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ARTICLE

Managing the shortfall risk of target date funds by overfunding

Giovanni Barone Adesi[1], Eckhard Platen[2] and Carlo Sala[3]

[1]Swiss Finance Institute at Università della Svizzera Italiana (USI), Institute of Finance, Lugano, Switzerland, [2]School of Mathematical and Physical Sciences and Finance Discipline Group, University of Technology Sydney, Sydney, Australia and [3]Department of Financial Management and Control, Universitat Ramon Llull, ESADE, Barcelona, Spain
Corresponding author: Carlo Sala; Email: carlo.sala@esade.edu

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Abstract
Is it possible to achieve almost riskless, nonfluctuating investment payoffs in the long run, at a fraction of the traditional funding requirement, using equity investments? What is their shortfall risk? These questions are motivated by the need to increase yields, while limiting the variability of investment results. We show how to use contingent claims, denominated in units of a stock index, to achieve an almost riskless investment outcome. To control the risk of the proposed hedge portfolios, we introduce an overfunded scheme and show its reliability using bootstrapping. Results show that a modest amount of overfunding is an effective risk-management approach that brings the probability of not achieving the target to less than 1 percent. Our results are based on the use of the minimal market model and a change of numeraire. Robustness tests support their validity under different market specifications.

Keywords: dynamic investment policy; hedging; locally riskless payout

JEL classification: G13; G14

1. Introduction

Many investors and pensioners who save for retirement may benefit from the fact that, in the long-term, the equity market tends to outperform the fixed income market (see Dimson *et al.* (2003)).[1] At retirement, the investors often prefer to target almost riskless payouts, where payouts in units of the savings account are not affected by a severe market drawdown close to retirement. A natural question is, therefore, whether it is possible to use equities to achieve almost riskless payouts of units of a savings account in the long-term. The growing interest in this question is motivated by the presence of low real interest rates, which reduces the real yields of investment policies followed traditionally by insurance and pension funds. Target date fund models show that it is possible to invest in equities less expensively than under risk-neutral assumptions to achieve almost riskless payouts of units of a savings account in the long-term, as is typical for defined-benefit pension lump-sum payouts. Arguably, pension funds should aim to achieve highly predictable payout by means of dynamic investment policies. Therefore, they should reduce the high variability typical of equity investments. The equity component of pension and insurance funds currently leads to highly variable outcomes. As an example, the returns of pension funds varied significantly in European countries in 2019, ranging from 14 percent in the United Kingdom to 4 percent in Switzerland. Furthermore, the performance of such funds has varied as well, as reported in the second and third columns of Table 1. Such

[1]The outperformance refers to the empirically evidence that the long-term average growth rate of the stock market index is larger than the average interest rate. There seems also to be no rational that would suggest that this empirical fact may reverse.

Table 1. Average performance values for the UK and Swiss pension systems for the period 2009–2019

Calendar year	United Kingdom	Switzerland
2019	14.0%	4.00%
2018	−6.2%	−2.81%
2017	15.7%	7.64%
2016	2.6%	3.58%
2015	5.8%	1.13%
2014	13.9%	7.31%
2013	10.8%	6.26%
2012	−4.6%	7.17%
2011	13.8%	−0.34%
2010	22.3%	2.94%
2009	−19.7%	10.31%

Source: Moneyfacts UK Personal Pension Trends Treasury Report/Lipper Reports for the UK performance, and Swiss Pension Fund Studies 2019, Swiss Canto Ltd, for the Swiss performance.

performance reflects the trade-off between risk (volatility) and expected return, which is roughly linear for optimal investment policies.

In this paper, we describe a dynamic approach to manage the shortfall risk of a portfolio that uses *equities* and risk-free securities to produce almost riskless *long-term* payouts. The proposed approach has the primary goal of achieving a target outcome of one unit of the risk-free security over a long time horizon by controlling for its variability at maturity. This is achieved by overfunding an innovative basic investment plan introduced by Fergusson and Platen (2014) that invests in the stock market heavily at inception, when investors have time to enjoy its long-term growth, and also whenever the stock market falls. We call our proposed approach 'the overfunded plan', and compare it with the 'basic investment plan', which is not overfunded. Overfunding and minimum expected shortfall (MES)[2] investment strategies are two distinct approaches to managing the risk of shortfalls in target date funds (TDFs). While both strategies aim to mitigate the risk of falling short of a TDF's target, they differ in their approach. In fact, while overfunding partially compensates for the shortfall arising in the implementation of the policy of Fergusson and Platen (2014) over a finite time horizon, the MES strategy is an optimization approach that optimizes the expected shortfall over a decision variable. As such, the two approaches differ both theoretically and in their implementations. The proposed overfunded plan is a mathematically founded formalization of the popular concept of investing in risky assets at a young age, and then moving to the risk-free security over time. From a theoretical viewpoint, the approaches we discuss here may be linked to the fact that there are rational asset price bubbles in the market, as theoretically predicted in, for example, Loewenstein and Willard (2000), Platen (2002), Fernholz *et al.* (2005) and Hugonnier (2012), and empirically confirmed in, for example, Baldeaux *et al.* (2015), Baldeaux *et al.* (2018) and Platen and Rendek (2020). The benchmark approach, as presented by Platen (2001), Platen (2002), and Platen and Heath (2010), allows a more general modeling world than the risk-neutral modeling world of Black and Scholes (1973) permits. This does not mean that there is any economically meaningful arbitrage under the benchmark approach. No portfolio can reach an infinite value in finite time. The pricing and hedging methodology we use has been only partially implemented in Fergusson and Platen (2014) and Fergusson and Platen (2023) in order to describe dynamic investment policies that target locally riskless payouts. In contrast to Fergusson and Platen (2014) and Fergusson and Platen (2023), we overfund the respective basic investment plan to achieve with a high probability the desired payoff at a given target date.

The desired final payoff (i.e., one unit of the risk-free security) becomes random when denominated in units of a stock index, which we call the benchmark. This fixed payoff denominated in the benchmark can then be viewed as a contingent claim, to be hedged dynamically in order to achieve

[2]See, e.g.: Acharya *et al.* (2012), Brownlees and Engle (2016) for some market data application and Bertsimas *et al.* (2004) for a review of the theoretical properties of expected shortfall.

the number of units of the stock index necessary to reach the targeted value at maturity. Intuitively, the investment portfolio initially allocates a relatively high weight to equities. By complying with the underlying model, the initially high weight is reduced progressively over time in line with the theoretical hedge ratio in favor of the risk-free security in order to achieve the desired target. The optimal policy is a dynamic hedging strategy that follows a similar logic to the Black and Scholes (1973) option hedging strategy, which has been used successfully to manage the risk for the payoff of short-term (usually up to three years) options.

Main drivers for the success of the proposed overfunded plan are the mean-reversion and the leverage effect, which are captured by the underlying model, and are missing under the Black–Scholes model. Supported by Summers (1986), Poterba and Summers (1988), Fama and French (1988), and Spierdijk *et al.* (2012), mean-reversion is the key feature in the Fergusson and Platen (2014) to model the long-term dynamics of the logarithm of the stock index. The presence of mean-reversion is debated in the literature. It is useful to remark that, while mean-reversion is necessary to asymptotically achieve the desired rate of return with certainty, this paper focuses on finite time horizons. Different robustness tests (Section 4.2) show that, while important, the mean-reversion is not the only driver of the good performance of our proposed overfunded model. Another important feature is the leverage effect, first documented by Black (1976). There is an empirical negative correlation between returns and volatility, which needs to be accommodated in a suitable model, and implies that there is an asymmetric reaction of volatility to bad and good news and, hence, to negative or positive return innovations, respectively. Economically, a negative return implies a drop in the firm's equity value. This increases its leverage, which in turn, leads to higher volatility of equity returns.

Aside from the importance of properly modeling these key features, which indeed could potentially be achieved theoretically using various stock index models, it is mostly the strict supermartingale property of the risk-free security denominated in units of the stock index, captured under the employed model, that ensures most of the cost savings of the proposed strategies. This property is of crucial importance for the effective management of long-term assets. More precisely, the supermartingale property states that the expected payoff of the risk-free bond, measured in current units of the stock index, tends to *decrease* – on average – for increasing time horizons, which is modeled realistically under the model.

