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A NOTE ON p-ADIC CARLITZ'S q-BERNOULLI NUMBERS

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In a recent paper I have shown that Carlitz's q-Bernoulli number can be represented as an integral by the q-analogue μ_q of the ordinary p-adic invariant measure. In the p-adic case, J. Satoh could not determine the generating function of q-Bernoulli numbers. In this paper, we give the generating function of q-Bernoulli numbers in the p-adic case.

1. INTRODUCTION

Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$ and $\mathbb{C}_p$ will respectively denote the ring of p-adic rational integers, the field of p-adic rational numbers and the completion of the algebraic closure of the $\mathbb{Q}_p$.

Let v_p be the normalised exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assume $|q| < 1$. If $q \in \mathbb{C}_p$, then we normally assume $|q-1|_p < p^{-1/(p-1)}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$. We use the notation:

$$[x] = [x:q] = \frac{1-q^x}{1-q}.$$

Hence, $\lim_{q\to 1}[x] = x$ for any x with $|x|_p \leq 1$ in the present p-adic case.

Carlitz's q-Bernoulli numbers $\beta_k = \beta_k(q)$ can be determined inductively by

$$\beta_0 = 1, \qquad q\big(q\beta(q)+1\big)^k - \beta_k = \begin{cases} 1 & \text{if } k=1 \\ 0 & \text{if } k>1, \end{cases}$$

with the usual convention of replacing β^i by β_i.

In complex case, J. Satoh (see [2]) constructed the generating function of the q-Bernoulli numbers $F_q(t)$ which is given by

$$F_q(t) = \sum_{n=0}^{\infty} q^n e^{[n]t}(1-q-q^n t) = \sum_{n=0}^{\infty} \frac{\beta_n}{n!} t^n.$$

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However, he could not explicitly determine $F_q(t)$ [**2**, p.347] in the p-adic case.

In this paper, we give a generating function $F_q(t)$ of q-Bernoulli numbers in the p-adic case.

Recently I have shown that Carlitz's q-Bernoulli number can be represented as an integral by the q-analogue μ_q of the ordinary p-adic invariant measure [**1**]. In this paper, we extend to the q-Bernoulli numbers using an integral by the q-analogue μ_q of the ordinary p-adic invariant measure and given some relation between the Carlitz's q-Bernoulli number and the q-Bernoulli numbers of order 2 in the p-adic case.

2. Generating function

Let d be a fixed integer and let p be a fixed prime number. We set

$$X = \varprojlim_N (\mathbb{Z}/dp^N\mathbb{Z}),$$

$$X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \left\{x \in X \mid x \equiv a \pmod{dp^N}\right\},$$

where $a \in \mathbb{Z}$ lies in $0 \le a < dp^N$.

It is known from [1] that

$$\beta_m(q) = \int_{\mathbb{Z}_p} [a]^m \, d\mu_q(a) = \int_X [a]^m \, d\mu_q(a)$$
$$= \frac{1}{(1-q)^m} \sum_{i=0}^{m} \binom{m}{i} (-1)^i \frac{i+1}{[i+1]},$$

where $\mu_q(a + dp^N\mathbb{Z}_p) = q^a/[dp^N]$.

Let $G_q(t)$ be the generating function of $\beta_i(q)$:

$$G_q(t) = \sum_{k=0}^{\infty} \beta_k(q) \frac{t^k}{k!}$$

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for $q \in \mathbb{C}_p$ with $|1-q| < p^{-1/p-1}$. Thus we have

$$\begin{aligned} G_q(t) &= \sum_{k=0}^{\infty} \beta_k(t) \frac{t^k}{k!} \\ &= \sum_{k=0}^{\infty} \left(\frac{1}{(1-q)^k} \sum_{i=0}^{k} \binom{k}{i} (-1)^i \frac{i+1}{[i+1]} \right) \frac{t^k}{k!} \\ &= \sum_{j=0}^{\infty} \frac{j+1}{[j+1]} (-1)^j \left(\frac{1}{1-q} \right)^j \frac{t^j}{j!} \sum_{i=0}^{\infty} \left(\frac{1}{1-q} \right)^i \frac{t^i}{i!} \\ &= e^{t/(1-q)} \cdot \sum_{j=0}^{\infty} \frac{j+1}{[j+1]} (-1)^j \left(\frac{1}{1-q} \right)^j \frac{t^j}{j!}. \end{aligned}$$

Therefore we obtain the following:

THEOREM 1. *For $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/p-1}$,*

$$G_q(t) = e^{t/(1-q)} \cdot \sum_{j=0}^{\infty} \frac{j+1}{[j+1]} (-1)^j \left(\frac{1}{1-q} \right)^j \frac{t^j}{j!}.$$

