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# THE SEMICENTRE OF A GROUP ALGEBRA

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We study the semicentre of a group algebra $K[G]$ where $K$ is a field of characteristic zero and $G$ is a polycyclic-by-finite group such that $\Delta(G)$ is torsion-free abelian. Several properties about the structure of this ring are proved, in particular as to when is the semicentre a UFD. Examples are constructed when this is not the case. We also prove necessary and sufficient conditions for every normal element of $K[G]$ which belongs to $K[\Delta(G)]$ to be the product of a unit and a semi-invariant.

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## Introduction

Let $K$ be a field of characteristic zero and $G$ a polycyclic-by-finite group such that $K[G]$ is a prime Noetherian ring. This type of group algebra is often compared with a universal enveloping algebra $U(L)$ of a finite dimensional Lie algebra $L$ over a field of characteristic zero. A number of basic properties are similar, but we show that there are also some striking differences between the semicentre of both types of rings. To be a little more concrete, in both cases the semicentre is a commutative domain graded by an abelian monoid. In the case of $U(L)$ it is well-known and obvious that this monoid is torsion-free abelian while for $K[G]$ we prove this is a finite group. We show that this difference in the grading monoid implies that the semicentre of the classical ring of quotients of $K[G]$ behaves quite differently from the case of $U(L)$ (cf. [7]). The main difference is perhaps the property of being a UFD: the semicentre of $U(L)$ is always a UFD (cf. [9, 13]) and we observe when it is a UFD in case of a group algebra. This is an immediate consequence of a result of M. Lorenz on rings of multiplicative invariants [10]. In computing the semicentre of $K[G]$ in a number of examples we show e.g., that the property of being a UFD of the semicentre does not depend only on $G$ but also on the field $K$.

Throughout this paper $K$ is a field of characteristic zero, $K^*$ denotes $K\backslash\{0\}$ and $\overline{K}$ is the algebraic closure of $K$. The group of elements of a group $G$ which have only finitely many conjugates is denoted by $\Delta(G)$ or shortly by $\Delta$. Then $K[G]$ is a prime ring if and only if $\Delta(G)$ is torsion-free abelian, by Connell's Theorem (see e.g., [19, 4.2.10]). Unless explicitly mentioned, all groups are polycyclic-by-finite such that $\Delta(G)$ is torsion-free abelian.

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## 1. Definitions and basic properties

In [**22**] M. Smith makes a distinction between semi-invariant ring and semicentre. To improve readability, we combine the definitions as mentioned in [**22**, p. 1283 and p. 1290].

**Definition 1.1.** Let $R$ be a $K$-algebra and $G$ a subgroup of units of $R$. Then $G$ acts on $R$ by inner automorphisms. If $r \in R$ and $g \in G$, denote $grg^{-1}$ by $r^g$.

(1) Let $0 \neq r \in R$ and $\lambda \in \operatorname{Hom}(G, K^*)$. Then $\alpha$ is said to be a *semi-invariant with weight* $\lambda$ if for each $g \in G$ $r^g = \lambda(g)r$.

(2) If $\lambda \in \operatorname{Hom}(G, K^*)$ we denote the set of semi-invariants with weight $\lambda$ together with 0 by $R_\lambda$. If $R_\lambda \neq 0$, $\lambda$ is a weight. The set of all weights is denoted by $\Lambda(G, K)$ (or shortly by $\Lambda(G)$).

(3) The *semicentre* of $R$, denoted by $\mathrm{Sz}R$, is defined as

$$\mathrm{Sz}R = \sum_{\lambda \in \Lambda(G)} R_\lambda.$$

Of course, if $R = K[G]$ is a group algebra, $G$ acts on $K[G]$ by inner automorphisms and the foregoing definition makes sense.

**Proposition 1.2.** (1) $\mathrm{Sz}K[G] = \oplus_{\lambda \in \Lambda(G)}(K[G])_\lambda$ *is a subring of* $K[\Delta(G)]$;

(2) $\mathrm{Sz}K[G]$ *is a commutative domain;*

(3) $\Lambda(G)$ *is an abelian cancellative monoid and* $\mathrm{Sz}K[G]$ *is a* $\Lambda(G)$*-graded ring;*

(4) *A semi-invariant is a normal element.*

**Proof.** (1) As mentioned in [**22**], a standard linear algebra argument shows that $\sum_{\lambda \in \Lambda(G)}(K[G])_\lambda$ is in fact a direct sum $\oplus_{\lambda \in \Lambda(G)}(K[G])_\lambda$. If $\alpha$ is a semi-invariant with weight $\lambda$, then $g\alpha g^{-1} = \lambda(g)\alpha$ for all $g \in G$, hence $\operatorname{supp} \alpha \subset \Delta(G)$. Therefore $\mathrm{Sz}K[G] \subset K[\Delta(G)]$.

(2) and (3) follow directly from (1) because $K[\Delta]$ is a commutative domain.

(4) If $\alpha$ is a semi-invariant with weight $\lambda$, then for all $g \in G$ $g\alpha = \alpha(\lambda(g)g)$ and $\alpha g = (\lambda(g)^{-1}g)\alpha$. So $K[G]\alpha = \alpha K[G]$, i.e., $\alpha$ is a normal element. □

**Lemma 1.3.** *If* $\lambda \in \Lambda(G)$, *then* $\lambda(G') = \lambda(C_G(\Delta)) = 1$.

**Proof.** (1) If $g, h \in G$, then $\lambda([g,h]) = \lambda(ghg^{-1}h^{-1}) = \lambda(g)\lambda(h)\lambda(g)^{-1}\lambda(h)^{-1} = 1$.

(2) Let $\alpha$ be a semi-invariant with weight $\lambda$. If $g \in C_G(\Delta)$, i.e., $gh = hg$ for all $h \in \Delta$, then $\alpha^g = g\alpha g^{-1} = \alpha$ since $\alpha \in K[\Delta]$. On the other hand $\alpha^g = \lambda(g)\alpha$. Thus $\lambda(g) = 1$. □

**Lemma 1.4.** (1) *$G/C_G(\Delta)$ is a finite group;*

(2) *$C_G(\Delta) = C_G(x)$ for some $x \in \Delta$.*

**Proof.** (1) Clearly $C_G(\Delta)$ is a normal subgroup of $G$. On the other hand, $\Delta$ is a subgroup of $G$, thus finitely generated, say by $x_1, \ldots, x_k$. Then $C_G(\Delta) = \bigcap_{i=1}^k C_G(x_i)$. Since $x_i \in \Delta$, $(G : C_G(x_i))$ is finite and hence $(G : C_G(\Delta))$ is finite by e.g., [**19**, Lemma 4.1.3].

(2) This is proved in [**16**, Lemma 2]. □

We now construct all semi-invariants having a certain weight $\lambda$. This is partially based on [**15**, Lemma 3]. First note that if $\alpha = \sum a_g g \in (K[G])_\lambda$ and $x \in \operatorname{supp} \alpha$ with (finite) conjugacy class $C_x$, then obviously $\alpha_{(x)} = \sum_{g \in C_x} a_g g$ is a semi-invariant having weight $\lambda$ and $\alpha$ is a sum of such $\alpha_{(x)}$. Therefore it suffices to construct semi-invariants $\alpha$ such that $\operatorname{supp} \alpha$ is precisely a conjugacy class.