The strategy we implement here mimics payouts of the risk-free security at maturity. To provide a broad and general analysis, we study different investment plans, with maturities ranging from 10 to 100 years. As a first step, we follow Platen (2001), Platen (2002), and Fergusson and Platen (2023), and present novel parametric investment plans. For all of the proposed investment plans, we describe how to dynamically set up the hedge portfolio by determining how much to invest in the stock index, depending on its value and time to maturity. Then, as a second step, we control the risk by *overfunding* the above strategies. The proposed overfunded plans are again based on the original basic investment plans. Thus, we invest the same fractions of the risky stock index as before, but have *higher* initial capital (the overfunded initial capital) invested in the hedge portfolio. This change carries relevant results, both theoretically and empirically, that could be important for whoever manages and invests in long-term assets (e.g., pension fund managers, life insurers, and other long-term investors).

In fact, overfunding the basic investment plan allows us to still benefit from the analytical tractability of the underlying model. Furthermore, we can overcome the well-known fact that it might not always be possible to achieve the targeted payoff, owing to model and hedge errors. Our empirical results refer to the US market and show that, using up to a century of data, and irrespective of the time horizon (from 10 to 100 years) and the initial time (from 1920 to 2020), an initial overfunding of 6 percent is enough to keep the probability of not achieving the target below 1 percent.

In theory, managing TDFs this way provides approximately the *same* benefits of a traditional plan, but at a *fraction* of the funding requirement. The benefit relative to traditional low-risk investment schemes is related to the equity risk-premium, and increases with the time horizon. It is modest at short maturities, but may become large at long maturities. Specifically, the benefit stems from the faster expected growth of the portion of funds invested in stocks. This is mitigated by the progressive, dynamic shift into the risk-free security, and determined theoretically by assuming appropriate dynamics for the stock index.

To bring the aforementioned theoretical results to fruition in the investment industry, it is necessary to evaluate their robustness and their margin of safety. As such, we test the basic investment plan and different overfunded schemes for 10-, 30-, 50-, and 100-year periods using bootstrapping based on US data for the period from 1920 to 2020. The long-term dynamic hedging strategy derived under the benchmark approach works similarly well, even for long time horizons close to 100 years. It appears that the proposed overfunded scheme can achieve the objectives of typical pension saving with the chosen margin of safety and less expensively than currently practiced.

The results presented in the paper are likely to raise questions for many readers which are only be partly answered in this paper and may require further research.[3] However, due to the potential of triggering a significant change in the practice of long-term asset management and providing a potential solution to many problems faced by social security, pension plans, and life insurance companies, the authors hope that the main message becomes disseminated and interest in the underlying theory raised. This should generate more interest and research on refined modeling and also on the related institutional aspects.

The remainder of the paper is organized as follows. In Section 2, we review the underpinning theory that allows us to manage and hedge the proposed investment strategies as a contingent claim. In Section 3, we show how to synthetically replicate a risk-free payoff using the stock index and the risk-free security, modeled as a roll-over short-term bond account. In Section 4, we present the overfunded plan and propose various alternatives. Then, we analyze which of the alternatives are best for short (10 year), medium (30 year), and very long (100 year) time horizons and consider different model robustness. Lastly, Section 5 concludes the paper.

2. Proposed investment strategy

To present our model as simply as possible, we denominate, at first, the securities in units of the risk-free security, a roll-over short-term bond account B_t, and consider only how to approximately hedge a target payoff $V_T = 1$ of one unit of the risk-free security using a total return stock index $S_t > 0$ and the risk-free security $B_t = 1$ for $[0, T]$, following Fergusson and Platen (2023). This avoids the need to involve a stochastic interest rate process. However, similar dynamic trading strategies can be derived for targeted payoffs of a currency unit or units of a consumer price index. Perhaps more importantly, the proposed strategies can also be extended and generalized to other long-term contingent claims including options.

Under the minimal market model (MMM) introduced in Platen (2001), the value S_t of the stock index follows a squared Bessel process of dimension four; see (12). By assuming the MMM with linear deterministic market time, Fergusson and Platen (2023) synthetically replicate a risk-free security payoff. As a main drawback, the authors implement a parametric model with an assumed linear market time. As is common for many parametric models, this provides analytical tractability, but might not fit well with real data (e.g., see Campbell *et al.* (1997), Chapter 2). As a possible resolution to this issue, we propose an overfunded scheme that adds a buffer to the dynamic hedging strategy, thus relaxing the issues linked to the parametric nature of the model, and making the overfunded approach of practical use in the real world. In fact, the overfunded approach allows us to still work with non-numerically intensive closed-form solutions while, at the same time, reducing the probability of not achieving the targeted final value. Future capital, for example, one unit of the risk-free security, becomes a random quantity $\widehat{V_T} = 1/S_T$ when measured in units of a well-diversified stock index S_T, the numeraire or benchmark. We call any security benchmarked when it is denominated in units of the benchmark. The benchmarked target $\widehat{V}_T$ can then be regarded as the payoff of a contingent claim, to be hedged over time in order to achieve its face value at maturity, where the hedge is

[3]The theory underpinning the basic investment strategy we implement is described in the monograph Platen and Heath (2010).

investing dynamically in the benchmarked stock index $\widehat{S}_t = 1$ and the benchmarked risk-free security $\widehat{B}_t$. It is common practice for financial planners to recommend starting investing in risky securities at a young age, and then shifting to risk-free assets over time, where the respective glide path remains rather subjective. The proposed overfunded strategies make this glide path theoretically rigorous and practically feasible by performing the hedging with benchmarked securities, whereas risk-neutral hedging is based on securities denominated in the risk-free security.

The proposed policies invest in the stock index S_t and a roll-over short-term bond account B_t, which we call the risk-free security. Dividends are reinvested in the stock index. To keep our presentation simple, we discount the values by the risk-free security, and plan to achieve one unit of the risk-free security (i.e., the discounted value $V_T = 1$) in, for example, 30 years ($T = 30$). For comparison, under the Law of One Price, a traditional pension policy that aims to produce one unit of the risk-free security would prescribe investing the present value of the desired payoff at the risk-free rate, which means it would simply buy and hold one unit of the risk-free security. However, as we mentioned at the beginning, the risk-free rate is, for economic reasons (on average), lower than the long-term average growth rate of a stock index. Thus, under classical assumptions, a pension that targets one unit of the risk-free security would have to buy and hold until maturity one unit of the risk-free security. However, this is rarely followed in practice because pension funds also invest in stocks. The practice of investing in stocks, even when one is targeting a risk-free payout, can be justified theoretically, as we describe in the following.

Under the benchmark approach, which generalizes the classical risk-neutral approach, one has the following fundamental result (see Theorem 10.3.1 in Platen and Heath (2010)): the expected value of the targeted payout $V_T = 1$, when denominated in the units of the benchmark (the stock index S_T) is, at all times $t < T < \infty$, equal to the investment portfolio value V_t denominated in units of the benchmark S_t. This is referred to as the real-world pricing formula, because the expectation is taken under the real-world probability measure. Therefore, we invest at time $t < T$ the amount V_t, where

$$\widehat{V}_t = \frac{V_t}{S_t} \geq \mathbb{E}_t\left(\frac{V_T}{S_T}\right) \tag{1}$$

for all $\leq t \leq T$. Here, $\mathbb{E}_t(\cdot)$ denotes the conditional expectation under the physical or real-word probability measure, given the information available at time t. The quantity $\widehat{V}_t$ defined in equation (1) is the investment measured in units of the stock index, also called the benchmarked investment. The benchmarked risk-free security can be interpreted as a stochastic discount factor in the sense of Cochrane (2001), if normalized to 1.0 at the initial time. However, under the benchmark approach, the benchmarked risk-free security is not required to form a true martingale, which would be required for risk-neutral prices to coincide with the prices under the real-world pricing formula. This allows for the less expensive production of the targeted payout than possible under the classical risk-neutral assumption, where the benchmarked risk-free security is assumed to follow a martingale. The price S_t of the benchmark, the stock index, fluctuates over time. Thus, we need to adjust dynamically, in a self-financing manner, the number of its units in the portfolio to keep equation (1) satisfied. If we can do that continuously over time, similarly to Black–Scholes option hedging, then we can replicate the benchmarked targeted value, exploiting the fact that:

$$\lim_{t \to T} \mathbb{E}_t\left(\frac{V_T}{S_T}\right) = \frac{V_T}{S_T}. \tag{2}$$

Therefore, by equation (1), the sequence of benchmarked holdings $\widehat{V}_t = V_t/S_t$ over time converges to $\widehat{V}_T = 1/S_T$. Thus, at maturity, we get $V_T = \widehat{V}_T \cdot S_T = 1$ unit of the roll-over short-term bond account, the risk-free security.