It is shown in [**1**] that

$$\beta_n(x,q) = \int_{\mathbb{Z}_p} [x+t]^n d\mu_q(t) = \left(q^x \beta + [x] \right)^n, \quad \text{for } n \geq 0.$$

Thus we have

$$\beta_n(x,q) = \frac{1}{(1-q)^n} \sum_{k=0}^{n} \binom{n}{k} \frac{k+1}{[k+1]} q^{kx} (-1)^k.$$

COROLLARY 2. *For $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/p-1}$, we have*

$$\begin{aligned} G_q(x,t) &= \sum_{n=0}^{\infty} \frac{\beta_n(x,t)}{n!} t^n \\ &= e^{t/(1-q)} \cdot \sum_{j=0}^{\infty} \frac{j+1}{[j+1]} (-1)^j \left(\frac{1}{1-q} \right)^j q^{jx} \frac{t^j}{j!}. \end{aligned}$$

REMARK. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$. For $q \in \overline{\mathbb{Q}} \cap \mathbb{C}_p$ with $|q| < 1$, we have

$$\begin{aligned} G_q(t) &= e^{t/(1-q)} \cdot \sum_{j=0}^{\infty} \frac{j+1}{[j+1]}(-1)^j \left(\frac{1}{1-q}\right)^j q^{jx} \frac{t^j}{j!} \\ &= \sum_{m=0}^{\infty} \left\{ \frac{1}{(1-q)^m} \left(\sum_{j=0}^{m} \binom{m}{j} (-1)^j j \frac{1}{[j+1]} + \sum_{j=0}^{m} \binom{m}{j} (-1)^j \frac{1}{[j+1]} \right) \right\} \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left\{ \frac{1}{(1-q)^{m-1}} \sum_{j=1}^{m} \binom{m}{j} (-1)^j j \sum_{n=0}^{\infty} q^{(j+1)n} \right. \\ &\qquad \left. + \frac{1}{(1-q)^{m-1}} \sum_{j=0}^{m} \binom{m}{j} (-1)^j \sum_{n=0}^{\infty} q^{(j+1)n} \right\} \frac{t^m}{m!} \\ &= \sum_{m=0}^{\infty} \left\{ -m \sum_{n=0}^{\infty} q^n [n]^{m-1} - (q-1)(m+1) \sum_{n=0}^{\infty} q^n [n]^m \right\} \frac{t^m}{m!}. \end{aligned}$$

Differentiating both side with respect to t and comparing coefficients, we have

$$\beta_m(q) = -m \sum_{n=0}^{\infty} q^n [n]^{m-1} - (q-1)(m+1) \sum_{n=0}^{\infty} q^n [n]^m,$$

that is,

$$-\frac{\beta_m(q)}{m} = \sum_{n=0}^{\infty} q^n [n]^{m-1} + (q-1) \frac{(m+1)}{m} \sum_{n=0}^{\infty} q^n [n]^m,$$

for $m \geq 0$.

3. EXTENDED q-BERNOULLI NUMBERS

For $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-1/p-1}$, we define the q-Bernoulli numbers of higher order by

$$(1) \quad \beta_m^{(h,k)} = \underbrace{\int_{\mathbb{Z}_p} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k \text{ times}} [x_1 + \cdots + x_k]^m q^{x_1(h-1)+\cdots+x_k(h-k)} d\mu_q(x_1) \cdots d\mu_q(x_k).$$

It has been proved in [1] that

$$\beta_m^{(h,k)} = \frac{1}{(1-q)^m} \sum_{j=0}^{m} \binom{m}{j} (-1)^j \frac{(j+h)_k}{[j+h]_k},$$

where $(j+h)_k = (j+h)(j+h-1)\cdots(j+h-k+1)$ and $[j+h]_k = [j+h]\cdots[j+h-k+1]$. Note that $\beta_m^{(1,1)}$ is Carlitz's q-Bernoulli number in the p-adic case (see [1]).

We also define the q-Bernoulli polynomial of higher order by

$$(2)\quad \beta_m^{(h,k)}(x) = \underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k \text{ times}} [x+x_1+\cdots+x_k]^m \cdot q^{x_1(h-1)+\cdots+x_k(h-k)} d\mu_q(x_1)\cdots d\mu_q(x_k).$$

LEMMA 3. *For $h, k \in \mathbb{Z}_+ = \{$the set of positive integers$\}$, we have*

$$\beta_m^{(h,k)} = \frac{1}{(1-q)^m}\sum_{j=0}^{m}\binom{m}{j}(-1)^j \frac{(j+h)_k}{[j+h]_k}$$

and

$$\beta_m^{(h,k)}(x) = \frac{1}{(1-q)^m}\sum_{j=0}^{m}\binom{m}{j}(-1)^j \frac{(j+h)_k}{[j+h]_k} q^{jx}.$$

