/B(n) \xrightarrow{P} 0$, so $(S_n-\alpha_n)/B(n) \xrightarrow{P} 0$ where $\alpha_n = n\int_{|u|\leq B(n)} u\,dF(u)$ by

Loève [6, p. 290]. Noting that $2^{r_k} V(x_k)/x_k^2 > \frac{1}{2}\zeta_k^{-3/4} \to +\infty$ shows that $B(2^{r_k})/x_k \to +\infty$, and so for k large enough,

$$\left|2^{r_k}\beta_k - \alpha_{2^{r_k}}\right| / B(2^{r_k}) = 2^{r_k}\left|\int_{x_k \le |u| \le B(2^{r_k})} u dF(u)\right| / B(2^{r_k})$$

$$\le 2^{r_k} P(|X| > x_k) \le \zeta_k^{\frac{1}{4}} \to 0 ,$$

which means we can replace $2^{r_k}\beta_k$ by $\alpha_{2^{r_k}}$ in (2) and deduce that

$\sum_{k \ge 1} P\left\{S_{2^{r_k}} - \alpha_{2^{r_k}} > aB(2^{r_k})\right\} < +\infty$ for $a > 1$. Since $(S_n - \alpha_n)/B(n) \xrightarrow{p} 0$, we can apply a version of Lévy's inequality (Lemma 2 below) to obtain from this that for some $k_0 \ge 1$,

$$\sum_{k \ge 1} P\left\{\max_{k_0 \le j \le 2^{r_k}} (S_j - \alpha_j) > aB(2^{r_k})\right\} < +\infty$$

for $a > 1$. By the Borel-Cantelli lemma, then,

$$\limsup_{k \to +\infty} \max_{k_0 \le j \le 2^{r_k}} (S_j - \alpha_j)/B(2^{r_k}) \le 1$$

almost surely. Now, given any $n \ge 1$, choose $k = k(n)$ so that $2^{r_{k-1}} < n \le 2^{r_k}$; then $B(n) = B(2^{r_k})$, and so we obtain half of what we want:

$$\limsup_{n \to +\infty} (S_n - \alpha_n)/B(n) \le \limsup_{n \to +\infty} \max_{k_0 \le j \le 2^{r_k}} (S_j - \alpha_j)/B(2^{r_k}) \le 1$$

almost surely.

Now let $a < 1$, $\varepsilon > 0$, $a + \varepsilon < 1$, $a - \varepsilon > 0$. Applying (1) for

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$r = r_k$ gives

$$P\left\{S^k_{2^{r_k}} - \alpha_{2^{r_k}} \in (a-\varepsilon,\ a+\varepsilon)\sqrt{2^{r_k} 2 \log kV(x_k)}\right\} \ge (2\pi)^{-\frac{1}{2}} \int_I e^{-\frac{1}{2}u^2} du - 2L^k_{r_k}$$

where $I = (a-\varepsilon,\ a+\varepsilon)\sqrt{2 \log k}$. Since $\sum_{k\ge 1} \int_I e^{-\frac{1}{2}u^2} du$ diverges for a and ε as defined, while $\sum_{k\ge 1} L^k_{r_k} < +\infty$, as shown earlier, we obtain (once again ignoring the truncation)

$$\sum_{k\ge 1} P\left\{S_{2^{r_k}} - \alpha_{2^{r_k}} \in (a-\varepsilon,\ a+\varepsilon)B\left(2^{r_k}\right)\right\} = +\infty .$$

An application of Lemma 1 now will give that $\left(S_n - \alpha_n\right)/B(n)$ is recurrent at a if $0 < a < 1$, providing we verify the conditions of the lemma. If $\mu_1 > 1$ and k_0 is large enough,

$$B\left[2^{r_k}\mu_1\right]/B\left(2^{r_k}\right) = B\left(2^{r_{k+1}}\right)/B\left(2^{r_k}\right) \ge \varepsilon^{-2}$$