The policy sketched above exploits the fact that our desired final benchmarked payoff corresponds to the price of the risk-free security measured in terms of the stock index value. This allows for the

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modeling of the investment payout as a contingent claim, which may be hedged analogously, as in Black and Scholes (1973), to an option payoff.

This approach entails several important differences to the risk-neutral approach. First, under the proposed approach, every security is denominated in units of the stock index, and it aims in expectation at maturity at one unit of the risk-free security, denominated in units of the stock index. Thus, this approach uses the risky asset, rather than the risk-free security, as its numeraire. Risk management evolves then around the stock index, which on average and in the long-term, grows faster than the risk-free security, the traditional numeraire. Second, the chosen model for the dynamics of the stock index, the MMM, captures rather realistically its real-world mean-reversion and the leverage effect. Finally, the physical or real-world probability measure is the chosen pricing measure, whereas risk-neutral option pricing uses the putative risk-neutral probability measure. As a result, targeted payoffs are hedged less expensively than when using the traditional method, as we show later on in the empirical section of the paper. Note that we use the stock index as a proxy for the growth-optimal portfolio, which in the long run, is theoretically the pathwise best performing portfolio for the given stock investment universe, see Kelly (1956) and Merton (1973), and equivalent to the numeraire portfolio, which was introduced in Long (1990) and is central to the benchmark approach; see Platen and Heath (2010).

Under the real-world pricing formula (1), with our chosen numeraire (the stock index), the investment portfolio value necessary to achieve the desired targeted payoff follows a martingale (when denominated in units of the numeraire, the benchmark) under the physical or real-world probability measure. This means, its current benchmarked value is equal to its expected future benchmarked values. The observed typical dynamics of the risk-free security denominated in units of the stock index can be modeled realistically using the MMM by a strict supermartingale, where the current benchmarked value is strictly *greater* than the expected future benchmarked values. Assuming here a martingale, would be the risk-neutral assumption. The parametrization of the stock index dynamics through the MMM is preferable for capturing strict supermartingale dynamics and enables the achievement of the benchmarked final payoff at a lower cost. Fergusson and Platen (2023) show how the physical or real-world probability measure replaces the risk-neutral measure. In addition, they show how the strict supermartingale property permits the less expensive production of a payoff approximating one unit of the risk-free security than achieved by buying and holding one unit of the risk-free security. We show in Figure 1 the benchmarked trajectories of the risk-free security and the respective hedge portfolio that targets at maturity one unit of the risk-free security, both denominated in units of the stock index.[4]

In this figure, the risk-free security, denominated in units of the stock index, clearly shows on average a declining behavior. In contrast, the hedge portfolio value, denominated in units of the index, looks more like a martingale. Note that if we would use for comparison any risk-neutral model for the parametrization of the index, the risk-free security denominated in units of the stock index would have to be theoretically interpreted as a true martingale, which would make V_t more expensive than under the MMM. We emphasize that this is true for any model that uses classical risk-neutral pricing. Under risk neutrality, by the Law of One Price, the theoretical value is equal to one unit of the risk-free security. From a hedging perspective, even in the presence of a working risk-neutral hedging scheme (in this case, a trivial buy and hold strategy), the risk-neutral price is equal to one unit of the risk-free security. This price is *higher* than that provided by the real-world pricing formula (1) under the MMM, owing to the strict supermartingale property of the benchmarked risk-free security.

3. Parametric synthetic bond strategy

In this section, we propose a parametric approach under the MMM, which we use to synthetically replicate a long-term bond with the discounted payoff $V_T = 1$ by investing dynamically, in a self-

[4]All details on how to estimate both quantities are presented in Section 3.

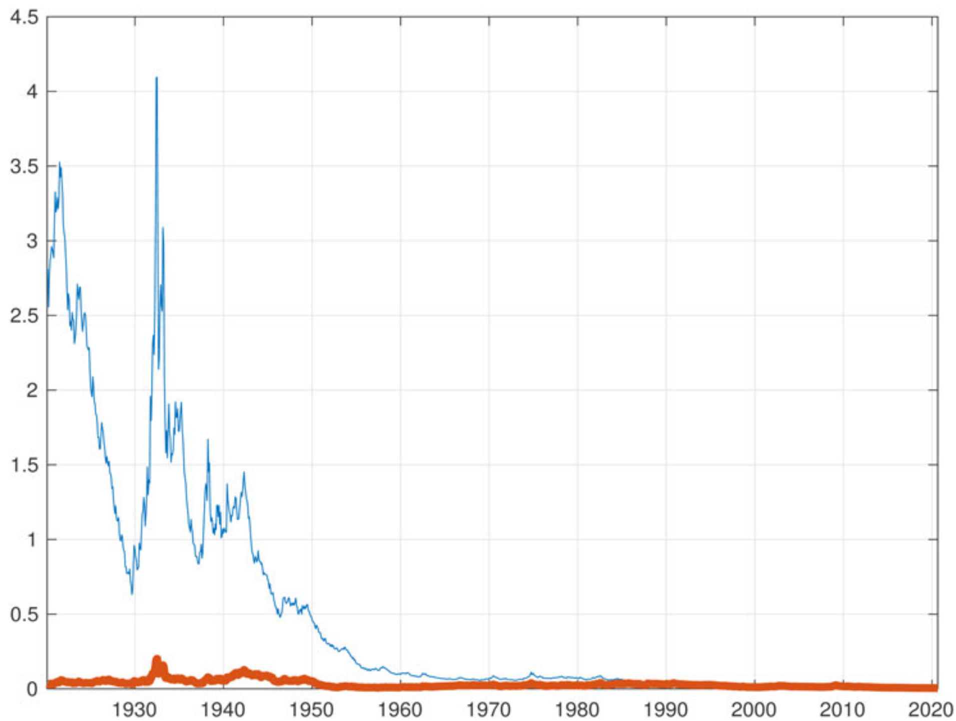

Figure 1. Risk-free security (in blue) and investment portfolio (thicker in red); both denominated in stock index.

financing manner, in a discounted stock index S_t and the discounted risk-free security. Throughout this paper, we identify the outcomes of the parametric bond strategy as those of the basic investment plan. This is to differentiate the basic investment plan from the overfunded investment plans presented in the next section.

The desired final risk-free value $V_T = 1$ is approached at time T because, under the MMM we describe below, the investment portfolio tends to glide gradually into the risk-free security when approaching the desired target. It is progressively reducing the fraction of the holdings in the stock index. This fraction is converging to zero, so that at maturity, one holds only units of the risk-free security. More details on this will follow. Fergusson and Platen (2023) provide an explicit formula for the proportion of the portfolio value to be invested in the index at any time. To do so, they assume the following stochastic differential equation (SDE) as the risk-free security discounted stock index process to follow under the MMM:

$$dS_t = \alpha_t dt + \sqrt{S_t \alpha_t} dW_t, \tag{3}$$

where $\alpha_t = S_t \theta_t^2 = \alpha \exp(\eta \cdot t)$, with θ_t denoting the volatility. Here, $\alpha > 0$ is a normalization parameter, $\eta > 0$ is the (net) growth rate of S_t, and $W = \{W_t \geq 0, t\}$ is the driving canonical Brownian motion. More details about the MMM are described in Appendix A.

