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A NEW HIGHER ORDER YANG–MILLS–HIGGS FLOW ON RIEMANNIAN 4-MANIFOLDS

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Abstract

Let (M, g) be a closed Riemannian 4-manifold and let E be a vector bundle over M with structure group G, where G is a compact Lie group. We consider a new higher order Yang–Mills–Higgs functional, in which the Higgs field is a section of $\Omega^0(\mathrm{ad}E)$. We show that, under suitable conditions, solutions to the gradient flow do not hit any finite time singularities. In the case that E is a line bundle, we are able to use a different blow-up procedure and obtain an improvement of the long-time result of Zhang ['Gradient flows of higher order Yang–Mills–Higgs functionals', *J. Aust. Math. Soc.* **113** (2022), 257–287]. The proof relies on properties of the Green function, which is very different from the previous techniques.

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1. Introduction

Let (M, g) be a closed Riemannian manifold of real dimension 4 and let E be a vector bundle over M with structure group G, where G is a compact Lie group. The Yang–Mills functional, defined on the space of connections of E, is given by

$$\mathcal{YM}(\nabla) = \frac{1}{2}\int_M |F_\nabla|^2\, d\mathrm{vol}_g,$$

where ∇ is a metric compatible connection, F_∇ denotes the curvature, and the pointwise norm $|\cdot|$ is given by g and the Killing form of $\mathrm{Lie}(G)$. The connection ∇ is called a Yang–Mills connection of E if it satisfies the Yang–Mills equation:

$$D_\nabla^* F_\nabla = 0.$$

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A solution of the Yang–Mills flow is given by a family of connections $\nabla_t := \nabla(x,t)$ such that

$$\frac{\partial \nabla_t}{\partial t} = -D^*_{\nabla_t} F_{\nabla_t} \quad \text{in } M \times [0,T).$$

The Yang–Mills flow was initially studied by Atiyah–Bott [2] to understand the topology of the space of connections by infinite dimensional Morse theory.

We consider the Yang–Mills–Higgs k-functional (or Yang–Mills–Higgs k-energy):

$$\mathcal{YMH}_k(\nabla, u) = \frac{1}{2}\int_M [|\nabla^{(k)} F_\nabla|^2 + |\nabla^{(k+1)} u|^2]\, d\mathrm{vol}_g, \tag{1.1}$$

where $k \in \mathbb{N} \cup \{0\}$, ∇ is a connection on E and u is a section of $\Omega^0(\mathrm{ad}E)$. In [13], we considered the case when u is a section of $\Omega^0(E)$. When $k = 0$, (1.1) is the Yang–Mills–Higgs functional studied in [4, 5]. In [4], Hassell proved the global existence of the Yang–Mills–Higgs flow in 3-dimensional Euclidean space. In [5], Hong–Tian studied the global existence of the Yang–Mills–Higgs flow in 3-dimensional hyperbolic space. Their results yield non-self-dual Yang–Mills connections on S^4. The Yang–Mills–Higgs flow has aroused much attention. For example, Li–Zhang [8] and Song–Wang [10] studied the asymptotic behaviour at time infinity of some Yang–Mills–Higgs flows.

The Yang–Mills–Higgs k-system, that is, the corresponding Euler–Lagrange equations of (1.1), is

$$\begin{cases} (-1)^k D^*_\nabla \Delta^{(k)}_\nabla F_\nabla + \displaystyle\sum_{v=0}^{2k-1} P_1^{(v)}[F_\nabla] + P_2^{(2k-1)}[F_\nabla] + \sum_{i=0}^{k} \nabla^{*(i)}(\nabla^{(k+1)} u * \nabla^{(k-i)} u) = 0, \\ \nabla^{*(k+1)} \nabla^{(k+1)} u = 0, \end{cases}$$

where $\Delta^{(k)}_\nabla$ denotes k iterations of the Bochner Laplacian $-\nabla^*\nabla$ and the notation P is defined in (2.1). A solution of the Yang–Mills–Higgs k-flow is given by a family of pairs $(\nabla(x,t), u(x,t)) := (\nabla_t, u_t)$ such that