for $k \ge k_0$, since it is clearly no loss of generality to assume $2^{r_{k+1}} - 2^{r_k} \to +\infty$. Thus $B\left[2^{r_k}\mu\right]/B\left(2^{r_k}\right) \ge \varepsilon^{-2}$ for $\mu \ge \mu_1$, $k \ge k_0$. If $1/\mu_1 = \mu_0 \le 1$ and k_0 is so large that $2^{r_{k-1}} < \left[2^{r_k}\mu_0\right] \le 2^{r_k}$ for $k \ge k_0$, $B\left[2^{r_k}\mu_0\right]/B\left(2^{r_k}\right) = 1$; thus $B\left[2^{r_k}\mu\right]/B\left(2^{r_k}\right) = 1$ for $\mu_0 \le \mu \le 1$, $k \ge k_0$. Thus taking $b_+(\mu) = 1$, $b_-(\mu) \ge \varepsilon^{-2}$, $\lambda_k = 2^{r_k}$, $k \ge k_0$, we deduce from Lemma 1 that $\left(S_n - \alpha_n\right)/B(n) \in (a-\varepsilon,\ a+\varepsilon)$ infinitely often with probability 1 , so $\left(S_n - \alpha_n\right)/B(n)$ is recurrent at all points of $(0,\ 1)$ and hence $\limsup_{n\to+\infty} \left(S_n - \alpha_n\right)/B(n) \ge 1$ almost surely. We proved earlier that $\limsup_{n\to+\infty} \left(S_n - \alpha_n\right)/B(n) \le 1$ almost surely, so $\limsup_{n\to+\infty} \left(S_n - \alpha_n\right)/B(n) = 1$ almost surely. Replacing X_i by $-X_i$ shows that $(S_n-\alpha_n)/B(n)$ is recurrent at all points of $(-1, 0)$, and $\liminf_{n\to+\infty} (S_n-\alpha_n)/B(n) = -1$ almost surely. The recurrence at 0 follows since $(S_n-\alpha_n)/B(n) \xrightarrow{p} 0$. This completes the proof of Theorem 1.

REMARKS. (i) The sequence $B(n)$ we used in Theorem 1 is defined differently to Kesten's $\gamma(n)$, and it has properties which are less desirable. Since it is constant on large subintervals, $n^{-\frac{1}{2}+\varepsilon}B(n)$ is not nondecreasing for $0 < \varepsilon < \frac{1}{2}$, whereas $n^{-\frac{1}{2}+\varepsilon}\gamma(n)$ is nondecreasing for $0 < \varepsilon < \frac{1}{2}$. If we try to use $\gamma(n)$ in the proof of Theorem 1, we can only obtain a partial result. In fact, if x_k and r_k are the sequences defined in the proof, define $\gamma^*(n)$ by

$$\gamma^*(n) = n^{\frac{1}{2}-\varepsilon}2^{\varepsilon r_k}\sqrt{2 \log kV(x_k)}\ , \text{ when } 2^{r_{k-1}} < n \le 2^{r_k}\ ,$$

where $0 < \varepsilon < \frac{1}{2}$ ($\gamma(n)$ has $(3/2) \log k$, instead of $2 \log k$). Then by the method of Theorem 1 we can show, under the same assumptions, that $(S_n-\alpha_n)/\gamma^*(n)$ is recurrent at all points of $[-1, 1]$, and thus that $\limsup_{n\to+\infty} (S_n-\alpha_n)/\gamma^*(n) \ge 1$ almost surely; but we only obtain $\limsup_{n\to+\infty} (S_n-\alpha_n)/\gamma^*(n) \le 2$ almost surely. It seems a reasonable conjecture that the result of Theorem 1 actually holds when $B(n)$ is replaced by $\gamma^*(n)$.

(ii) If (λ_j) is a sequence of integers we introduce the notation $(\lambda_i,\ i \ge j)$ to mean the collection $\{\lambda_j,\ \lambda_{j+1},\ \ldots\}$, and we define

$$P\{S_{\lambda_j}-\alpha_{\lambda_j} \in I_{\lambda_j} \text{ infinitely often}\} = \lim_{j\to+\infty} P\{S_n-\alpha_n \in I_n \text{ for some } n \in (\lambda_i,\ i \ge j)\}$$

where I_n are any intervals.

Lemma 1 is a modification of Theorem 2 of [3] in which we allow for centering of S_n, a wide class of norming sequences, and the fact that recurrence can be deduced from behaviour on a subsequence. We only need to consider nondecreasing norming sequences.