In particular, for $h=1$, we see

$$(3)\quad \begin{aligned} \beta_m^{(h,1)}(x) &= \int_{\mathbb{Z}_p}[x+x_1]^m q^{(h-1)x_1} d\mu_q(x_1) \\ &= \sum_{j=0}^{m}\binom{m}{j}[x]^{n-j} q^{jx}\beta_j^{(h,1)} \\ &= \left(q^x\beta^{(h,1)}+[x]\right)^m, \quad \text{for } m \geq 1. \end{aligned}$$

It is easy to see that

$$(4)\quad \begin{aligned} &q^h\beta_m^{(h,1)}(x+1) - \beta_m^{(h,1)}(x) \\ &= \frac{q^h}{(1-q)^m}\sum_{i=0}^{m}\binom{m}{i}(-1)^i\frac{i+h}{[i+h]}q^{i(x+1)} - \frac{1}{(1-q)^m}\sum_{i=0}^{m}\binom{m}{i}(-1)^i q^{ix}\frac{i+h}{[i+h]} \\ &= q^x m[x]^{m-1} + h(q-1)[x]^m. \end{aligned}$$

By (3) and (4), we see

$$q^h\left(q\beta^{(h,1)}+1\right)^m - \beta_m^{(h,1)} = \begin{cases} 1 & \text{if } m=1 \\ 0 & \text{if } m>1. \end{cases}$$

Note that

$$\beta_0^{(h,1)} = \int_{\mathbb{Z}_p} q^{(h-1)x} d\mu_q(x) = \frac{h}{[h]}.$$

EXAMPLE.

$$\beta_0^{(2,1)} = \frac{2}{[2]}, \quad \beta_1^{(2,1)} = -\frac{2q+1}{[2][3]},$$
$$\beta_2^{(2,1)} = \frac{2q^2}{[3][4]}, \quad \beta_3^{(2,1)} = -\frac{q^2(q-1)\big(2[3]+q\big)}{[3][4][5]}, \dots$$

Let $G_q^{(h,1)}(t)$ be the generating function of $\beta_i^{(h,1)}(q)$:

$$G_q^{(h,1)}(t) = \sum_{k=0}^{\infty} \beta_k^{(h,1)}(q)\frac{t^k}{k!}, \quad \text{for } q \in \mathbb{C}_p \text{ with } |1-q|_p < p^{-1/p-1},$$

then $G_q^{(h,1)}(t)$ is given by

$$G_q^{(h,1)}(t) = \lim_{\rho\to\infty} \frac{1}{[p^\rho]} \sum_{i=0}^{p^\rho-1} q^i e^{[i]t} q^{(h-1)t} = \int_{\mathbb{Z}_p} e^{[x]t} q^{(h-1)x} d\mu_q(x)$$
$$= e^{t/(1-q)} \cdot \sum_{j=0}^{\infty} \frac{j+h}{[j+h]}(-1)^j \left(\frac{1}{1-q}\right)^j \frac{t^j}{j!}.$$

REMARK. For $q \in \overline{\mathbb{Q}} \cap \mathbb{C}_p$, we have

$$G_q^{(h,1)}(t) = e^{t/(1-q)} \cdot \sum_{j=0}^{\infty} \frac{j+h}{[j+h]}(-1)^j \left(\frac{1}{1-q}\right)^j \frac{t^j}{j!}$$
$$= \sum_{m=0}^{\infty} \left\{ \frac{1}{(1-q)^m} \left(\sum_{j=0}^{m} \binom{m}{j} (-1)^j j \frac{j}{[j+h]} + \sum_{j=0}^{m} \binom{m}{j} (-1)^j \frac{h}{[j+h]} \right) \right\} \frac{t^m}{m!}$$
$$= \sum_{m=0}^{\infty} \left\{ \frac{1}{(1-q)^{m-1}} \sum_{j=1}^{m} \binom{m}{j} (-1)^j j \sum_{n=0}^{\infty} q^{(j+h)n} \right.$$
$$\left. + \frac{h}{(1-q)^{m-1}} \sum_{j=0}^{m} \binom{m}{j} (-1)^j \sum_{n=0}^{\infty} q^{(j+h)n} \right\} \frac{t^m}{m!}$$
$$= \sum_{m=0}^{\infty} \left\{ -m \sum_{n=0}^{\infty} q^{hn}[n]^{m-1} - (q-1)(m+h) \sum_{n=0}^{\infty} q^{hn}[n]^m \right\} \frac{t^m}{m!}.$$

Differentiating both side with respect to t and comparing coefficients, we have

$$\beta_m^{(h,1)}(q) = -m \sum_{n=0}^{\infty} q^{hn}[n]^{m-1} - (q-1)(m+h) \sum_{n=0}^{\infty} q^{hn}[n]^m,$$

that is,

$$-\frac{\beta_m^{(h,1)}(q)}{m} = \sum_{n=0}^{\infty} q^{hn}[n]^{m-1} + (q-1)\frac{m+h}{m} \sum_{n=0}^{\infty} q^{hn}[n]^m,$$

for $m \geq 0$.