$$\begin{cases} \dfrac{\partial \nabla_t}{\partial t} = (-1)^{(k+1)} D^*_{\nabla_t} \Delta^{(k)}_{\nabla_t} F_{\nabla_t} + \displaystyle\sum_{v=0}^{2k-1} P_1^{(v)}[F_{\nabla_t}] \\ \qquad\qquad + P_2^{(2k-1)}[F_{\nabla_t}] + \displaystyle\sum_{i=0}^{k} \nabla_t^{*(i)}(\nabla_t^{(k+1)} u_t * \nabla_t^{(k-i)} u_t), \\ \dfrac{\partial u_t}{\partial t} = -\nabla_t^{*(k+1)} \nabla_t^{(k+1)} u_t, \quad \text{in } M \times [0,T). \end{cases} \tag{1.2}$$

When $k = 0$, the flow (1.2) is a Yang–Mills–Higgs flow [5].

From an analytic point of view, the Yang–Mills–Higgs k-flow (1.2) admits similar properties to the case in which the Higgs field takes values in $\Omega^0(E)$. In fact, by the approach in [13], we can prove the following theorem.

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THEOREM 1.1. *Let E be a vector bundle over a closed Riemannian 4-manifold (M, g) and k be an integer with $k > 1$. For every smooth initial data (∇_0, u_0), there exists a unique smooth solution (∇_t, u_t) to the Yang–Mills–Higgs k-flow (1.2) in $M \times [0, +\infty)$.*

Our motivation for considering such flows comes from recent work of Waldron who proved long-time existence for the Yang–Mills flow [12], thereby settling a long standing conjecture. In the context of the Yang–Mills–Higgs flow, it is still unknown whether the flow exists for all times on a Riemannian 4-manifold. The above theorem shows that provided $k > 1$, the Yang–Mills–Higgs k flow does obey long time existence on a 4-manifold. A question that arises at this point is to understand what is the optimum value for k. By assuming our bundle E is a line bundle, we are able to make progress on this question and show that long-time existence holds for all positive k.

THEOREM 1.2. *Let E be a line bundle over a closed Riemannian 4-manifold (M, g) and k be an integer with $k > 0$. For every smooth initial data (∇_0, u_0), there exists a unique smooth solution (∇_t, u_t) to the Yang–Mills–Higgs k-flow (1.2) in $M \times [0, +\infty)$.*

At present, we do not know if this theorem is optimal, meaning that we cannot rule out long-time existence occurring for $k = 0$.

The proof of Theorem 1.1 involves local L^2 derivative estimates, energy estimates and blow-up analysis. An interesting aspect of this work is that by using a different blow-up procedure, we are able to obtain a proof of Theorem 1.2, which may be of independent interest. Another interesting aspect is that the proof of long-time existence obstruction (see Theorem 3.7) relies on properties of the Green function, which is very different from the previous techniques in [6, 9, 13].

2. Preliminaries

In this section, we introduce the basic setup and notation that will be used throughout the paper. We follow the notation of [6, 9, 13].

Let E be a vector bundle over a smooth closed manifold (M, g) of real dimension n. The set of all smooth unitary connections on E will be denoted by $\mathcal{A}_E$. A given connection $\nabla \in \mathcal{A}_E$ can be extended to other tensor bundles by coupling with the corresponding Levi–Civita connection ∇_M on (M, g).

Let D_∇ be the exterior derivative, or skew symmetrisation of ∇. The curvature tensor of E is denoted by

$$F_\nabla = D_\nabla \circ D_\nabla.$$

We set ∇^*, D_∇^* to be the formal L^2-adjoints of ∇, D_∇, respectively. The Bochner and Hodge Laplacians are given respectively by

$$\Delta_\nabla = -\nabla^*\nabla, \quad \Delta_{D_\nabla} = D_\nabla D_\nabla^* + D_\nabla^* D_\nabla.$$