**Lemma 1.5.** *Let $\lambda \in \operatorname{Hom}(G, K^*)$.*

(1) *If $C_G(x) \subset \ker \lambda$ for some $x \in \Delta(G)$ and $T$ denotes a left transversal for $C_G(x)$ in $G$, then $\alpha = \sum_{t \in T} \lambda(t)^{-1} x^t$ is a semi-invariant with weight $\lambda$ (note that $\operatorname{supp} \alpha = C_x$).*

(2) *Conversely, if $\alpha \in (K[G])_\lambda$ such that $\operatorname{supp} \alpha$ equals precisely a conjugacy class of an element $x$, then $\alpha = a(\sum_{t \in T} \lambda(t)^{-1} x^t)$ where $a \in K^*$ and $T$ is a left transversal for $C_G(x)$ in $G$.*

**Proof.** (1) This is proved in [**15**, Lemma 3], up to a slight difference in notation.

(2) Conversely, let $\alpha \in (K[G])_\lambda$ such that $\operatorname{supp} \alpha = C_x$ for some $x$. Let $T$ be a left transversal for $C_G(x)$ in $G$. Write $T = \{t_1 = 1, t_2, \ldots, t_n\}$ and $\alpha = \sum_{i=1}^n a_i x^{t_i}$. If $j \neq 1$, then

$$\alpha^{t_j} = \sum_{i=1}^n a_i x^{t_j t_i} \text{ and } \alpha^{t_j} = \lambda(t_j)\alpha = \sum_{i=1}^n a_i \lambda(t_j) x^{t_i}.$$

In particular, $a_1 = a_j \lambda(t_j)$ or $a_j = \lambda(t_j^{-1}) a_1$. Therefore

$$\alpha = a_1 \left( \sum_{i=1}^n \lambda(t_i^{-1}) x^{t_i} \right).$$

□

**Proposition 1.6.** $\Lambda(G) \cong \operatorname{Hom}(G/C_G(\Delta), K^*)$ *and thus $\Lambda(G)$ is a finite abelian group.*

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**Proof.** Since $\lambda(C_G(\Delta)) = 1$ if $\lambda \in \Lambda(G)$ by Lemma 1.3, it is straightforward to check that the map

$$\Lambda(G) \to \mathrm{Hom}(G/C_G(\Delta), K^*) : \lambda \mapsto \bar{\lambda} : G/C_G(\Delta) \to K^* \quad \bar{g} \mapsto \lambda(g)$$

is a well-defined injective homomorphism of monoids. To prove the surjectivity, let $\lambda \in \mathrm{Hom}(G/C_G(\Delta), K^*)$, then with

$$\underbrace{G \to G/C_G(\Delta) \xrightarrow{\lambda} K^*}_{\mu},$$

$\mu \in \mathrm{Hom}(G, K^*)$ such that $\mu(C_G(\Delta)) = 1$. Combining Lemma 1.4(2) and Lemma 1.5(1), there exists a semi-invariant $\alpha$ with weight $\mu$. Thus $\mu \in \Lambda(G)$ and $\bar{\mu} = \lambda$. In particular this shows that $\Lambda(G)$ is a finite abelian group. □

**Corollary 1.7.** $\mathrm{Sz}K[G] = ZK[G]$ *if and only if* $\mathrm{Hom}(G/C_G(\Delta), K^*) = \{1\}$.

Of course, if $G$ is nilpotent (and such that $G$ is polycyclic-by-finite and $\Delta(G)$ is torsion-free abelian), then $C_G(\Delta) = G$ since $\Delta(G) = Z(G)$ in this case (cf. e.g., [**19**, Lemma 11.4.3]). However, since $\Delta(G) = Z(G)$, we immediately have $K[Z(G)] \subset Z(K[G]) \subset \mathrm{Sz}K[G] \subset K[\Delta(G)] = K[Z(G)]$ and thus $\mathrm{Sz}K[G] = ZK[G]$. Example 6.4 shows however that $\mathrm{Sz}KG$ can also be equal to $ZK[G]$ if $G$ is not nilpotent.

The next proposition shows that every finite abelian group can be the group of weights of some group algebra.

**Proposition 1.8.** *Every finite abelian group is the group of weights of some group algebra $K[G]$ where $K$ is algebraically closed.*

**Proof** (due to D. S. Passman; the original proof of the author was longer). Let $A$ be an infinite cyclic group and $H$ a finite abelian group. Let $G$ be the wreath product of $A$ by $H$, denoted $G = A \wr H$. Then $G = W \times_\sigma H$, the semidirect product of $W$ and $H$, where $W$ is a direct product of copies of $A$ indexed by $H$. Using the fact that $A$ is abelian and infinite and that $H$ is finite abelian, it is straightforward to conclude that $\Delta(G) = W = C_G(W)$. Thus $G/C_G(\Delta) \cong H$ and using Proposition 1.6 we obtain that $\Lambda(G) \cong \mathrm{Hom}(H, K^*) \cong H$ because $K$ is algebraically closed. □

**Remark 1.9.** Proposition 1.8 does not hold if $K$ is not algebraically closed. For example, let $K = \mathbb{R}$; then the cyclic group of order 4 cannot be the group of weights of some group algebra $\mathbb{R}[G]$. For if such a group exists and $\alpha$ is a semi-invariant with weight $\lambda$, then for all $g \in G$ $\lambda(g^4) = \lambda(g)^4 = 1$ since $\lambda \in C_4$. Within $\mathbb{R}$ this means $\lambda(g)$ is either 1 or $-1$. Thus $\lambda^2 = 1$ within $\Lambda(G)$. Hence $\Lambda(G)$ is not isomorphic to $C_4$.

## 2. The semicentre of $Q_{cl}(K[G])$

**2.1.** Since $K[G]$ is prime Noetherian, it has a classical ring of quotients $Q_{cl}(K[G])$ which is simple Artinian. We will denote this ring shortly by $Q$. The group $G$ is obviously contained in the units of $Q$ and so the definition of semi-invariant and semicentre of $Q$ makes sense by Definition 1.1. In this case we will denote the set of weights by $\Lambda_Q(G, K)$ (or shortly $\Lambda_Q(G)$).

**Remarks 2.2.** (1) As mentioned in proposition 1.2 and its proof, $\sum_{\lambda\in\Lambda_Q(G)} Q_\lambda$ is in fact a direct sum $\oplus_{\lambda\in\Lambda_Q(G)} Q_\lambda$.

(2) Let $\lambda \in \Lambda_Q(G)$ and $0 \neq \alpha \in Q_\lambda$. By definition of a semi-invariant $Q\alpha = \alpha Q$ and hence $\alpha$ is invertible. In particular $\Lambda_Q(G)$ is a monoid. From $g\alpha g^{-1} = \lambda(g)\alpha$ one obtains $g\alpha^{-1}g^{-1} = \lambda(g)^{-1}\alpha^{-1}$ for all $g \in G$, i.e., $\alpha^{-1}$ is a semi-invariant with weight $\lambda^{-1}$. Thus $\Lambda_Q(G)$ is an (abelian) group. Clearly $\Lambda(G) \subset \Lambda_Q(G)$; we will show later on that these two groups coincide.

(3) By (1) and (2) Sz$Q$ is a $\Lambda_Q(G)$-graded ring and $(\mathrm{Sz}Q)_1 = Z(Q_{cl}(K[G]))$ which equals $Q_{cl}(ZK[G])$ by [**19**, Theorem 4.4.5].

The following lemma is just a basic observation.

**Lemma 2.3.** *Let $\alpha \in Q_\lambda$. If $u, v \in K[G]$, $u$ regular, then $(u^{-1}v)\alpha = \alpha(\lambda^{\#}(u)^{-1}\lambda^{\#}(v))$ where $\lambda^{\#}(\sum u_g g) = \sum u_g \lambda(g) g$* (cf. [**15**, p. 397] *for the notation $\lambda^{\#}$*).