LEMMA 1. *Suppose* α_n *and* $B(n)$ *are constants, with* $B(n) > 0$, $B(n) \uparrow +\infty$, *for which (i) of Lemma 2 holds. Let* (λ_j) *be a sequence of integers satisfying* $\lambda_j/\lambda_i \geq q_{j-i}$, *where* $q_j \to +\infty$ *as* $j \to +\infty$, *for which* $B[\mu\lambda_j]/B(\lambda_j) \geq 1/b_+(\mu)$ *for* $\mu \in (\mu_0, 1]$ *and* $B[\mu\lambda_j]/B(\lambda_j) \geq b_-(\mu)$ *for* $\mu \geq 1/\mu_0 > 1$, *for some* $\mu_0 \in (0, 1)$ *and some real valued functions* b_+ *and* b_- . *Then if* $b > a \geq 0$, *(i) implies (ii) and (ii) implies (iii):*

(i) $P\{S_{\lambda_j} - \alpha_{\lambda_j} \in (a, b)B(\lambda_j)$ *infinitely often*$\} = 1$;

(ii) $\sum_{j\geq 0} P\{S_{\lambda_j} - \alpha_{\lambda_j} \in (a, b)B(\lambda_j)\} = +\infty$;

(iii) $P\Big\{S_n - \alpha_n \in \big(a-\varepsilon-b/b_-(\varepsilon^{-1}), bb_+(1-\varepsilon)+\varepsilon\big)B(n)$ *infinitely often*$\Big\}$
$= 1$ *for every* $\varepsilon \in (0, 1-\mu_0)$.

Proof of Lemma 1. It is easy to see that *(i)* implies *(ii)*, so let *(ii)* hold. Fix $\varepsilon > 0$, $\varepsilon < 1 - \mu_0$, and let $I = (a, b)$, $I' = \big(a-\varepsilon-b/b_-(\varepsilon^{-1}), bb_+(1-\varepsilon)+\varepsilon\big)$. If $s \geq 1$ is an integer, *(ii)* implies

$$\infty = \sum_{j\geq s} P\{S_{\lambda_j} - \alpha_{\lambda_j} \in IB(\lambda_j)\}$$
$$= \sum_{t=0}^{s-1} \sum_{j\geq 1} P\{S_{\lambda_{js+t}} - \alpha_{\lambda_{js+t}} \in IB(\lambda_{js+t})\}$$

so there is a $t \in [0, s)$ for which

$$\sum_{j\geq 1} P\{S_{\lambda_{js+t}} - \alpha_{\lambda_{js+t}} \in IB(\lambda_{js+t})\} = +\infty . \tag{4}$$

Fix s so large that $q_s - 1 \geq \varepsilon^{-1}$, and define the disjoint sets

$$E_j = \{S_n - \alpha_n \notin IB(n) \text{ for } n \in (\lambda_{(i+1)s+t}, i \geq j), S_{\lambda_{js+t}} - \alpha_{\lambda_{js+t}} \in IB(\lambda_{js+t})\}$$

so that

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$$P(E_j) = \int_{u\in I} P\{S_n-\alpha_n \notin IB(n) \text{ for } n \in (\lambda_{(i+1)s+t},\ i \ge j) \mid$$
$$S_{\lambda_{js+t}}-\alpha_{\lambda_{js+t}} = uB(\lambda_{js+t})\}dP\{S_{\lambda_{js+t}}-\alpha_{\lambda_{js+t}} < uB(\lambda_{js+t})\} .$$

By independence and stationarity, the probability in the integrand equals

$$(5) \quad P\{S_{n-\lambda_{js+t}}-\alpha_{n-\lambda_{js+t}}+(\alpha_{n-\lambda_{js+t}}-\alpha_n+\alpha_{\lambda_{js+t}}) \notin IB(n)-uB(\lambda_{js+t})$$
$$\text{for } n \in (\lambda_{(i+1)s+t},\ i \ge j)\} .$$

Now when n is one of the numbers $\lambda_{(i+1)s+t}$, $i \ge j$,

$\lambda_{js+t} \le q_s^{-1}\lambda_{(j+1)s+t} \le \varepsilon n$, and

$$B(n-\lambda_{js+t}) = B(n(1-\lambda_{js+t}/n)) \ge B[(1-\varepsilon)n] \ge B(n)/b_+(1-\varepsilon) ,$$

which means $bB(n) - uB(\lambda_{js+t}) \le bb_+(1-\varepsilon)B(n-\lambda_{js+t})$ when $u \ge a > 0$.