Let ξ, η be p-forms valued in E or $\mathrm{End}(E)$. Let $\xi * \eta$ denote any multilinear form obtained from a tensor product $\xi \otimes \eta$ in a universal way. That is to say, $\xi * \eta$ is obtained by starting with $\xi \otimes \eta$, taking any linear combination of this tensor, taking any number

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of metric contractions and switching any number of factors in the product. We then have

$$|\xi * \eta| \leq C|\xi||\eta|.$$

Denote

$$\nabla^{(i)} = \underbrace{\nabla \cdots \nabla}_{i \text{ times}}.$$

We will also use the P notation, as introduced in [7]. Given a tensor ξ, we denote

$$P_{v}^{(k)}[\xi] := \sum_{w_1+\cdots+w_v=k} (\nabla^{(w_1)}\xi) * \cdots * (\nabla^{(w_v)}\xi) * T, \tag{2.1}$$

where $k, v \in \mathbb{N}$ and T is a generic background tensor dependent only on g.

3. Long-time existence obstruction

We can use De Turck's trick to establish the local existence of the Yang–Mills–Higgs k-flow. We refer to [6, 9, 13] for more details. As the proof is standard, we will omit the details.

THEOREM 3.1 (Local existence). *Let E be a vector bundle over a closed Riemannian manifold (M, g). There exists a unique smooth solution (∇_t, u_t) to the Yang–Mills–Higgs k-flow (1.2) in $M \times [0, \epsilon)$ with smooth initial value (∇_0, u_0).*

Following [6, 9], we can derive estimates of Bernstein–Bando–Shi type, similar to [13, Proposition 4.10].

PROPOSITION 3.2. *Let $q \in \mathbb{N}$, $\gamma \in C_c^\infty(M)$ $(0 \leq \gamma \leq 1)$ and (∇_t, u_t) be a solution to the Yang–Mills–Higgs k-flow (1.2) defined on $M \times I$. Suppose $Q = \max\{1, \sup_{t\in I} |F_{\nabla_t}|\}$, $K = \max\{1, \sup_{t\in I} |u_t|\}$ and $s \geq (k+1)(q+1)$. For $t \in [0, T) \subset I$ with $T < 1/(QK)^4$, there exists a positive constant $C_q := C_q(\dim(M), \mathrm{rk}(E), G, q, k, s, g, \gamma) \in \mathbb{R}_{>0}$ such that*

$$\|\gamma^s \nabla_t^{(q)} F_{\nabla_t}\|_{L^2}^2 + \|\gamma^s \nabla_t^{(q)} u_t\|_{L^2}^2 \leq C_q t^{-q/(k+1)} \sup_{t\in[0,T)} (\|F_{\nabla_t}\|_{L^2}^2 + \|u_t\|_{L^2}^2).$$

The following corollary is a direct consequence of the above proposition and will be used in the blow-up analysis. The proof relies on the Sobolev embedding, $W^{p,2} \subset C^0$ provided $p > n/2$, and Kato's inequality $|d|u_t|| \leq |\nabla_t u_t|$. More details can be found in Kelleher's paper [6, Corollary 3.14].

COROLLARY 3.3. *Suppose (∇_t, u_t) solves the Yang–Mills–Higgs k-flow (1.2) defined on $M \times [0, \tau]$. Set $\bar{\tau} := \min\{\tau, 1\}$. Suppose $Q = \max\{1, \sup_{t\in[0,\bar{\tau}]} |F_{\nabla_t}|\}$, $K = \max\{1, \sup_{t\in[0,\bar{\tau}]} |u_t|\}$. Assume $\gamma \in C_c^\infty(M)$ $(0 \leq \gamma \leq 1)$. For $s, l \in \mathbb{N}$ with $s \geq (k+1)(l+1)$, there exists $C_l := C_l(\dim(M), \mathrm{rk}(E), K, Q, G, s, k, l, \tau, g, \gamma) \in \mathbb{R}_{>0}$ such that*

$$\sup_M (|\gamma^s \nabla_{\bar{\tau}}^{(l)} F_{\nabla_{\bar{\tau}}}|^2 + |\gamma^s \nabla_{\bar{\tau}}^{(l)} u_{\bar{\tau}}|^2) \leq C_l \sup_{M\times[0,\bar{\tau})} (\|F_{\nabla_t}\|_{L^2}^2 + \|u_t\|_{L^2}^2).$$

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From Corollary 3.3, we deduce the following corollary, which can be used for finding obstructions to long-time existence.