LEMMA 4. *For $q \in \overline{\mathbb{Q}} \cap \mathbb{C}_p$, we have*

$$-\frac{\beta_m^{(h,1)}(q)}{m} = \sum_{n=0}^{\infty} q^{hn}[n]^{m-1} + (q-1)\frac{m+h}{m}\sum_{n=0}^{\infty} q^{hn}[n]^m.$$

For $m > 1$, we see

$$\begin{aligned}\beta_m^{(h,1)}(1-x, q^{-1}) &= \int_{\mathbb{Z}_p} [1 - x + x_1 : q^{-1}]^m q^{-(h-1)x_1} d\mu_{q^{-1}}(x_1) \\ &= \lim_{N\to\infty} \frac{1}{[p^N : q^{-1}]} \sum_{x_1=0}^{p^N-1} [1 - x + x_1 : q^{-1}]^m q^{-hx_1} \\ &= q^{m+h-1}(-1)^m \beta_m^{(h,1)}(x).\end{aligned}$$

Thus we have

$$\beta_m^{(h,1)}(0, q^{-1}) = (-1)^m q^{m+h-1} \beta_m^{(h,1)}(1) = (-1)^m q^{m-1} \beta_m^{(h,1)} \quad \text{(by (4))}.$$

Therefore we obtain the following:

PROPOSITION 5. *For $h, m\ (>1) \in \mathbb{Z}_+$, we have*

$$\beta_m^{(h,1)}(1-x, q^{-1}) = (-1)^m q^{m+h-1} \beta_m^{(h,1)}(1).$$

Moreover,

$$\beta_m^{(h,1)}(0, q^{-1}) = (-1)^m q^{m+h-1} \beta_m^{(h,1)}(1) = (-1)^m q^{m-1} \beta_m^{(h,1)}.$$

By the definition of $\beta_m^{(h,k)}$, we see

$$\begin{aligned}\beta_m^{(k,k)}(x) &= \underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k \text{ times}} [x + x_1 + \cdots + x_k]^m q^{x_1(k-1)+\cdots+x_{k-1}} d\mu_q(x_1)\cdots d\mu_q(x_k) \\ &= \sum_{j=0}^{m} \binom{m}{j} q^{xj} \underbrace{\int_{\mathbb{Z}_p}\int_{\mathbb{Z}_p}\cdots\int_{\mathbb{Z}_p}}_{k \text{ times}} [x_k]^j [x_1 + \cdots + x_{k-1}]^{m-j} \\ &\qquad\qquad \cdot q^{(k+i-1)x_1+\cdots+(i+1)x_{k-1}} d\mu_q(x_1)\cdots d\mu_q(x_k) \\ &= \sum_{j=0}^{m} \binom{m}{j} q^{xj} \beta_j^{(1,1)} \beta_{m-j}^{(k+i,k-1)}.\end{aligned}$$

Therefore we obtain the following:

Theorem 6. *For $k \in \mathbb{Z}_+$, we have*

$$\beta_m^{(k,k)} = \sum_{i=0}^{m} \binom{m}{i} \beta_i^{(1,1)} \beta_{m-i}^{(i+k,k-1)}.$$

Moreover,

$$\beta_m^{(2,2)} = \sum_{j=0}^{m} \binom{m}{j} \beta_j \beta_{m-j}^{(j+2,1)}.$$

We see

$$\beta_m^{(h+1,1)} = \int_{\mathbb{Z}_p} [x]^m q^{hx} d\mu_q(x) = \int_{\mathbb{Z}_p} [x]^m \big((q-1)[x]+1\big)^h d\mu_q(x)$$
$$= \sum_{j=0}^{h} \binom{h}{j} (q-1)^j \beta_{m+j}.$$

Corollary 7.

$$\beta_m^{(2,2)} = \sum_{j=0}^{m} \binom{m}{j} \beta_j \sum_{i=0}^{j+1} (q-1)^i \binom{j+1}{i} \beta_{m-j+i}$$

Remark. If $q \to 1$, then we have

$$B_m^{(2)} = \sum_{i=0}^{m} \binom{m}{i} B_i B_{m-i},$$

where $B_m, B_m^{(2)}$ denote the mth Bernoulli number and the mth Bernoulli number of order 2.

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