**Proposition 2.4.** *Sz$Q$ is the localisation of Sz$K[G]$ at the nonzero central elements of $K[G]$, i.e., $\mathrm{Sz}Q = (\mathrm{Sz}K[G])_{ZK[G]\setminus\{0\}}$.*

**Proof.** Obviously $(\mathrm{Sz}K[G])_{ZK[G]\setminus\{0\}}$ is contained in Sz$Q$. We show the converse inclusion. Let $\alpha \in Q_\lambda$. Denote $I_l = \{u \in K[G] | u\alpha \in K[G]\}$ and $I_r = \{u \in K[G] | \alpha u \in K[G]\}$. By Lemma 2.3 $I_l$ and $I_r$ are nonzero twosided ideals of $K[G]$. Thus $\alpha \in Q_s(K[G])$, the symmetric Martindale ring of quotients of $K[G]$. If $K[G]$ is prime and $G$ is polycyclic-by-finite, then $Q_s(K[G]) = K[G]_{ZK[G]\setminus\{0\}}$ by [**21**, Theorem 11.12] (or [**20**, Corollary 7.8] for a detailed proof). Thus $z\alpha \in K[G]$ for some nonzero central element $z$ in $K[G]$. Obviously $z\alpha \in (K[G])_\lambda$ which shows the result. □

**Corollary 2.5.** (1) *$\Lambda_Q(G) = \Lambda(G)$; in particular, $\Lambda_Q(G)$ is a finite abelian group;*

(2) *Sz$Q$ is a commutative domain;*

(3) *$\mathrm{Sz}Q \cong F^t[\Lambda(G)]$, a twisted group algebra, where $F = Q_{cl}(ZK[G])$;*

(4) *$\mathrm{Sz}Q = Q(\mathrm{Sz}K[G])$, the field of fractions of Sz$K[G]$.*

**Proof.** (1) and (2) follow immediately from Proposition 2.4.

(3) Let $\lambda \in \Lambda(G)$ and choose a nonzero element $\alpha_\lambda \in Q_\lambda$. Since $\alpha_\lambda^{-1} \in Q_{\lambda^{-1}}$ (cf.

Remarks 2.2 (2)) $Q_\lambda\alpha_\lambda^{-1} \subset F$ and thus $Q_\lambda \subset F\alpha_\lambda$. The converse inclusion is trivial and therefore $Q_\lambda = F\alpha_\lambda$. This shows that $\mathrm{Sz}Q = \oplus_{\lambda\in\Lambda(G)} F\alpha_\lambda$ which is obviously isomorphic to $F^t[\Lambda(G)]$.

(4) By (2) and (3) $\mathrm{Sz}Q$ is a commutative domain and finite dimensional over $F$; a classical argument shows that $\mathrm{Sz}Q$ is a field. Moreover,

$$\mathrm{Sz}K[G] \subset \mathrm{Sz}Q = (\mathrm{Sz}K[G])_{ZK[G]\setminus\{0\}} \subset Q(\mathrm{Sz}K[G]).$$

Since $\mathrm{Sz}Q$ is a field, every nonzero element of $\mathrm{Sz}K[G]$ is invertible in $\mathrm{Sz}Q$ and thus $\mathrm{Sz}Q = Q(\mathrm{Sz}K[G])$. □

The foregoing proposition and corollary show that $\mathrm{Sz}Q$ behaves quite differently from $\mathrm{Sz}D(L)$, where $D(L)$ denotes the division ring of quotients of a universal enveloping algebra of a finite dimensional Lie algebra $L$. In case of $D(L)$ one has $\mathrm{Sz}D(L) \cong ZD(L)[\Lambda_D(L)]$ [17], a group algebra over the torsion-free abelian group of weights $\Lambda_D(L)$. In particular $\mathrm{Sz}D(L)$ is finite dimensional over $ZD(L)$ only in case $\Lambda_D(L)$ is trivial. This is also the only case in which $\mathrm{Sz}D(L)$ is a field.

## 3. Centralizers of semi-invariants

We prove the analogue of [7, Prop. 1.15 and Cor. 1.16] in the case of a group algebra.

Denote $G_\Lambda = \bigcap_{\lambda\in\Lambda(G)} \ker \lambda$.

**Proposition 3.1.** *Let $\alpha$ be a semi-invariant of $K[G]$ with weight $\lambda$. Denote $H = \ker \lambda$. Then*

(1) $C_{K[G]}(\alpha) = K[H]$;

(2) $C_Q(\alpha) = Q_{cl}(K[H])$.

**Proof.** Since $H$ has finite index in $G$ we have $\Delta(H) \subset \Delta(G)$; conversely $\Delta(G) \subset C_G(\Delta) \subset \ker \lambda = H$ by Lemma 1.3. Thus $\Delta(H) = \Delta(G)$. Also note that by Proposition 1.2 $\alpha \in K[\Delta(G)] = K[\Delta(H)] \subset K[H]$.

(1) Clearly $K[H] \subset C_{K[G]}(\alpha)$. To prove the converse inclusion, let $\{g_1 = 1, g_2, \ldots, g_n\}$ be a transversal for $H$ in $G$. Then $K[G] = \oplus_{i=1}^n K[H]g_i$ and each element $u \in K[G]$ can be written in a unique way as $u = \sum_{i=1}^n u_i g_i$ where $u_i \in K[H]$ for all $i$. Let $u \in C_{K[G]}(\alpha)$ and write $u = \sum_{i=1}^n u_i g_i$ as before. Using the fact that $\alpha \in (K[G])_\lambda$, and that $\alpha$ and all $u_i$ belong to $K[H]$, we obtain

$$
\begin{aligned}
0 &= u\alpha - \alpha u \\
&= \sum u_i g_i \alpha - \sum \alpha u_i g_i \\
&= \sum u_i \lambda(g_i)\alpha g_i - \sum u_i \alpha g_i \\
&= \sum u_i \alpha(\lambda(g_i) - 1) g_i
\end{aligned}
$$

and thus $u_i\alpha(\lambda(g_i) - 1) = 0$ for all $i$. If $i \neq 1$, then $\lambda(g_i) \neq 1$ since $g_i \notin \ker \lambda = H$. Thus $u_i\alpha = 0$ and hence $u_i = 0$ because $\alpha$ is a regular element. This shows that $u = u_1 \in K[H]$.

(2) Clearly $Q_{cl}(K[H]) \subset C_Q(\alpha)$. Conversely, let $u \in C_Q(\alpha)$. By [**19**, Lemma 13.3.5]. $Q_{cl}(K[G]) = \{\beta^{-1}\gamma | \gamma \in K[G], \beta$ a regular element of $K[H]\}$. Thus $\beta u = \gamma$ for some $\gamma \in K[G]$ and some regular element $\beta \in K[H]$. Since $\beta$ and $u$ commute with $\alpha$, the same holds for $\gamma$ and by (1) $\gamma \in K[H]$. Therefore $u \in Q_{cl}(K[H])$. □

**Proposition 3.2.** (1) $C_{K[G]}(\mathrm{Sz}K[G]) = K[G_\Lambda]$;

(2) $C_Q(\mathrm{Sz}K[G]) = Q_{cl}(K[G_\Lambda])$.

**Proof.** (1) $C_{K[G]}(\mathrm{Sz}K[G]) = \bigcap_{\alpha \text{ semi-invariant}} C_{K[G]}(\alpha) = \bigcap_{\lambda \in \Lambda(G)} K[\ker \lambda] = K[G_\Lambda]$;

(2) This is shown in the same way as (1), using again [**19**, Lemma 13.3.5]. □

## 4. Structure of Sz$K[G]$

We already know by Proposition 1.2 that Sz$K[G]$ is a commutative domain. In this section we show that Sz$K[G]$ has a much richer structure. To simplify notations we will denote $C_G(\Delta)$ in this section by $C$.