COROLLARY 3.4. *Suppose* (∇_t, u_t) *solves the Yang–Mills–Higgs k-flow (1.2) defined on* $M \times [0, T)$ *for* $T \in [0, +\infty)$. *Suppose*

$$Q = \max\{1, \sup_{t\in[0,T)} |F_{\nabla_t}|, \sup_{t\in[0,T)} \|F_{\nabla_t}\|_{L^2}\}$$

and

$$K = \max\{1, \sup_{t\in[0,T)} |u_t|, \sup_{t\in[0,T)} \|u_t\|_{L^2}\}$$

are finite. Assume $\gamma \in C_c^\infty(M)$ $(0 \le \gamma \le 1)$. *Then, for* $t \in [0, T)$ *and* $s, l \in \mathbb{N}$ *with* $s \ge (k + 1)(l + 1)$, *there exists* $C_l := C_l(\nabla_0, u_0, \dim(M), \mathrm{rk}(E), K, Q, G, s, k, l, g, \gamma) \in \mathbb{R}_{>0}$ *such that*

$$\sup_{M\times[0,T)} (|\gamma^s \nabla_t^{(l)} F_{\nabla_t}|^2 + |\gamma^s \nabla_t^{(l)} u_t|^2) \le C_l.$$

We will use Corollary 3.4 to show that the only obstruction to long-time existence of the Yang–Mills–Higgs k-flow (1.2) is a lack of a supremal bound on $|F_{\nabla_t}| + |\nabla_t u_t|$. Before doing so, we need the following proposition, which is similar to [13, Proposition 4.15].

PROPOSITION 3.5. *Suppose* (∇_t, u_t) *is a solution to the Yang–Mills–Higgs k-flow (1.2) defined on* $M \times [0, T)$ *for* $T \in [0, +\infty)$. *Suppose that for all* $l \in \mathbb{N} \cup \{0\}$, *there exists* $C_l \in \mathbb{R}_{>0}$ *such that*

$$\max\left\{\sup_{M\times[0,T)} \left|\nabla_t^{(l)}\left[\frac{\partial \nabla_t}{\partial t}\right]\right|, \sup_{M\times[0,T)} \left|\nabla_t^{(l)}\left[\frac{\partial u_t}{\partial t}\right]\right|\right\} \le C_l.$$

Then $\lim_{t\to T}(\nabla_t, u_t) = (\nabla_T, u_T)$ *exists and is smooth.*

The following proposition is straightforward.

PROPOSITION 3.6. *Suppose* (∇_t, u_t) *is a solution to the Yang–Mills–Higgs k-flow (1.2) defined on* $M \times [0, T)$. *We have*

$$\sup_{t\in[0,T)} \|u_t\|_{L^2} < +\infty.$$

Using Propositions 3.5 and 3.6, we are ready to prove the main result in this section.

THEOREM 3.7. *Assume* E *is a line bundle. Suppose* (∇_t, u_t) *is a solution to the Yang–Mills–Higgs k-flow (1.2) for some maximal* $T < +\infty$. *Then,*

$$\sup_{M\times[0,T)} (|F_{\nabla_t}| + |\nabla_t u_t|) = +\infty.$$

PROOF. Suppose to the contrary that

$$\sup_{M\times[0,T)} (|F_{\nabla_t}| + |\nabla_t u_t|) < +\infty,$$

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which means that

$$\sup_{M\times[0,T)} |F_{\nabla_t}| < +\infty, \qquad \sup_{M\times[0,T)} |\nabla_t u_t| < +\infty.$$