In the long-term, when using only constant volatility, as is the case in the Black–Scholes model with log-normal index increments, we would not capture the stock index dynamics sufficiently well. This is because the variance of the logarithm of its value would theoretically continue to increase linearly with time, even after long periods. This is not the case in reality, as can be seen in Figures 2 and 3. Here, we display the monthly log-closing prices and the monthly closing prices, respectively, of the discounted S&P500 total return index for the period from January 1920 until August 2020, denominated in units

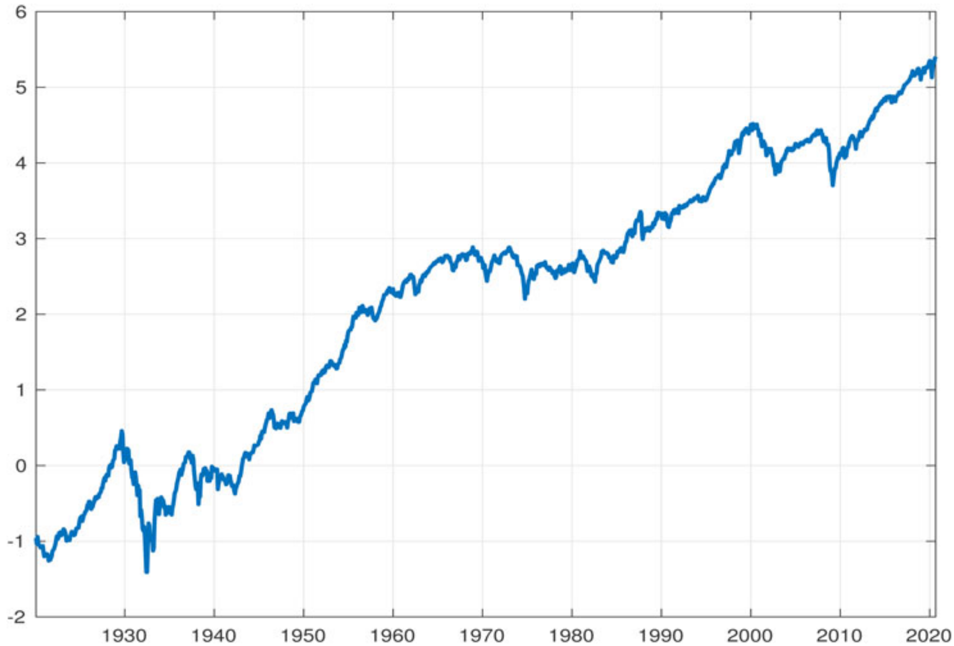

Figure 2. Discounted S&P500 total return log-index: time series of monthly closing log-prices for the period January 1920 to August 2020.

of the risk-free security. The risk-free security is here taken as the three-month US T-Bill roll-over account. All monthly data are obtained from Global Financial Data.

Note that the quadratic variation $[X.]_t$ of a process X is approximated by the sum of the squares of its increments. The MMM can be conveniently fitted to historical stock index data, by exploiting the fact that the quadratic variation of the square root of the index, denoted by $\left[\sqrt{S.}\right]_t$, satisfies the equation $\left[\sqrt{S.}\right]_t = 0.25\alpha(e^{\eta t} - 1)/\eta$ owing to the Ito formula. Thus, when α and η are estimated appropriately, theoretically under the MMM, the value $\tau_t = \ln\left(4\eta\left[\sqrt{S.}\right]_t/\alpha + 1\right)/\eta$ of the so-called market time should form an approximately straight line, with slope one. Figure 4 shows the resulting graph and the respective trendline, which obtained a maximum R^2 of 0.9933 when fitting the MMM for the period from January 1920 until August 2020, with $\alpha = 0.024$ and $\eta = 0.048$, with standard errors equal to 0.000613 and 0.002968, respectively. Appendix B describes the estimation approach used to obtain these values. The rather high R^2 value indicates that the MMM parametrization makes good sense. Defined as the expected growth rate minus the short rate, the estimated net growth rate of $\eta = 0.048$ is aligned with Table 23 of Damodaran (2020).[5] We use these parameter estimates for the MMM in the remainder of the paper.

As presented in Fergusson and Platen (2023), it follows that at any time $t \in [0, T)$ under the MMM, we have the following discounted investment portfolio value:

$$V_t = 1 - \exp\left\{-\frac{2\eta S_t}{\alpha(\exp\{\eta T\} - \exp\{\eta t\})}\right\}. \tag{4}$$

Note that $\widehat{V}_T = (1/S_T)$ is the benchmarked payoff. Here, the benchmarked value $\widehat{V}_t = V_t/S_t = \mathbb{E}_t(1/S_T)$ of the investment portfolio V_t is its value denominated in units of the

[5]In the 11th edition of his survey on the risk-premium, using different approaches, Damodaran (2020) presents different estimates for the United States. All are between 3.20 percent and 5.20 percent.

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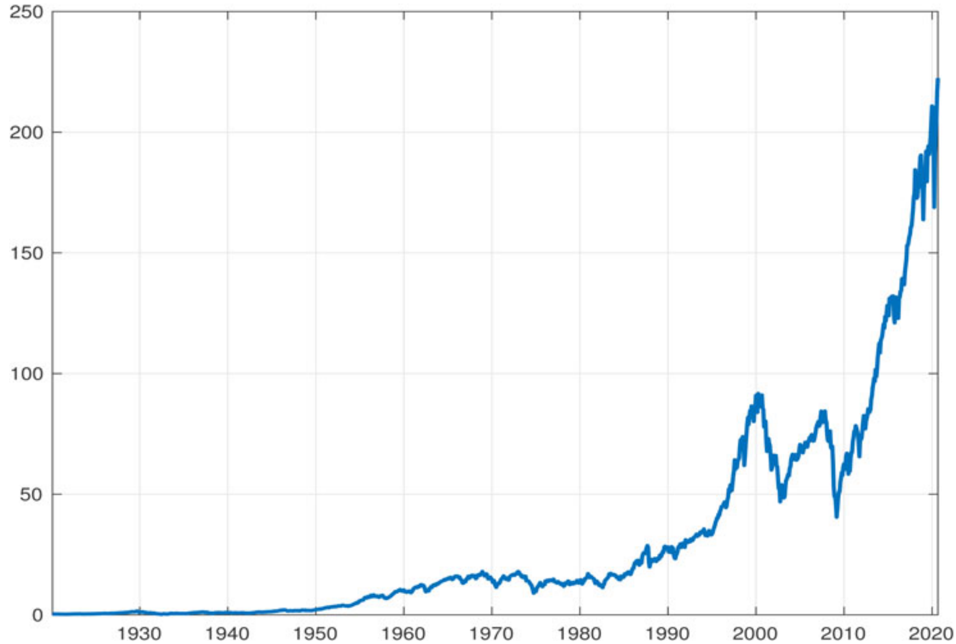

Figure 3. Discounted S&P500 total return index: time series of discounted monthly closing prices for the period January 1920 to August 2020.

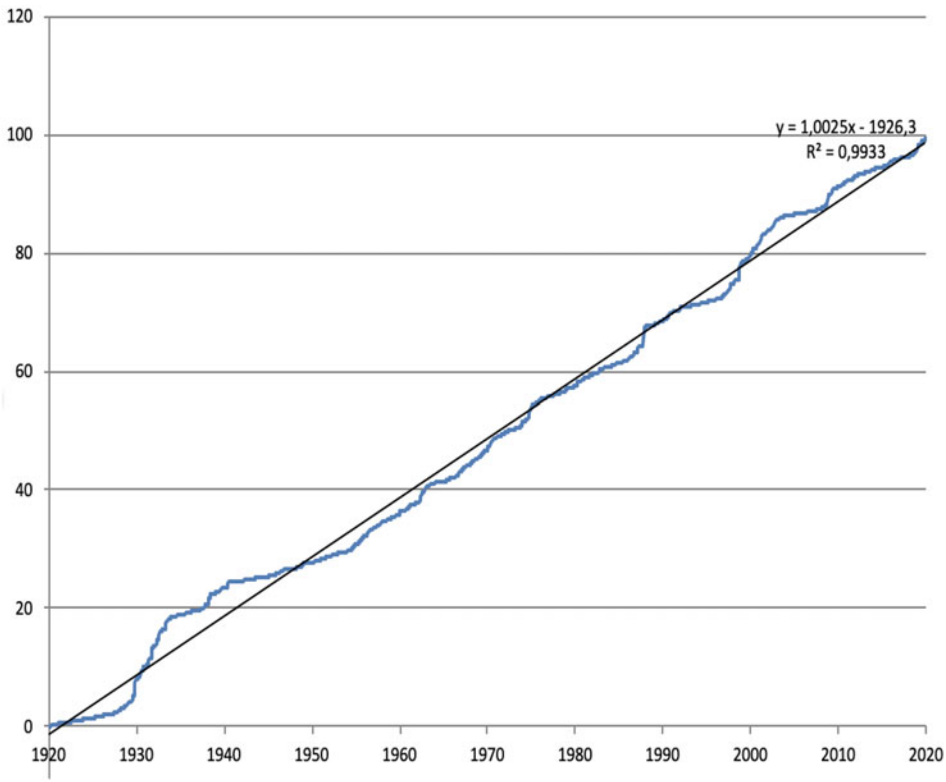

Figure 4. Resulting intrinsic time τ_t and trendline under the MMM with R^2 of 0.9933 for the period January 1920 to August 2020.

benchmark S_t, the numeraire and stock index, and forms a martingale. In Figure 5, we show the theoretical discounted value of V_t that pays in August 2020 one unit of the risk-free security (the continuous red line). Note that it fluctuates initially, similarly to the index, and approaches the risk-free security when close to maturity. Most importantly, its value is significantly less than one for many years, which makes it *less expensive* than the risk-free security. Only close to maturity is its value not much different to that of the risk-free security.