**Lemma 4.1.** *$ZK[G] = K[\Delta]^G = K[\Delta]^{\overline{G}}$ where $\overline{G} = G/C$.*

**Proposition 4.2.** *$\mathrm{Sz}K[G] = K[\Delta]^{G_\Lambda} = K[\Delta]^{\overline{G}_\Lambda}$ where $\overline{G}_\Lambda = G_\Lambda/C$.*

**Proof.** The proof is a slight change of the proof of [**22**, Lemma 3]; for the sake of completeness we include the main details. First note that $G/G_\Lambda$ is a finite abelian group because $G' \subset G_\Lambda$ and $C_G(\Delta) \subset G_\Lambda$. If $G_\Lambda = G$, then $\lambda(G_\Lambda) = \lambda(G) = 1$ for all $\lambda \in \Lambda(G)$, i.e., $\lambda = 1$. In particular $\Lambda(G) = 1$ and thus $\mathrm{Sz}K[G] = ZK[G] = K[\Delta]^G = K[\Delta]^{G_\Lambda}$ by Lemma 4.1.

If $G_\Lambda \neq G$, we claim that $G/G_\Lambda$ is $K$-complete. Let $x \in G \backslash G_\Lambda$; then for some $\lambda \in \Lambda(G)$ we have $\lambda(x) \neq 1$. Since $\lambda(G_\Lambda) = 1$, the map $\lambda : G \to K^*$ can be lifted to $\overline{\lambda} : G/G_\Lambda \to K^*$. So $\overline{\lambda}(xG_\Lambda) = \lambda(x) \neq 1$. Thus $G/G_\Lambda$ is $K$-complete.

By definition of $G_\Lambda$ it is obvious that $\mathrm{Sz}K[G] \subset K[\Delta]^{G_\Lambda}$. To prove the converse inclusion, let $a \in \Delta$; if $C(a)$ denotes the centralizer of $a$ in $G$, then $C = C_G(\Delta) \subset C(a)$.

Let $Q$ be a (finite) transversal for $C$ in $G_\Lambda$. Define $s_a = \sum_{q\in Q} a^q$; clearly $s_a$ is independent of the choice of $Q$; moreover $K[\Delta]^{G_\Lambda}$ is spanned by elements of the form $s_a$. Let $\lambda \in \Lambda(G)$ and $U$ be any transversal for $C$ in $G$. Define $s_\lambda = \sum_{\sigma\in U} \lambda(\sigma^{-1})a^\sigma$. Then $s_\lambda$ is independent of the choice of $U$ and $s_\lambda \in K[G]_\lambda$. In particular, if $P = \{p_1 = 1, \ldots, p_n\}$ is a transversal for $G_\Lambda$ in $G$ and $Q = \{q_1, \ldots, q_m\}$ is defined as before, then $PQ$ is easily seen to be a transversal for $C$ in $G$. A straightforward calculation as in [**22**, Lemma 3] shows that

$$s_\lambda = \sum_{i=1}^{n} \lambda(p_i^{-1}) s_a^{p_i}.$$

As mentioned before, $\lambda$ can be lifted to $\bar{\lambda} : G/G_\Lambda \to K^*$. Since $G/G_\Lambda$ is $K$-complete, by [**19**, Lemma 4.3.3] there exist $\bar{\lambda}_1, \ldots, \bar{\lambda}_n \in \mathrm{Hom}(G/G_\Lambda, K^*)$ such that $\det(\lambda_i(p_j^{-1})) \neq 0$. By elementary linear algebra each $s_a^{p_i}$ is a linear combination of $s_{\lambda_1}, \ldots, s_{\lambda_n}$, thus belongs to $\mathrm{Sz}K[G]$. In particular $s_a = s_a^{p_1} \in \mathrm{Sz}K[G]$. This shows that $K[\Delta]^{G_\Lambda} \subset \mathrm{Sz}K[G]$. □

In the case that the field of coefficients is algebraically closed, M. Smith proved that the result of Proposition 4.2 can be sharpened – and simplified – by replacing $G_\Lambda$ by $G'$.

**Proposition 4.3.** (1) $\mathrm{Sz}\bar{K}[G] = \bar{K}[\Delta]^{G'} = \bar{K}[\Delta]^{G'C/C}$;

(2) $\mathrm{Sz}\bar{K}[G] \cap K[G] = K[\Delta]^{G'} = K[\Delta]^{G'C/C}$.

**Proof.** (1) As mentioned, this is proved in [**22**, Lemma 3].

(2) By (1) $\mathrm{Sz}\bar{K}[G] \cap K[G] = \bar{K}[\Delta]^{G'} \cap K[G] = K[\Delta]^{G'}$. □

Example 6.4 shows that it is possible that $\mathrm{Sz}K[G] \subsetneq (\mathrm{Sz}\bar{K}[G] \cap K[G])$. In some cases however, equality holds.

**Lemma 4.4.** *Let $K$ and $L$ be fields with $K \subset L$. Write $L = \oplus_{i\in I} a_i K$ for some index set $I$ and choose $a_1 = 1$. If $\Lambda = \Lambda(G, K) = \Lambda(G, L)$ (i.e., $\lambda(G) \subset K^*$ for all $\lambda \in \Lambda(G, L)$), then*

(1) *for each $\lambda \in \Lambda$ $(L[G])_\lambda = \oplus_{i\in I} a_i (K[G])_\lambda$;*

(2) $\mathrm{Sz}L(G) \cap K[G] = \mathrm{Sz}K[G]$.

**Proof.** Obvious. □

**Corollary 4.5.** *If $\Lambda(G, K) = \Lambda(G, \bar{K})$, then* $\mathrm{Sz}K[G] = \mathrm{Sz}(\bar{K}[G]) \cap K[G] = K[\Delta]^{G'}$.

**Corollary 4.6.** *Given a field $K$ there exists a finite extension $L$ of $K$ such that* $\mathrm{Sz}L[G] = L[\Delta]^{G'}$.

**Proof.** Consider $\Lambda(G,\overline{K})$, which is a finite group by Proposition 1.6. Write $\Lambda(G,\overline{K})=\{\lambda_1,\ldots,\lambda_m\}$ for some $m$. Also $G$ is finitely generated, say by $g_1,\ldots,g_n$ for some $n$. Let $L$ be the field generated by $K$ and all $\lambda_i(g_j)$, where $1\le i\le m$ and $1\le j\le n$. Since each $\lambda_i(g_j)\in\overline{K}^*$, it is clear that $L$ is a finite dimensional extension of $K$. By construction of $L$ we have $\lambda_i(G)\subset L^*$ for all $i$. Thus $\Lambda(G,\overline{K})=\Lambda(G,L)$. The result follows now from Corollary 4.5. □

**Proposition 4.7.** *$ZK[G]$, $\mathrm{Sz}K[G]$ and $\mathrm{Sz}\overline{K}[G]\cap K[G]$ are finitely generated K-algebras and Noetherian Krull domains.*

**Proof.** By 4.1, 4.2 and 4.3 the rings mentioned above are fixed rings of $K[\Delta]$ under a finite group. Noether's theorem implies that these rings are finitely generated $K$-algebras and thus Noetherian. Since

$$ZK[G]=K[\Delta]^G=K[\Delta]\cap L^G,$$

where $L$ is the field of fractions of $K[\Delta]$, it is trivial to see that $ZK[G]$ is a Krull domain. For the two other rings, the proof is similar. □

In contrast to Lie algebras, an example due to the author which appeared in [**17**] shows that $ZU(L)$ and $\mathrm{Sz}U(L)$ need not be Noetherian. We have shown now that $ZK[G]$, $\mathrm{Sz}K[G]$ and $\mathrm{Sz}(\overline{K}[G])\cap K[G]$ are Krull domains. A natural question is whether these rings are also *UFD*'s, and if not in general when are they a *UFD*? Note that in the case of a universal enveloping algebra $U(L)$ of a finite dimensional Lie-algebra $L$, the semicentre $\mathrm{Sz}U(L)$ is always a *UFD*, as is well-known [**12**, **9**].