Denote by $G_t(x, y)$ the Green function associated to the operator Δ_{∇_t}. Then for any fixed $x \in M$, $\|\nabla_0 G_t(x, \cdot)\|_{L^\infty(M)} \le C_G$ for a constant C_G from [1, Appendix A]. Note that $\nabla_t G_t - \nabla_0 G_t = [\nabla_t - \nabla_0, G_t] = 0$. We conclude that $\|\nabla_t G_t\|_{L^\infty(M)}$ is also uniformly bounded. Therefore, using the properties of the Green function in [1, Appendix A],

$$\begin{aligned}\left|u_t(x) - \frac{1}{\mathrm{Vol}(M)}\int_M u_t(y)\,dy\right| &= \left|\int_M \Delta_{\nabla_t} G_t(x,y)u_t(y)\,dy\right| \\ &= \left|\int_M \nabla_t G_t(x,y)\nabla_t u_t(y)\,dy\right| \\ &< +\infty,\end{aligned}$$

which together with Proposition 3.6 implies

$$\sup_{M\times[0,T)} |u_t| < +\infty.$$

For all $t \in [0, T)$ and $l \in \mathbb{N} \cup \{0\}$, by Corollary 3.4, $\sup_M(|\nabla_t^{(l)} F_{\nabla_t}|^2 + |\nabla_t^{(l)} u_t|^2)$ is uniformly bounded and so by Proposition 3.5, $\lim_{t\to T}(\nabla_t, u_t) = (\nabla_T, u_T)$ exists and is smooth. However, by local existence (Theorem 3.1), there exists $\epsilon > 0$ such that (∇_t, u_t) exists over the extended domain $[0, T + \epsilon)$, which contradicts the assumption that T is maximal. Thus, we prove the theorem. □

4. Blow-up analysis

In this section, we will address the possibility that the Yang–Mills–Higgs k-flow admits a singularity given no bound on $|F_{\nabla_t}| + |\nabla_t u_t|$. To begin with, we first establish some preliminary scaling laws for the Yang–Mills–Higgs k-flow.

PROPOSITION 4.1. *Suppose (∇_t, u_t) is a solution to the Yang–Mills–Higgs k-flow (1.2) defined on $M \times [0, T)$. Define the 1-parameter family ∇_t^ρ with local coefficient matrices given by*

$$\Gamma_t^\rho(x) := \rho\Gamma_{\rho^{2(k+1)}t}(\rho x),$$

where $\Gamma_t(x)$ is the local coefficient matrix of ∇_t. Define the ρ-scaled Higgs field u_t^ρ by

$$u_t^\rho(x) := \rho u_{\rho^{2(k+1)}t}(\rho x).$$

Then $(\nabla_t^\rho, u_t^\rho)$ is also a solution to the Yang–Mills–Higgs k-flow (1.2) defined on $[0, T/\rho^{2(k+1)})$.

Next we will show that if the curvature coupled with a Higgs field is blowing up as one approaches the maximal time, then one can extract a blow-up limit. The proof will closely follow the arguments in [6, Proposition 3.25] and [13, Theorem 5.2].

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THEOREM 4.2. *Assume E is a line bundle. Suppose (∇_t, u_t) is a solution to the Yang–Mills–Higgs k-flow (1.2) defined on some maximal time interval $[0, T)$ with $T < +\infty$. Then there exists a blow-up sequence (∇_t^i, u_t^i) which converges pointwise to a smooth solution $(\nabla_t^\infty, u_t^\infty)$ to the Yang–Mills–Higgs k-flow (1.2) defined on the domain $\mathbb{R}^n \times \mathbb{R}_{<0}$.*

PROOF. From Theorem 3.7,

$$\lim_{t\to T} \sup_M (|F_{\nabla_t}| + |\nabla_t u_t|) = +\infty.$$

Therefore, we can choose a sequence of times $t_i \nearrow T$ within $[0, T)$ and a sequence of points x_i, such that

$$|F_{\nabla_{t_i}}(x_i)| + |\nabla_{t_i} u_{t_i}(x_i)| = \sup_{M\times[0,t_i]} (|F_{\nabla_t}| + |\nabla_t u_t|).$$