For illustration, recall Figure 1, which depicts the trajectories of the benchmarked risk-free security (in red) and the benchmarked investment portfolio (in blue). Note that the benchmarked risk-free security (red line) seems to fit a strict supermartingale, and the benchmarked investment portfolio (blue line) seems to fit a martingale.

The so-called 'hedge ratio', needed for the replication of the desired final target value is, similarly to the Black-Scholes hedge ratio for options, the partial derivative of the investment portfolio value with respect to changes in the stock index value

$$\frac{\partial V_t}{\partial S_t} = \exp\left\{\frac{-2\eta S_t}{\alpha(\exp\{\eta T\} - \exp\{\eta t\})}\right\} \frac{2\eta}{\alpha(\exp\{\eta T\} - \exp\{\eta t\})}. \tag{5}$$

Note that the strategy buys stocks when the index value S_t falls, and vice versa. This is not what many investment funds do when the stock markets decline severely. However, this is optimal under the given model, and also rational when believing that the stock index will revert back to its average long-term growth behavior.

The classical Law of One Price assigns a discounted price of one to the payout of one unit of the risk-free security. Thus, in theory, managing pension funds the proposed way using hedging, provides with high probability the *desired benefits* but at a *fraction* of the traditional (risk-neutral based) funding requirement when targeting one unit of the risk-free security. The benefit relative to choosing a traditional pension scheme under classical risk-neutral assumptions increases with the length of the time horizon. It is modest at short maturities, but becomes substantial at maturities commonly of interest, such as 30 years. For example, Figure 5 shows an approximate value of 0.66 for January 1990, representing a saving of around 33 percent. This is a consequence of the faster expected growth of the portion of funds invested in stocks. Closer to maturity, the progressive shift out of the stock

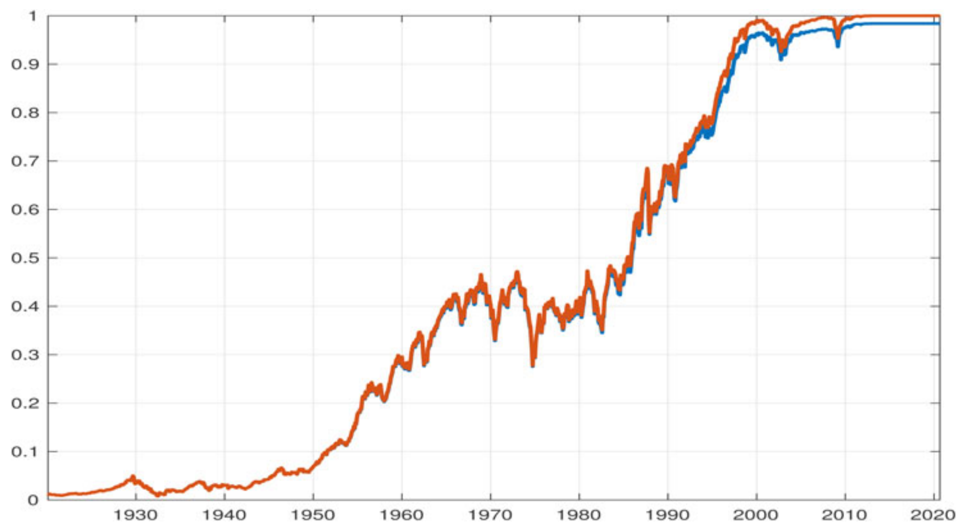

Figure 5. Theoretical discounted value of V_t that, in August 2020, pays one unit of the risk-free security (red line) and the self-financing discounted hedge portfolio (blue line).

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index into the risk-free security (equation (5)), necessary to achieve the risk-free target, reduces the growth of the investment portfolio. When the time horizon is extremely long, say 100 years, the saving increases to about 98 percent, as shown in Figure 5, with $V_0 = 0.012$ in January 1920.

The fraction $\pi_t = S_t(dV_t/dS_t)/V_t$ invested in the stock index at time t, which uses (12), is shown in Figure 6.

Note that, initially, almost all wealth is invested in the index, before gradually shifting to the risk-free security. This appears to be consistent with the traditional popular financial planning advice, where one should invest in equities when young, and then shift to a risk-free security when getting close to retirement. Next, we create a self-financing discounted hedge portfolio, where we reinvest the wealth monthly according to the fraction π_t. Figure 5 shows the value of the hedge portfolio (blue continuous line). Note that it evolves similarly to the theoretical value V_t (red continuous line), with some minor hedge error at maturity. This dynamic asset allocation is very similar to the well-established risk-neutral hedging of options. A major difference is that it is required to perform successfully over a period of 100 years in this case, whereas most options are hedged over only a few years. In practice, this represents a major mitigation of risk over time, which may be widely exploited in industry to reduce production costs significantly. For illustration, we later examine the hedge error under the MMM for a period of up to 100 years.

Concerning the cost of the strategy, the institution that hedges according to the strategy is providing highly desirable liquidity to the market because it buys more units of the index when the index is falling and vice versa. This can be inferred by the number of units held in the index. Importantly, brokers are usually very supportive to such liquidity provider, and do not charge major transaction costs. As such, we do not add any frictions in our model to keep it mathematically cleaner and easier to follow. Clearly, we are not claiming that the proposed strategy is cost-less, just that its performance and the main message of the paper will not change due to its cost. It is also worth noticing that, from a regulatory and macroeconomic perspective, the strategy also 'stabilizes' the market dynamics, because when the index falls and many investors want to sell in a crash, the institution that hedges the bond creates demand.

4. The overfunded plan

The previously presented hedging scheme has been shown to work well under a variety of alternative diffusion models for the index S_t by Gnoatto *et al.* (2018). However, its implementation in the pension industry requires a careful assessment of its possible shortcomings. For example, an important drawback of Fergusson and Platen (2023) and Gnoatto *et al.* (2018) is that their models rely on ad-hoc diffusion processes. Because parametric models are often not realistic, the obtained outcomes might lead

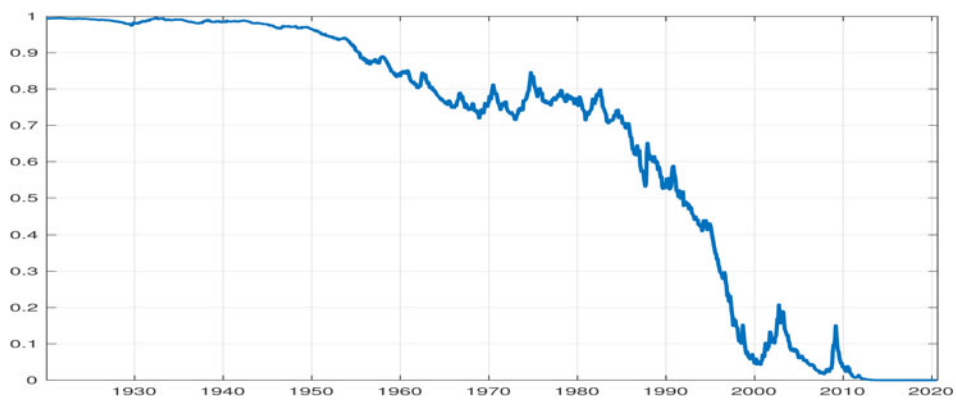

Figure 6. Fraction of stock index over time, defined as $\pi_t = S_t(dV_t/dS_t)/V_t$.

to under/overestimated final hedging outcomes. To overcome this problem, we propose an overfunded plan that adds extra funds to the hedging portfolio. As a result, it is possible to control the underestimation risk of the final targeted payout. In this section, we set up the overfunded plan and explain empirically how it works.

The overfunded plan still aims to achieve a long-term payoff $V_T = 1$ using a stock index and a risk-free security, but with a simple overfunding approach. For simplification purposes, we extract Brownian motion increments under the MMM, and bootstrap these to simulate alternative trajectories under the MMM. Let us introduce the discrete observation times $t_i = i/12$, for $i = 0, 1, \ldots$. We use the Euler scheme (see e.g., Kloeden and Platen (1992)) to discretize the SDE (12) in Appendix B. We do so for two reasons. First, we use it to extract the postulated Brownian motion increments under the MMM, shown in Figure 7, from the observed discounted index increments.