Therefore S. Montgomery asked whether the semicentre of a prime group algebra is a *UFD*, as is mentioned in the introduction of [**22**]. M. Smith answers this question in the negative sense. In the same paper [**22**], M. Smith states that it may be of interest to determine necessary and sufficient conditions for $\mathrm{Sz}K[G]$ to be a *UFD*. In particular she asks whether $\mathrm{Sz}K[G]$ is a *UFD* in case $K[G]$ is a *UFR* in the sense of Chatters and Jordan [**6**]. We show that the example of a group algebra $K[G]$ – given by M. Smith in [**22**] – such that $\mathrm{Sz}K[G]$ is not a *UFD*, is such that $G$ is polycyclic-by-finite and $K[G]$ is a prime *UFR* (cf. Example 6.3).

Quite recently, M. Lorenz described the class group of a ring of multiplicative invariants [**10**]. To be a little more precise, recall that $\mathrm{Sz}K[G]=K[\Delta]^{\overline{G}_\Lambda}$ where $\overline{G}_\Lambda=G_\Lambda/C$ is a finite group which acts on the finitely generated free abelian group $\Delta$. By identifying $\Delta$ with $\mathbb{Z}^d$ for some $d$, $\overline{G}_\Lambda$ becomes a finite subgroup of $GL_d(\mathbb{Z})\cong GL(\Delta)$. In particular $SL(\Delta)\cong SL_d(\mathbb{Z})$ are the elements of $GL_d(\mathbb{Z})$ having determinant 1. If $N$ denotes the (normal) subgroup of $\overline{G}_\Lambda$ generated by all the reflections in $\overline{G}_\Lambda$ and $D$ the (normal) subgroup generated by the reflections that are diagonalisable over $\mathbb{Z}$, then the following result holds.

**Proposition 4.8** (M. Lorenz [**10**]). *The class group of* $\mathrm{Sz}K[G]$ *is isomorphic to*

$$Cl(\mathrm{Sz}K[G]) \cong \mathrm{Hom}(\overline{G}_\Lambda/N, K^*) \oplus H^1(\overline{G}_\Lambda/D, \Delta^D). \qquad \square$$

By replacing $\overline{G}_\Lambda$ by $\overline{G}$ (resp. $G'C/C$) one obtains a similar result for the class group of $ZK[G]$ and $\mathrm{Sz}(\overline{K}[G]) \cap K[G]$. The formula in Proposition 4.8 already indicates that the property of being a *UFD* will not only depend on $G$ but also on the field $K$.

The following result is due to K. A. Brown and M. Lorenz [**5**] and is somewhat weaker than Proposition 4.8, but will turn out to be more practical in some concrete cases.

**Proposition 4.9** (K. A. Brown, M. Lorenz [**5**]). (1) *$Cl(ZK[G])$ is a subgroup of* $\mathrm{Hom}(\overline{G}, K^*) \times H^1(\overline{G}, \Delta)$;

(2) *$Cl(\mathrm{Sz}K[G])$ is a subgroup of* $\mathrm{Hom}(\overline{G}_\Lambda, K^*) \times H^1(\overline{G}_\Lambda, \Delta)$;

(3) *$Cl(\mathrm{Sz}(\overline{K}[G]) \cap K[G])$ is a subgroup of* $\mathrm{Hom}(G'C/C, K^*) \times H^1(G'C/C, \Delta)$;

(4) *if $K$ is algebraically closed, then $Cl(\mathrm{Sz}K[G])$ is a subgroup of* $\mathrm{Hom}(G'C/C, K^*)\times H^1(G'C/C, \Delta)$.

**Corollary 4.10.** (1) *If* $\mathrm{Hom}(\overline{G}, K^*) = \{1\}$ *and* $H^1(\overline{G}, \Delta) = \{1\}$, *then* $\mathrm{Sz}K[G] = ZK[G]$ *and is a UFD;*

(2) *If $(G : C)$ is odd and $H^1(\overline{G}, \Delta) = \{1\}$, then $\mathrm{Sz}\mathbb{R}[G] = Z\mathbb{R}[G]$ and is a UFD.*

**Proof.** (1) This is obvious by Corollary 1.7 and Proposition 4.9(1).

(2) An elementary calculation shows that $\mathrm{Hom}(\overline{G}, \mathbb{R}^*) = \{1\}$ because $|\overline{G}|$ is odd. $\square$

## 5. Normal elements versus semi-invariants

If $L$ is a finite dimensional Lie-algebra, then $u \in U(L)$ is a normal element if and only if $u$ is a semi-invariant (see [7, Proposition 1.8] or [**21**, Corollary 13.8]). In case of a group algebra $K[G]$ this is no longer true, because any unit in $K[G]$ is trivially a normal element but a unit $u$ is only a semi-invariant if $u = kg$ where $k \in K^*$ and $g \in Z(G)$. Therefore the best we can hope is that every normal element is the product of a unit and a semi-invariant. In general this will not be the case. In Theorem 5.3 we will prove necessary and sufficient conditions such that every normal element which belongs to $K[\Delta]$ is the product of an element of $\Delta$ and a semi-invariant. In case $K[G]$ is a *UFR* in the sense of Chatters and Jordan, the restriction to normal elements belonging to $K[\Delta]$ won't be a real restriction.

**Lemma 5.1.** (1) *If $\alpha$ is a normal element of $K[G]$, then for each $g \in G$ there exists a unit $v_g$ of $K[G]$ such that $\alpha^g = g\alpha g^{-1} = \alpha v_g$.*

(2) *If $\alpha$ is a normal element belonging to $K[\Delta]$, then $v_g = k_g u_g$ where $k_g \in K^*$ and $u_g \in \Delta$.*

**Proof.** (1) Let $g \in G$; since $\alpha$ is normal

$$(g\alpha g^{-1})K[G] = g\alpha K[G] = gK[G]\alpha = K[G]\alpha = \alpha K[G]$$

and the fact that $\alpha$ and $g\alpha g^{-1}$ are regular implies that $g\alpha g^{-1} = \alpha v_g$ for some unit $v_g$ of $K[G]$.

(2) If $\alpha$ is normal and $\alpha \in K[\Delta]$, then $v_g \in K[\Delta]$ because $\alpha^g \in K[\Delta]$. The fact that $v_g$ is a unit in $K[\Delta]$ and $\Delta$ is torsion-free abelian implies that $v_g = k_g u_g$ for some $k_g \in K^*$ and $u_g \in \Delta$. □

Up to a slight difference in notation, the following lemma is proved in [**15**, Lemma 2] and [**16**, Lemma 1(i)].

**Lemma 5.2.** *Let $\sigma \in \operatorname{Aut} G$ centralise a subgroup of finite index. Let $W = C_G(\sigma) = \{g \in G \mid \sigma(g) = g\}$ and $T$ be a left transversal of $W$ in $G$. Denote*

$$\alpha = \sum_{t \in T} t^{\sigma} t^{-1}.$$

*Then $\alpha$ is a normal element of $K[G]$ belonging to $K[\Delta]$ such that $g^{\sigma}\alpha g^{-1} = \alpha$ for all $g \in G$.*

**Theorem 5.3.** *The following conditions are equivalent:*

(1) *every normal element of $K[G]$ belonging to $K[\Delta]$ can be written as us where $u \in \Delta$ and s is a semi-invariant;*

(2) *if $\sigma \in \operatorname{Aut} G$ centralise a subgroup $H$ of finite index, then $\sigma$ is an inner automorphism of $G$;*

(3) $H^1(G/C_G(\Delta), \Delta) = \{1\}$.