Let $\{\rho_i\} \subset \mathbb{R}_{>0}$ be constants to be determined. Define $\nabla_t^i(x)$ by

$$\Gamma_t^i(x) = \rho_i^{1/2(k+1)} \Gamma_{\rho_i t+t_i}(\rho_i^{1/2(k+1)} x + x_i)$$

and

$$u_t^i(x) = \rho_i^{1/2(k+1)} u_{\rho_i t+t_i}(\rho_i^{1/2(k+1)} x + x_i).$$

By Proposition 4.1, (∇_t^i, u_t^i) are also solutions to the Yang–Mills–Higgs k-flow (1.2) and the domain for each (∇_t^i, u_t^i) is $B_0(\rho_i^{-1/2(k+1)}) \times [-t_i/\rho_i, (T - t_i)/\rho_i)$. We observe that

$$F_t^i(x) := F_{\nabla_t^i}(x) = \rho_i^{1/(k+1)} F_{\nabla_{\rho_i t+t_i}}(\rho_i^{1/2(k+1)} x + x_i),$$

which means that

$$\begin{aligned}
&\sup_{t\in[-t_i/\rho_i, T-t_i/\rho_i)} (|F_t^i(x)| + |\nabla_t^i u_t^i(x)|) \\
&\quad= \rho_i^{1/(k+1)} \sup_{t\in[-t_i/\rho_i, T-t_i/\rho_i)} (|F_{\nabla_{\rho_i t+t_i}}(\rho_i^{1/2(k+1)} x + x_i)| + |\nabla_{\rho_i t+t_i} u_{\rho_i t+t_i}(\rho_i^{1/2(k+1)} x + x_i)|) \\
&\quad= \rho_i^{1/(k+1)} \sup_{t\in[0,t_i]} (|F_{\nabla_t}(x)| + |\nabla_t u_t(x)|) \\
&\quad= \rho_i^{1/(k+1)} (|F_{\nabla_{t_i}}(x_i)| + |\nabla_{t_i} u_{t_i}(x_i)|).
\end{aligned}$$

Therefore, setting

$$\rho_i = (|F_{\nabla_{t_i}}(x_i)| + |\nabla_{t_i} u_{t_i}(x_i)|)^{-(k+1)}$$

gives

$$1 = |F_0^i(0)| + |\nabla_0^i u_0^i(0)| = \sup_{t\in[-t_i/\rho_i, 0]} (|F_t^i(x)| + |\nabla_t^i u_t^i(x)|). \tag{4.1}$$

Now, we are ready to construct smoothing estimates for the sequence (∇_t^i, u_t^i). Let $y \in \mathbb{R}^n$, $\tau \in \mathbb{R}_{\le 0}$. For any $s \in \mathbb{N}$,

$$\sup_{t\in[\tau-1,\tau]} (|\gamma_y^s F_t^i(x)| + |\gamma_y^s \nabla_t^i u_t^i(x)|) \le 1.$$

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Note that E is a line bundle and, similar to the proof of Theorem 3.7, it suffices to use Corollary 3.3. Then for all $q \in \mathbb{N}$, one may choose $s \geq (k+1)(q+1)$ so that there exists a positive constant C_q such that

$$\begin{aligned}\sup_{x \in B_y(1/2)} &(|(\nabla^i_\tau)^{(q)} F^i_\tau(x)| + |(\nabla^i_\tau)^{(q)} u^i_\tau(x)|)\\ &\leq \sup_{x \in B_y(1)} (|\gamma^s_y (\nabla^i_\tau)^{(q)} F^i_\tau(x)| + |\gamma^s_y (\nabla^i_\tau)^{(q)} u^i_\tau(x)|) \leq C_q.\end{aligned}$$

Then by the Coulomb gauge theorem of Uhlenbeck [11, Theorem 1.3] (see also [5]) and the gauge patching theorem [3, Corollary 4.4.8], passing to a subsequence (without changing notation) and in an appropriate gauge, $(\nabla^i_t, u^i_t) \to (\nabla^\infty_t, u^\infty_t)$ in C^∞. □

5. Proof of Theorem 1.2

The following energy estimates are similar to the ones in [13, Section 6].