Second, we use it to simulate alternative trajectories for the index using the bootstrapped Brownian motion increments. For the SDE (12), the Euler scheme is of the form

$$S_{t_{i+1}} - S_{t_i} = \alpha e^{\eta t_i}(t_{i+1} - t_i) + \sqrt{S_{t_i}\alpha e^{\eta t_i}}(W_{t_{i+1}} - W_{t_i}). \tag{6}$$

We rely on a simple, but effective bootstrap approach. However, more complex approaches are indeed applicable (e.g., the filtered historical simulation approach of Barone-Adesi *et al.* (1999)).

Recall that Figure 3 represents the monthly observed time series of the S&P500 total return index, denominated in units of a monthly rolled-over three-month US T-Bill portfolio, for the period from January 1920 until August 2020. Being a total return index, the time series is computed by reinvesting all dividends paid over time. We first obtain the bootstrapped time series by extracting the historical increments of the Brownian motion under the MMM; (see Figure 7). Then, we generate 10,000 Brownian motion-type trajectories by bootstrapping the extracted increments. Specifically, the historical Brownian motion increments are extracted to provide a pool of about 100·12 = 1,200 monthly Brownian motion increments. These are then randomly resampled with replacement to generate

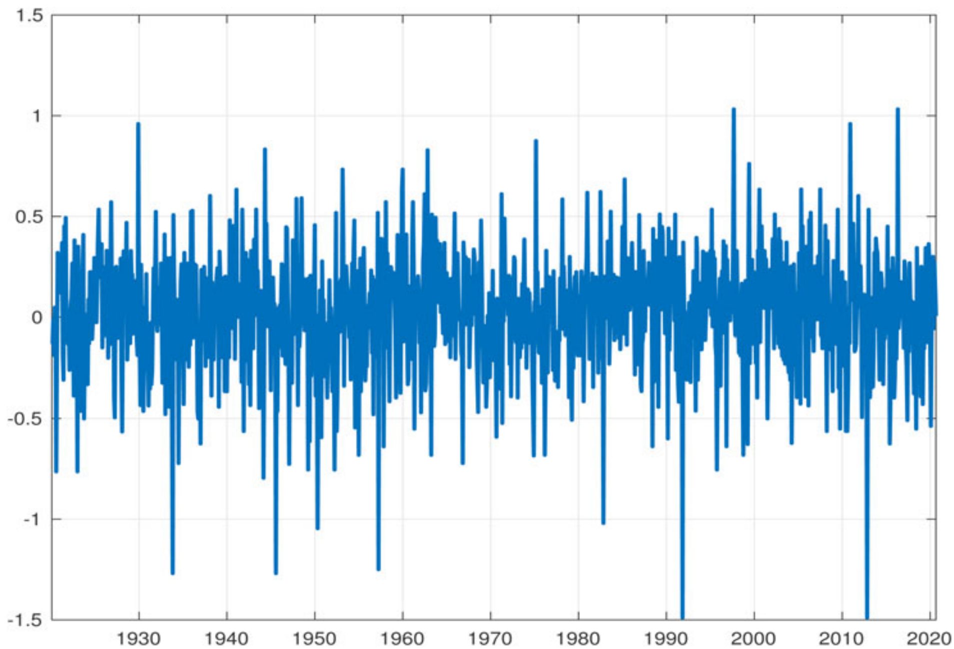

Figure 7. Increments of the MMM extracted Brownian motion: time series of monthly increments for the period January 1920 to August 2020.

the 10,000 simulated (bootstrapped) Brownian motion-type trajectories for the observation period, January 1920 to August 2020. Needless to say, the number of simulations can be increased with almost no limit. These simulated Brownian motion-type trajectories are then used in the Euler scheme (6) to generate the trajectories of an index under the MMM that resemble the path of the discounted S&P500 total return index. The logarithms of 50 of these simulated trajectories are shown in Figure 8.

Note the long-term mean-reversion of these trajectories. Using a straightforward generalization of the Black-Scholes model to bootstrap the observed S&P500 returns would not have generated such mean-reversion.

Following equation (10), we first compute the discounted investment portfolio value, for which only 50 trajectories are depicted for clarity in Figure 9.

We set the initial value of the hedge portfolio equal to the initial value of the discounted investment portfolio ($V_0 = 0.012$). Then, following equation (5), and for each of the 10,000 simulated index trajectories, we compute the trajectory of the hedge portfolio in a self-financing manner until maturity, with a monthly reallocation, using the fraction invested in the index involving the hedge ratio (5). From January 1920 until August 2020, the time horizon of all simulations is fixed at 100 years. Indeed, to compute the hedge portfolio, we need to compute both the hedge ratio (dV_t/dS_t) and the fraction of the investment portfolio invested in the stock index over time (π_t); see Figure 10.

Once the hedge ratio and the quantity π_t are known, we can compute the investment plan H_t, which is defined as follows:

$$H_{t_{i+1}} = H_{t_i} \cdot \left[1 + \pi_{t_i} \cdot \left(\frac{S_{t_{i+1}}}{S_{t_i}} - 1\right)\right]. \tag{7}$$

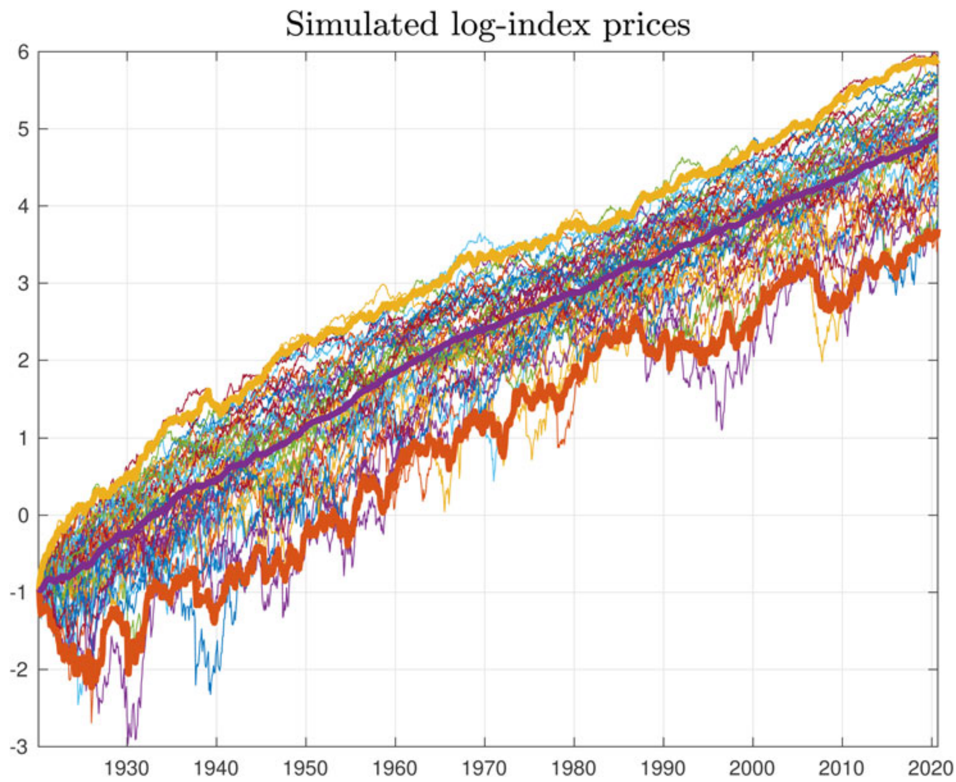

Figure 8. 50 trajectories of the bootstrapped logarithms of the S&P500 total return index for the period January 1920 to August 2020. Superimposed: the mean and the 5% and 95% confidence intervals.