*In the case that these conditions are satisfied, the decomposition of a normal element into a product of an element of $\Delta$ and a semi-invariant is unique up to a central element of $G$.*

**Proof.** (1) ⇒ (2): Let $\sigma \in \operatorname{Aut} G$ be such that $\sigma$ centralises a subgroup of finite index. If $W = C_G(\sigma)$, then $(G : W)$ is finite. Using the result and the notations as in Lemma 5.2, $\alpha = \sum_{t \in T} t^{\sigma} t^{-1}$ is a normal element of $K[G]$ belonging to $K[\Delta]$. By (1) $\alpha$ can be written as $\alpha = us$, where $u \in \Delta$ and $s \in (K[G])_{\lambda}$ for some weight $\lambda$. For all $g \in G$

$$\alpha^g = u^g s^g = \lambda(g) u^g s = \lambda(g) u^g u^{-1} \alpha. \qquad (*)$$

By Lemma 5.2

$$\alpha^g = g(g^{-1})^{\sigma} g^{\sigma} \alpha g^{-1} = g(g^{-1})^{\sigma} \alpha. \qquad (**)$$

A combination of (*) and (**), using the fact that $\alpha$ is regular, yields

$$g(g^{-1})^{\sigma} = \lambda(g)u^{g}u^{-1}.$$

For a start this implies that $\lambda(G) = \{1\}$, i.e., $s$ is a central element. Secondly $g(g^{-1})^{\sigma} = u^{g}u^{-1}$ implies $(g^{-1})^{\sigma} = ug^{-1}u^{-1}$, i.e., $\sigma$ is an inner automorphism.

(2) $\Leftrightarrow$ (3): This holds even more generally, as shown by K. A. Brown in [**4**, p. 85].

(3) $\Rightarrow$ (1): Let $\alpha$ be a normal element of $K[G]$ belonging to $K[\Delta]$. By Lemma 5.1(2), for all $g \in G$ we have $g\alpha g^{-1} = k_g u_g \alpha$ for some $k_g \in K^*$ and $u_g \in \Delta$. Note that if $g \in C$ then $g\alpha g^{-1} = \alpha$, so $k_g = 1$ and $u_g = 1$. Hence the following map is well-defined:

$$G/C \to K[G] : \overline{g} \mapsto \alpha^{g} = g\alpha g^{-1}.$$

For all $g, h \in G, (gh)\alpha(gh)^{-1} = k_{gh}u_{gh}\alpha$ and $(gh)\alpha(gh)^{-1} = g(h\alpha h^{-1})g^{-1} = g(k_h u_h \alpha)g^{-1} = k_g k_h u_g u_h^{g}\alpha$, which implies that $k_{gh} = k_g k_h$ and $u_{gh} = u_g u_h^{g}$. Therefore

$$f_{\alpha} : G/C \to \Delta : \overline{g} \mapsto u_g$$

is a 1-cocycle. By (3) $f_{\alpha}$ is a 1-coboundary, so there exists $v \in \Delta$ such that $u_g = f_{\alpha}(\overline{g}) = v^{-1}v^{\overline{g}}$ for all $\overline{g} \in G/C$. Let $s = v^{-1}\alpha$. Using the fact that $\Delta$ is abelian a straightforward calculation shows that $s$ is a semi-invariant with weight $k$. Since $\alpha = vs$, this shows (1).

Finally suppose $\alpha = us = u's'$, where $u, u' \in \Delta$ and $s$ (resp. $s'$) are semi-invariants with weight $\lambda$ (resp. $\lambda'$). Then $s = u''s'$ where $u'' = u^{-1}u' \in \Delta$. Using the fact that $s$ and $s'$ are semi-invariants we obtain

$$\lambda(g)s = s^{g} = u''^{g}s'^{g} = \lambda'(g)u''^{g}s' = \lambda'(g)u''^{g}u''^{-1}s,$$

which implies that $\lambda = \lambda'$ and $u''^{g} = u''$ for all $g \in G$, i.e., $u'' \in Z(G)$. □

**Proposition 5.4.** *Let $K[G]$ be a UFR such that $H^1(G/C, \Delta) = \{1\}$.*

(1) *Every height one prime ideal $P$ of $K[G]$ is generated by a semi-invariant.*

(2) *A normal element of $K[G]$ is the product of a unit of $K[G]$ and a semi-invariant.*

(3) *Every semi-invariant can uniquely (up to an element of $K^*$ and of $Z(G)$) be written as a product of irreducible semi-invariants.*

**Proof.** Note first that a Noetherian *UFR* is a maximal order [**6**, Theorem 2.4]. In this case a divisorial prime ideal is the same as a height one prime ideal. Moreover the group of divisorial ideals is a free abelian group generated by the height one prime ideals.

(1) This is clear from the fact that $K[G]$ is a *UFR*, that $P$ is generated by a normal element belonging to $K[\Delta]$ [**3**, Theorem B] and Theorem 5.3.

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(2) Let $p$ be a normal element of $K[G]$. If $p$ is not a unit of $K[G]$ then $K[G]p$ is a divisorial ideal of $K[G]$. Using the fact that each height one prime ideal $P$ is generated by a semi-invariant $s$, we have

$$p = us_1 \ldots s_n$$

for some $n, u$ a unit of $K[G]$ and each $s_i$ a semi-invariant.

(3) This is proved in the same way as (2), using the fact that in the expression $s = us_1 \ldots s_n$ the unit $u$ of $K[G]$ belongs to $K[\Delta]$ and is a semi-invariant, thus $u \in K^* \cdot Z(G)$. The uniqueness of this decomposition follows from the fact that $K[G]$ is a $UFR$. □

## 6. Examples

In this section we give a number of examples which either illustrate some of the results in the foregoing section or either illustrate some differences with the semicentre of a universal enveloping algebra of a Lie-algebra, especially concerning the question of when the semicentre is a $UFD$.

**6.1 Example 1.** Let $G$ be the group generated by $x$ and $y$ such that $yxy^{-1} = x^{-1}$. Clearly $G$ is torsion-free poly-infinite cyclic, so $K[G]$ is a Noetherian domain which is a maximal order by [3, Theorem F] and even a $UFR$, because $K[G]$ is clearly a $PI$-ring, by [3, Theorems C and D]. As can easily be checked $\Delta = \Delta(G) = C_G(\Delta) = C = \langle x, y^2 \rangle$ and $G' = \langle x^2 \rangle$. For any field $K$ we have $\mathrm{Hom}(G/C, K^*)$ is cyclic of order two; thus by Corollary 4.5 $\mathrm{Sz}K[G] = K[\Delta]^{G'} = K[\Delta]$, because $G' \subset C$.

Using the fact and notation that $G/C \cong \langle \bar{y} \rangle$ where $\bar{y}^2 = 1$, we obtain by direct calculation that a map $\varphi$ from $G/C$ to $\Delta$ is a 1-cocycle if $\varphi(\bar{y}) = x^i$ ($i \in \mathbb{Z}$) and $\varphi$ is a 1-coboundary if $\varphi(\bar{y}) = x^{2i}$ ($i \in \mathbb{Z}$). Thus $H^1(G/C, \Delta) \cong C_2$. By Theorem 5.3, not every normal element of $K[G]$ which belongs to $K[\Delta]$ can be written as the product of an element of $\Delta$ and a semi-invariant. A concrete example is the following: let $p = 1 + x$; as is directly checked $p$ is a normal element belonging to $K[\Delta]$. Using the fact that $ypy^{-1} = x^{-1}p$, the map $\varphi_p : G/C \to \Delta$ is such that $\varphi_p(\bar{y}) = x^{-1}$ is a 1-cocycle but not a 1-coboundary. A straightforward calculation shows that a normal element $p$ which can be written as $us$ where $u \in \Delta$ and $s$ a semi-invariant induces a 1-coboundary $\varphi_p$.