Similarly, $n \ge \lambda_{(j+1)s+t}$ means $n - \lambda_{js+t} \ge (q_s-1)\lambda_{js+t} \ge \varepsilon^{-1}\lambda_{js+t}$, so

$$B(n-\lambda_{js+t}) \ge B\left(\varepsilon^{-1}\lambda_{js+t}\right) \ge b_-(\varepsilon^{-1})B(\lambda_{js+t})$$

which means

$$aB(n) - uB(\lambda_{js+t}) \ge aB(n-\lambda_{js+t}) - bB(\lambda_{js+t}) \ge (a-b/b_-(\varepsilon^{-1}))B(n-\lambda_{js+t}) .$$

By *(i)* of Lemma 2, there are integers n_0, j_0 , for which

$$|\alpha_n-\alpha_{\lambda_{js+t}}-\alpha_{n-\lambda_{js+t}}| \le \varepsilon B(n)/b_+(1-\varepsilon) \le \varepsilon B(n-\lambda_{js+t})$$

if $n > \lambda_{js+t} > j_0$, $n > n_0$, and n is one of the numbers $\lambda_{(i+1)s+t}$, $i \ge j$. Thus we see that if s and j are larger than some fixed integers, the probability in (5) is

$$\ge P\{S_{n-\lambda_{js+t}}-\alpha_{n-\lambda_{js+t}} \notin I'B(n-\lambda_{js+t}) \text{ for } n \in (\lambda_{(i+1)s+t},\ i \ge j)\}$$
$$\ge P\{S_{n-\lambda_{js+t}}-\alpha_{n-\lambda_{js+t}} \notin I'B(n-\lambda_{js+t}) \text{ for } n \ge \lambda_{(j+1)s+t}\}$$
$$\ge P\{S_n-\alpha_n \notin I'B(n) \text{ for } n \ge \lambda_{(j+1)s+t}-\lambda_{js+t}\}$$
$$\ge P\{S_n-\alpha_n \notin I'B(n) \text{ for } n \ge q_s-1\} .$$

This means

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$$P(E_j) \quad P\{S_n-\alpha_n \notin I'B(n) \text{ for } n \ge q_s-1\} \int_{u\in I} dP\{S_{\lambda_{js+t}}-\alpha_{\lambda_{js+t}} < uB(\lambda_{js+t})\}$$
$$= P\{S_n-\alpha_n \notin I'B(n) \text{ for } n \ge q_s-1\} P\{S_{\lambda_{js+t}}-\alpha_{\lambda_{js+t}} \in IB(\lambda_{js+t})\}$$

and since the E_j are disjoint, summing over j gives

$$1 \ge P\{S_n-\alpha_n \notin I'B(n) \text{ for } n \ge q_s-1\} \sum_{j\ge 1} P\{S_{\lambda_{js+t}}-\alpha_{\lambda_{js+t}} \in IB(\lambda_{js+t})\}$$

which implies, by (4), that $P\{S_n-\alpha_n \notin I'B(n) \text{ for } n \ge q_s-1\} = 0$. Since $q_s \to +\infty$ as $s \to +\infty$, we now have $P\{S_n-\alpha_n \in I'B(n) \text{ infinitely often}\} = 1$, as required.

REMARKS. (i) Lemma 1 simplifies if, for example, the $B(n)$ are assumed to be regularly varying with positive index, that is, $B[n\mu]/B(n) \to \mu^{\beta} (n \to +\infty)$ for $\mu > 0$ for some $\beta > 0$. Then λ_j can be taken to be the geometric subsequence $[\lambda^j]$, where λ is any number greater than 1 , and *(iii)* simplifies by omitting b_+ and b_- altogether (equivalently, putting $b_-(\varepsilon^{-1}) = +\infty$, $b_+(1-\varepsilon) = 1$, which can be achieved at the expense only of replacing ε by 2ε in *(iii)*).

(ii) The restriction $a \ge 0$ can be easily removed from Lemma 1, but it does not seem that *(ii)* can be replaced by the slightly weaker condition

$$(ii') \quad \sum_{j\ge 0} P\{S_n-\alpha_n \in (a, b)B(n) \text{ for some } n \in [\lambda_j, \lambda_{j+1})\} = +\infty$$

unless the sequence λ_j grows at most as rapidly as a geometric sequence.