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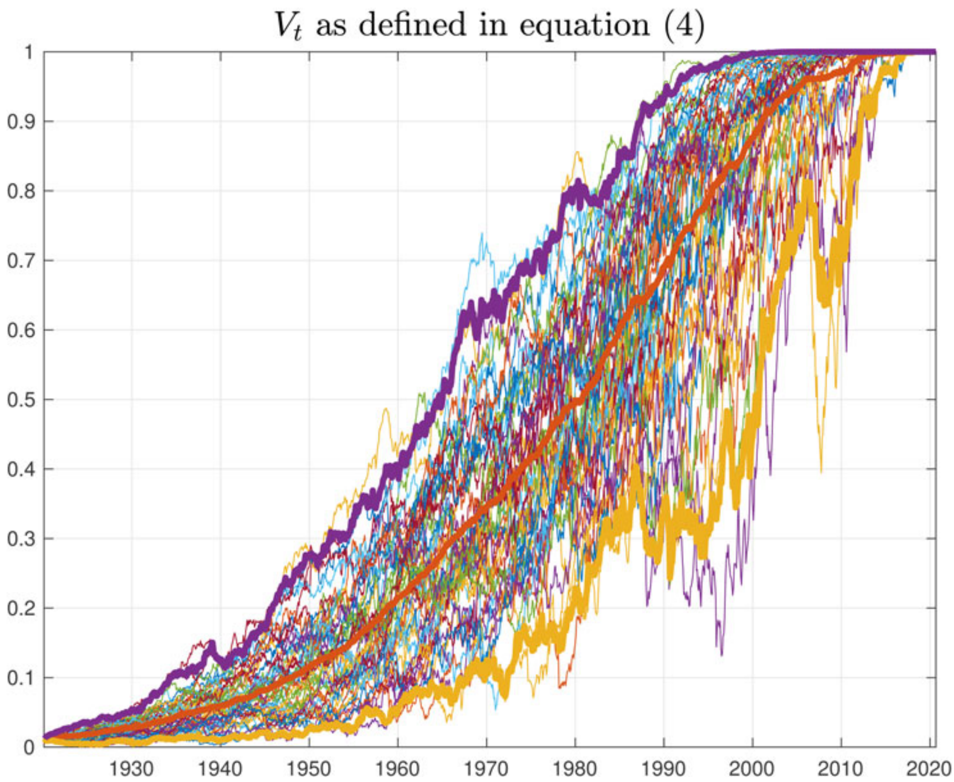

Figure 9. Discounted investment portfolio value for the simulated total return index for the period January 1920 to August 2020, computed using equation (10). Superimposed: the mean and the 5% and 95% confidence intervals.

We perform the hedge simulation for all bootstrapped index scenarios; 50 trajectories are shown in Figure 11. We interpret this dynamic asset allocation as the basic investment plan. Note that most hedges end up rather close to 1.0 at maturity.

Taken from another viewpoint, Figure 12 shows the histogram values of the 10,000 bootstrapped investment plans at maturity. As expected, the histogram is centered around one and strongly non-normal.

Figures 11 and 12 both show that model errors and discretization errors in the hedge implementation for the proposed basic investment plan mean it is not possible to exactly achieve the target of one unit of the risk-free security at maturity. The main reasons for the hedging errors are that we have no continuous hedging and that the model is not perfect. To overcome this problem, we propose *overfunding* the basic investment plan. The overfunded plan is defined by following the investment strategy, with the same fractions as before. The only change is that we have a higher initial capital $V_0^x = xV_0$, for $x > 1$, invested in the hedge portfolio, which yields the following overfunded hedge portfolio value:

$$V_t^x = xV_t, \tag{8}$$

for $0 \leq t \leq T$, where $100 \cdot (x - 1)$ determines the percentage of the overfunding. Note that the overfunded plan simply multiplies the entire basic investment plan by x because the self-financing hedging portfolio evolves such that (8) holds. The returns remain the same, but the initial value is larger. Figure 13 depicts the original basic investment plan with no overfunding $x = 1$ (top left), and the three overfunded strategies, $x_1 = 1.02$ (top right), $x_2 = 1.04$ (bottom left), and $x_3 = 1.06$ (bottom

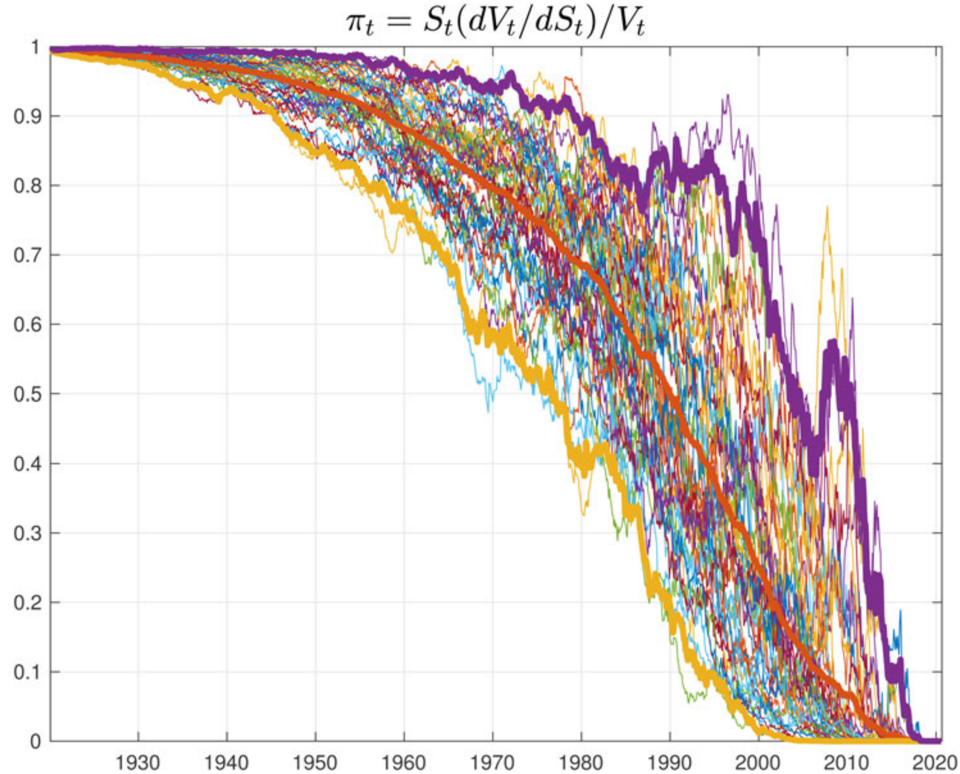

Figure 10. Fraction of investment portfolio invested in the stock index over time for the period January 1920 to August 2020, defined as $\pi_t = S_t(dV_t/dS_t)/V_t$. Superimposed: the mean and the 5% and 95% confidence intervals.

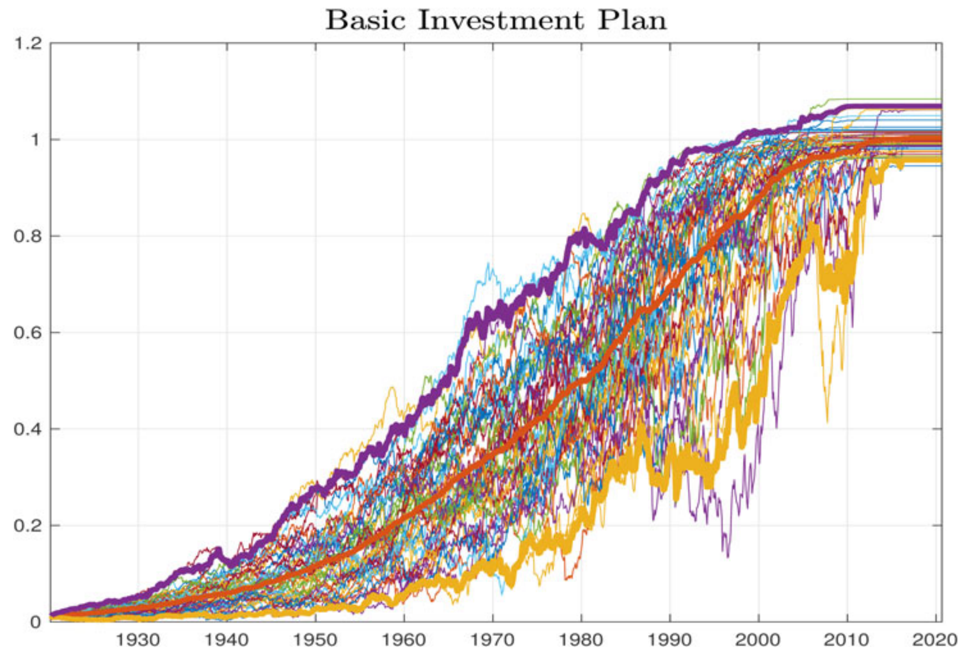

Figure 11. Bootstrapped basic investment plan for the simulated total return index for the period January 1920 to August 2020. Superimposed: the mean and the 5% and 95% confidence intervals.