A useful property in enveloping algebras of finite dimensional Lie-algebras is the fact that $u$ and $v$ are semi-invariants if $uv$ is a semi-invariant [13]. This need not hold anymore for group algebras $K[G]$. Consider this group $G$. Let $u = (1 + x)y$; then $u^2 = (2 + x + x^{-1})y^2 \in ZK[G]$. Let $v = 2 + x$ and $w = 2 + x^{-1}$, then $vw = 5 + 2(x + x^{-1}) \in ZK[G]$. In particular $u^2$ and $vw$ are semi-invariants but neither $u, v$ or $w$ is a semi-invariant, in fact $v$ and $w$ are not even normal elements.

**6.2 Example 2.** This example can be found in [**11**, pp. 383–384]. Let $A$ be the free abelian group with basis $a, b, c$ and $d$, and let $\langle z\rangle$ be an infinite cyclic group acting on $A$ via an automorphism $\varphi_z$, where $\varphi_z(a) = a$, $\varphi_z(b) = b^{-1}a$, $\varphi_z(c) = cb$ and $\varphi_z(d) = d^{-1}c$. Let $G$ be the semidirect product of $A$ and $\langle z\rangle$. As mentioned in [**11**], $G$ is torsion-free nilpotent-by-finite and by [**11**, Proposition 5.4] every nonzero ideal of $K[G]$ intersects $ZK[G]$ nontrivially. By [**3**, Theorems C and D], $K[G]$ is a $UFR$. One has $\Delta = \Delta(G) = \langle a, b\rangle$ and $C = C_G(\Delta) = \langle A, z^2\rangle$; thus $G/C \cong \langle\bar{z}\rangle$ and $\bar{z}^2 = 1$, which implies that $\Lambda(G, K) \cong \mathrm{Hom}(G/C, K^*) \cong C_2$ for all fields $K$. On the other hand, $\Delta \subset G' = \langle\Delta, c^{-1}d^2\rangle \subset C$. Corollary 4.5 implies that $\mathrm{Sz}K[G] = K[\Delta]^{G'} = K[\Delta]$ for any field $K$. A straightforward calculation shows that every 1-cocycle from $G/C$ to $\Delta$ is a 1-coboundary. Thus $H^1(G/C, \Delta) = \{1\}$ and the results of Theorem 5.3 and Proposition 5.4 apply.

**6.3 Example 3.** This example appears in [**22**, Example 1], in which the author proves that the semicentre is not a $UFD$ in case $K$ is algebraically closed. We will show precisely for which fields the semicentre is a $UFD$; in the other case we will compute the class group of the semicentre. The next two examples are variants of this construction.

Let $A$ be free abelian on $a_1, b_1, c_1, a_2, b_2, c_2$ and let $H$ be generated by $\sigma$ and $\tau$ such that $\sigma\tau\sigma^{-1} = \tau^{-1}$ ($H$ is thus the group used in Example 1). Let $\varphi : H \to \mathrm{Aut}\, A$ be a homomorphism defined by $\varphi(\sigma)$ (denoted in brief by $\varphi_\sigma$) for which $\varphi_\sigma(a_i) = b_i$, $\varphi_\sigma(b_i) = a_i$, $\varphi_\sigma(c_i) = c_i$ ($i \in \{1, 2\}$) and $\varphi_\tau$ is defined by cyclic permutation of $a_1, b_1, c_1$ and $a_2, b_2, c_2$. Let $G$ be the semidirect product of $A$ and $H$. Clearly $G$ is torsion-free poly-infinite cyclic. Then $\Delta = \Delta(G) = \langle A, \sigma^2, \tau^3\rangle$. Thus $(G : \Delta)$ is finite, $G$ is abelian-by-finite and $K[G]$ is a $PI$-ring. Using [**3**, Theorems C and D], $K[G]$ is a $UFR$. Moreover $C = C_G(\Delta) = \Delta$ and $G/C$ is isomorphic to the symmetric group of degree 3. Then $\Lambda(G, K) \cong \mathrm{Hom}(G/C, K^*) \cong C_2$ for any field $K$. Then $\overline{G}_\Lambda \cong \langle\bar{\tau}\rangle$ is cyclic of order three and we have $\overline{G}_\Lambda \subset SL(\Delta)$, as is readily checked (the notation $SL(\Delta)$ is mentioned just before Proposition 4.8). In particular $\overline{G}_\Lambda$ contains no reflections, since a reflection has determinant $-1$ (see e.g., [**10**]). To show that $H^1(\overline{G}_\Lambda, \Delta) = \{1\}$, let $f$ be a 1-cocycle defined by $f(\bar{\tau}) = p\sigma^{2i}\tau^{3j}$, where $p \in A$ and $i, j, \in \mathbb{Z}$. Using the fact that $\bar{\tau}^3 = 1$, direct calculation shows that $i = j = 0$ and $f(\bar{\tau}) = p = (a_1c_1^{-1})^{\alpha_1}(b_1c_1^{-1})^{\beta_1}(a_2c_2^{-1})^{\alpha_2}(b_2c_2^{-1})^{\beta_2}$ for some $\alpha_1, \alpha_2, \beta_1, \beta_2 \in \mathbb{Z}$. Let $u = b_1^{-\beta_1}c_1^{\alpha_1}b_2^{-\beta_2}c_2^{\alpha_2}$; then $f(\bar{\tau}) = u^{-1}u^{\bar{\tau}}$, i.e., $f$ is a 1-coboundary. Proposition 4.8 implies that $Cl(\mathrm{Sz}K[G]) \cong \mathrm{Hom}(\overline{G}_\Lambda, K^*)$. Obviously $\mathrm{Sz}K[G]$ is a $UFD$ if and only if $K$ does not contain a primitive third root of unity. In the other case $Cl(\mathrm{Sz}K[G]) \cong C_3$.

**6.4 Example 4.** This example is a slight variation of the foregoing example. Let again $A$ be the free abelian group on $a_1, b_1, c_1, a_2, b_2, c_2$ and $H = \langle\tau\rangle$ be infinite cyclic. Let $\varphi : H \to \mathrm{Aut}\, A$ be defined by $\varphi(\tau)$ which permutes $\{a_1, b_1, c_1\}$ and $\{a_2, b_2, c_2\}$ cyclically. Let $G$ be the semidirect product of $A$ and $H$. Then $\Delta = \Delta(G) = \langle A, \tau^3\rangle = C_G(\Delta) = C$ and $G' = \langle a_i^{-1}b_i, b_i^{-1}c_i \mid i \in \{1, 2\}\rangle$. Note that $G' \subset C$. Since $G/C \cong C_3$, we have $\Lambda(G, K) \cong \mathrm{Hom}(G/C, K^*) = \{1\}$ if $K$ does not contain a primitive third root of unity; in the other case $\mathrm{Hom}(G/C, K^*) \cong C_3$. A computation similar to the one in Example 6.3 shows that $H^1(G/C, \Delta) = \{1\}$.

This leads to the following result. If $K$ does not contain a primitive third root of unity, then Corollary 4.10 implies that $\mathrm{Sz}K[G] = ZK[G]$ is a $UFD$. If $K$ does contain a primitive third root of unity $\Lambda(G, K) = \Lambda(G, \overline{K})$, so $\mathrm{Sz}K[G] = K[\Delta]^{G'} = K[\Delta]$ by Corollary 4.5 and the fact that $G' \subset \Delta$. Since $\overline{G} = G/C \cong C_3$ and, as in Example 6.3, $\overline{G} \subset SL(\Delta)$, so that $\overline{G}$ contains no reflections. Proposition 4.8 used for $ZK[G]$ shows that $Cl(ZK[G]) \cong \mathrm{Hom}(\overline{G}, K^*) \oplus H^1(\overline{G}, \Delta) \cong C_3$. In this example $\mathrm{Sz}K[G]$ is a $UFD$ for all fields while $ZK[G]$ is not. Finally, $\mathrm{Sz}K[G] \subsetneq \mathrm{Sz}(\overline{K}[G]) \cap K[G]$ if $K$ does not contain a primitive third root of unity.