LEMMA 2. *Suppose* $(S_n-\alpha_n)/B(n) \xrightarrow{P} 0$, *where* α_n *and* $B(n)$ *are constants with* $B(n) > 0$, $B(n) \uparrow +\infty$. *Then for every* $\varepsilon > 0$, $\varepsilon < 1/6$, *there are constants* $n_0(\varepsilon)$, $k_0(\varepsilon)$, $n_0 > k_0$, *for which* $n \ge n_0$ *implies for every real* x ,

(i) $\max_{k_0\le k<n} |\alpha_n-\alpha_k-\alpha_{n-k}| \le 4\varepsilon B(n)$, *and*

(ii) $(1-\varepsilon)P\{\max_{k_0\le k\le n} (S_k-\alpha_k) \ge xB(n)\} \le P\{S_n-\alpha_n \ge (x-6\varepsilon)B(n)\}$.

Proof of Lemma 2. Let $0 < \varepsilon < 1/6$ and consider

$$P\{\alpha_n-\alpha_k-\alpha_{n-k} > 4\varepsilon B(n)\}$$
$$= P\{(S_n-\alpha_n)-(S_k-\alpha_k)-(S_{n-k}-\alpha_{n-k})-S_n+S_k+S_{n-k} < -4\varepsilon B(n)\}$$
$$\le P\{S_n-\alpha_n < -\varepsilon B(n)\} + P\{S_k-\alpha_k > \varepsilon B(n)\}$$
$$+ P\{S_{n-k}-\alpha_{n-k} > \varepsilon B(n)\} + P\{T_{nk} > \varepsilon B(n)\}$$

where $T_{nk} = S_n - S_k - S_{n-k}$. Since $(S_n-\alpha_n)/B(n) \xrightarrow{p} 0$ there is a $k_0(\varepsilon) \ge 1$ for which $P\{|S_k-\alpha_k| > \frac{1}{2}\varepsilon B(k)\} < \varepsilon$ if $k \ge k_0$. Hence when $n \ge k \ge k_0$, the first two probabilities in the last expression are each less than ε. The third probability is also, if $n - k \ge k_0$, less than ε, while if $n - k \le k_0$, it can be made less than ε by taking $n \ge n_0$ for some $n_0(\varepsilon) \ge 1$ (whatever the value of $k \le n$) since only a finite number of the X_i are being summed. Finally by stationarity

$$P\{T_{nk} > \varepsilon B(n)\} = P\left\{\sum_{i=n-k+1}^{n} X_i - \sum_{i=1}^{k} X_i > \varepsilon B(n)\right\} \le 2P\{|S_k-\alpha_k| > \tfrac{1}{2}\varepsilon B(n)\} < 2\varepsilon$$

if $n > k \ge k_0$. Thus we have shown that $P\{\alpha_n-\alpha_k-\alpha_{n-k} > 4\varepsilon B(n)\} < 6\varepsilon < 1$ for $k_0 \le k < n$, $n \ge n_0$, so this probability is actually zero. A symmetrical argument gives similarly $\alpha_n - \alpha_k - \alpha_{n-k} \ge -4\varepsilon B(n)$, and these two together prove *(i)*.

To prove *(ii)* we proceed as in the proof of Lévy's inequality. Using *(i)*, we have for $n \ge n_0$,

$$P\{S_n-\alpha_n \ge (x-6\varepsilon)B(n)\}$$
$$\ge P \bigcup_{j=k_0}^{n} \{\max_{k_0\le i\le j-1} (S_i-\alpha_i) < xB(n),\ S_j-\alpha_j \ge xB(n)\} \cap \{S_n-S_j-\alpha_{n-j} \ge -2\varepsilon B(n)\}$$
$$= \sum_{j=k_0}^{n} P(A_j)P(B_j)\ , \text{ say.}$$

Note that

$$P(B_j) = P\{S_{n-j}-\alpha_{n-j} \ge -2\varepsilon B(n)\} \ge P\{S_{n-j}-\alpha_{n-j} \ge -2\varepsilon B(n-j)\} \ge 1 - \varepsilon$$

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if $n - j \geq k_0$, while if $j \leq n \leq k_0 + j$, $P\{S_{n-j}-\alpha_{n-j} \geq -2\varepsilon B(n)\} \geq 1 - \varepsilon$ for $n \geq n_0$ since only a finite number of the X_i are being summed.

Hence

$$P\{S_n-\alpha_n \geq (x-6\varepsilon)B(n)\} \geq (1-\varepsilon) \sum_{j=k_0}^{n} P(A_j) = (1-\varepsilon)P\{\max_{k_0 \leq k \leq n} (S_k-\alpha_k) \geq xB(n)\} ,$$

completing the proof. (See [7] for another version of Lévy's inequality.)

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