PROPOSITION 5.1. *Suppose (∇_t, u_t) is a solution to the Yang–Mills–Higgs k-flow (1.2) defined on $M \times [0, T)$. Then the Yang–Mills–Higgs k-energy (1.1) is decreasing along the flow (1.2).*

PROPOSITION 5.2. *Suppose (∇_t, u_t) is a solution to the Yang–Mills–Higgs k-flow (1.2) defined on $M^4 \times [0, T)$ with $T < +\infty$. Then the Yang–Mills–Higgs energy*

$$\mathcal{YMH}(\nabla_t, u_t) = \frac{1}{2} \int_M [|F_{\nabla_t}|^2 + |\nabla_t u_t|^2]\, d\mathrm{vol}_g$$

is bounded along the flow (1.2).

Next, we will complete the proof of Theorem 1.2. To accomplish this, we first show that the L^p-norm controls the L^∞-norm by blow-up analysis.

PROPOSITION 5.3. *Assume E is a line bundle. Suppose (∇_t, u_t) is a solution to the Yang–Mills–Higgs k-flow (1.2) defined on $M^4 \times [0, T)$ and*

$$\sup_{t \in [0,T)} (\|F_{\nabla_t}\|_{L^p} + \|\nabla_t u_t\|_{L^p}) < +\infty.$$

If $p > 2$, then

$$\sup_{t \in [0,T)} (\|F_{\nabla_t}\|_{L^\infty} + \|\nabla_t u_t\|_{L^\infty}) < +\infty.$$

PROOF. To obtain a contradiction, assume

$$\sup_{t \in [0,T)} (\|F_{\nabla_t}\|_{L^\infty} + \|\nabla_t u_t\|_{L^\infty}) = +\infty.$$

As we did in Theorem 4.2, we can construct a blow-up sequence (∇_t^i, u_t^i), with blow-up limit $(\nabla_t^\infty, u_t^\infty)$. Noting (4.1), by Fatou's lemma and the natural scaling law,

$$\begin{aligned}\|F_{\nabla_t^\infty}\|_{L^p}^p + \|\nabla_t^\infty u_t^\infty\|_{L^p}^p &\leq \lim_{i\to+\infty} \inf(\|F_{\nabla_t^i}\|_{L^p}^p + \|\nabla_t^i u_t^i\|_{L^p}^p)\\ &\leq \lim_{i\to+\infty} \rho_i^{(2p-4)/(2k+2)}(\|F_{\nabla_t}\|_{L^p}^p + \|\nabla_t u_t\|_{L^p}^p).\end{aligned}$$

Since $\lim_{i\to+\infty} \rho_i^{(2p-4)/(2k+2)} = 0$ when $p > 2$, the right-hand side of the above inequality tends to zero, which is a contradiction since the blow-up limit has nonvanishing curvature. □

Now we are ready to give the proof of Theorem 1.2.

PROOF OF THEOREM 1.2. By the Sobolev embedding theorem, we can solve for p such that $W^{k,2} \subset L^p$, when $k > 0$. Using the interpolation inequalities [7, Corollary 5.5]:

$$\begin{aligned}\|F_{\nabla_t}\|_{L^p} + \|\nabla_t u_t\|_{L^p} &\leq CS_{k,p} \sum_{j=0}^{k} (\|\nabla_t^{(j)} F_{\nabla_t}\|_{L^2}^2 + \|\nabla_t^{(j)} u_t\|_{L^2}^2 + 1)\\ &\leq CS_{k,p}(\|\nabla_t^{(k)} F_{\nabla_t}\|_{L^2}^2 + \|F_{\nabla_t}\|_{L^2}^2 + \|\nabla_t^{(k+1)} u_t\|_{L^2}^2 + \|u_t\|_{L^2}^2 + 1)\\ &\leq CS_{k,p}(\mathcal{YMH}_k(\nabla_t, u_t) + \mathcal{YMH}(\nabla_t, u_t) + \|u_t\|_{L^2}^2 + 1),\end{aligned}$$

then using Propositions 5.1, 3.6 and 5.2, we conclude that the flow exists smoothly for all time. □

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