**6.5 Example 5.** Let $A$ be the free abelian group on $x_1, y_1, x_2, y_2$ and let $H = \langle \sigma \rangle$ be infinite cyclic. Let $\varphi : H \to \mathrm{Aut}\, A$ be defined by $\varphi(\sigma)(x_i) = y_i^{-1}$ and $\varphi(\sigma)(y_i) = x_i$ where $i \in \{1, 2\}$. Let $G$ be the semidirect product of $A$ and $H$. Then $\Delta = \Delta(G) = \langle A, \sigma^4 \rangle = C_G(\Delta) = C$ and $G' = \langle x_1y_1, x_1y_1^{-1}, x_2y_2, x_2y_2^{-1} \rangle$ ($\subset \Delta$). Since $G/C$ is cyclic of order 4, we have $\mathrm{Hom}(G/C, K^*) \cong C_2$ if $K$ does not contain a primitive 4th root of unity and $\mathrm{Hom}(G/C, K^*) \cong C_4$ if $K$ does contain a primitive 4th root of unity. In the last case, $\Lambda(G, K) = \Lambda(G, \overline{K})$ and thus by Corollary 4.5 $\mathrm{Sz}K[G] = K[\Delta]^{G'} = K[\Delta]$, which clearly is a $UFD$. If $K$ does not contain a primitive 4th root of unity, we claim that $\mathrm{Sz}K[G]$ is not a $UFD$. Since $\Lambda(G, K) \cong C_2$, we have $\overline{G}_\Lambda = G_\Lambda/C \cong \langle \sigma^2 \rangle \cong C_2$. Now $\overline{G}_\Lambda \subset SL(\Delta)$:

$$\overline{G}_\Lambda \to SL(\Delta)$$
$$\overline{\sigma^2} \mapsto \begin{pmatrix} -1 & & & & 0 \\ & -1 & & & \\ & & -1 & & \\ & & & -1 & \\ 0 & & & & 1 \end{pmatrix}$$

and therefore $\overline{G}_\Lambda$ contains no reflections. By Proposition 4.8 $Cl(\mathrm{Sz}K[G]) \cong \mathrm{Hom}(\overline{G}_\Lambda, K^*) \oplus H^1(\overline{G}_\Lambda, \Delta)$ and $\mathrm{Hom}(\overline{G}_\Lambda, K^*)$ is always nontrivial, thus so is $Cl(\mathrm{Sz}K[G])$ and $\mathrm{Sz}K[G]$ is not a $UFD$. By direct computation we have

$$\mathrm{Sz}K[G] = K[\sigma^4, \sigma^{-4}]\langle x_i^k y_j^\ell + x_i^{-k} y_j^{-\ell} \mid i, j \in \{1, 2\}, k, \ell \in \mathbb{Z} \rangle.$$

Comparing Examples 6.3 and 6.5 leads to the following observation. Let $G$ be the group as in Example 6.3; then $\mathrm{Sz}\mathbb{R}[G]$ is a $UFD$ but $\mathrm{Sz}\mathbb{C}[G]$ is not a $UFD$. In Example 6.5 the converse happens: $\mathrm{Sz}\mathbb{R}[G]$ is not a $UFD$ while $\mathrm{Sz}\mathbb{C}[G]$ is a $UFD$.

**6.6 Example 6.** If $L$ is a finite dimensional Lie algebra, then $\mathrm{Sz}U(L)$ is never trivial, i.e., equal to $K$, because every non-zero ideal of $U(L)$ contains a semi-invariant. This example shows that for a group algebra $K[G]$ the semicentre can be trivial. This example appears e.g., in [**1**, p. 195]. Let $A$ be a free abelian group with basis $y$ and $z$ and $H = \langle x \rangle$ be infinite cyclic. Let $H$ act on $A$ by

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$$H \to \text{Aut } A : x \mapsto \varphi_x$$

where

$$\varphi_x(y) = y^e z^g \text{ and } \varphi_x(z) = y^f z^h,$$

such that the matrix $\begin{pmatrix} e & f \\ g & h \end{pmatrix}$ has no integer eigenvalues. Let $G$ be the semidirect product of $A$ by $H$. A straightforward calculation shows that $\Delta(G) = \{1\}$, thus $\text{Sz}K[G] = ZK[G] = K$. As mentioned in [1], $K[G]$ is not a $UFR$ in the sense of Chatters and Jordan.

## REFERENCES


**1.** G. Q. Abbasi, S. Kobayashi, H. Marubayashi, and A. Ueda, Noncommutative unique factorization rings, *Comm. Algebra* **19** (1991), 167–198.

**2.** D. F. Anderson, Graded Krull domains, *Comm. Algebra* **7** (1979), 79–106.

**3.** K. A. Brown, Height one primes of polycyclic group rings, *J. London Math. Soc.* **32** (1985), 426–438.

**4.** K. A. Brown, Class groups and automorphism groups of group rings, *Glasgow Math. J.* **28** (1986), 79–86.

**5.** K. A. Brown and M. Lorenz, Grothendieck groups of invariant rings and of group rings, *J. Algebra* **166** (1994), 423–454.

**6.** A. W. Chatters and D. A. Jordan, Non-commutative unique factorization rings, *J. London Math. Soc* **33** (1986), 22–32.

**7.** L. Delvaux, E. Nauwelaerts and A. I. Ooms, On the semi-centre of a universal enveloping algebra, *J. Algebra* **94** (1985), 324–346.

**8.** D. F. Farkas, Multiplicative invariants, *Enseign. Math.* **30** (1984), 141–157.

**9.** L. Le Bruyn and A. I. Ooms, The semicenter of an enveloping algebra is factorial, *Proc. Amer. Math. Soc.* **93** (1985), 397–400.

**10.** M. Lorenz, Class groups of multiplicative invariants, *J. Algebra* **177** (1995), 242–254.

**11.** M. Lorenz and D. S. Passman, Centers and prime ideals in group algebras of polycyclic-by-finite groups, *J. Algebra* **57** (1979), 355–386.

**12.** M. P. Malliavin, *Ultraproduit d'algèbres de Lie* (LNM **924**, 1982), 157–166.

**13.** C. Moeglin, Factorialité dans les algèbres enveloppantes, *C. R. Acad. Sci. Paris Ser. A* **282** (1976), 1269–1272.

**14.** C. Moeglin, Idéaux bilatères dans les algèbres enveloppantes, *Bull. Soc. Math. France* **108** (1980), 143–186.

**15.** S. Montgomery and D. S. Passman, $X$-inner automorphisms of group rings, *Houston J. Math.* **7** (1981), 395–402.

**16.** S. Montgomery and D. S. Passman, $X$-inner automorphisms of group rings II, *Houston J. Math.* **8** (1982), 537–544.

**17.** E. Nauwelaerts and A. I. Ooms, Weights of semi-invariants of the quotient division ring of an enveloping algebra, *Proc. Amer. Math. Soc.* **104** (1988), 13–19.

**18.** A. I. Ooms and P. Wauters, Primitive extensions of an enveloping algebra, *J. Algebra* **161** (1993), 392–405.

**19.** D. S. Passman, *The algebraic structure of group rings* (Wiley-Interscience, New York, 1977).

**20.** D. S. Passman, Computing the symmetric ring of quotients, *J. Algebra* **105** (1987), 207–235.

**21.** D. S. Passman, *Infinite crossed products* (Academic Press Inc., San Diego, 1989).

**22.** M. Smith, Semi-invariant rings, *Comm. Algebra* **13** (1985), 1283–1298.

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