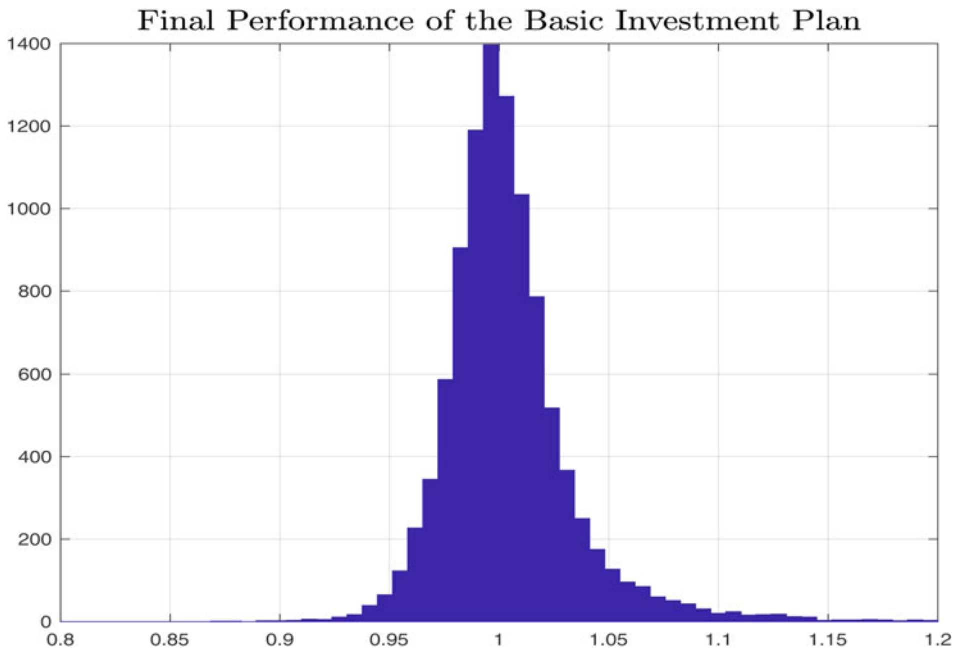

Figure 12. Histogram of the basic investment plan at maturity. The histogram captures the last value of the investment computed for the period January 1920 to August 2020.

right). This figure shows how almost all trajectories with $x_3 = 1.06$ lead to final values greater than the targeted payoff of one.

Once more, and to better understand the final outcomes of the overfunded plans, we focus on their final payouts. Figure 14 shows (in clockwise order) histograms of the final values of the basic investment plan and the overfunded plans $V_T^x = xV_T$, with x_1, x_2, and x_3 representing an overfunding of 1.02, 1.04, and 1.06, respectively.

To better understand the magnitude of the still remaining chance of underachieving, Table 2 collects the basic summary statistics of the final values of the overfunded plans, where the first column (overfunding weight x) defines the level of overfunding. Note how the arithmetic means of all schemes confirm equation (8), and the 25 percent and 75 percent quantiles together with the 99.9 percent percentile values confirm the increasing success of the overfunded plans over the basic plan. This is further emphasized by the rate of failure (Ratio), computed as the percentage of the total number of simulations that the plan does not reach the target of one, and the expected shortfall (ES), which accounts for the expected loss in units of the risk-free security when the target has not been achieved. While the basic investment plan misses the target about half the time (50.72%), producing an expected shortfall of 0.0083 units of the risk-free security, overfunding the basic investment plan quickly decreases both the rate of failure and the expected shortfall of the investment. More precisely, with 6 percent overfunding, both the probability of missing the target and the expected shortfall become almost negligible (0.87% and 0.0002, respectively).

To test the robustness of the overfunded plan, we repeat the analysis for three different time horizons of the investment period. Specifically, we analyze three 30-year periods: 1920–1950, 1950–1980, and 1990–2020. For each study, we use the simulated 10,000 stock index trajectories, changing the starting times and maturities for the pricing and hedging accordingly. Note that all hedge portfolios are initialized using their respective theoretical bond values, which are determined fully by the simulated random value of the stock index at that time. This makes the initial value of the hedge portfolio quite random, an important effect that may impact the accuracy of the hedge. However, similarly to

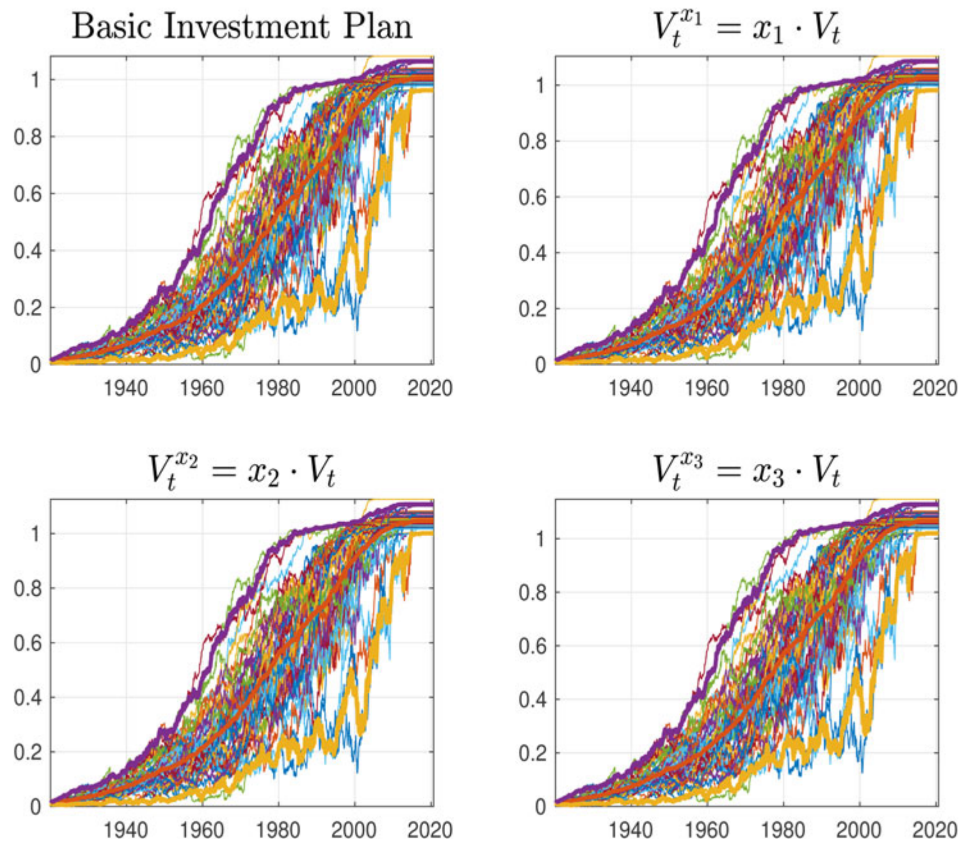

Figure 13. Bootstrapped overfunded plans: The four panels capture the basic investment plan (top left) with no overfunding ($x = 1$) and the three overfunded schemes, $x_1 = 1.02$ (top right), $x_2 = 1.04$ (bottom left), and $x_3 = 1.06$ (bottom right). Superimposed in all panels: the mean and the 5% and 95% confidence intervals.

Black–Scholes option hedging, if the model is correct, the hedge should theoretically generate the targeted payout independently of the particular scenario for the stock index. Comparing Tables 2 and 3, the basic investment plan performs slightly worse over 30 years than over 100 years in terms of the percentage of failure, but with a lower expected shortfall. Furthermore, the percentage of failure is almost the same for each of the three periods and close to 60 percent. However, the magnitude of the expected shortfall decreases when the stock index value at the initialization of the hedge becomes more random. This result can be explained by the fact that there is more averaging over the stationary density available to diversify over-time the stock index value. Interestingly, overfunding the basic investment plan by 6 percent is again enough to make the likelihood of failure and the expected loss almost negligible.

A similar trend is confirmed by keeping the maturity date fixed as August 2020 and considering plans of 10, 20, 30, and 50 years. Table 4 reports the summary statistics of the basic investment plan and the overfunded plans, where the plans end in August 2020 and start in January 2010 (10-year plan), January 2000 (20-year plan), and January 1970 (50-year plan).

From Table 4, interesting results emerge. First, the 6 percent overfunding rule still applies. Almost irrespective of the time length and period of observations, 6 percent overfunding appears to be high enough to achieve about 99 percent of the targeted value. Second, as the time window widens, the likelihood of failure and the expected shortfall both appear to increase. This is mostly the result of hedging errors that tend to build up over time. However, note that this trend is not monotonic. In fact, as soon

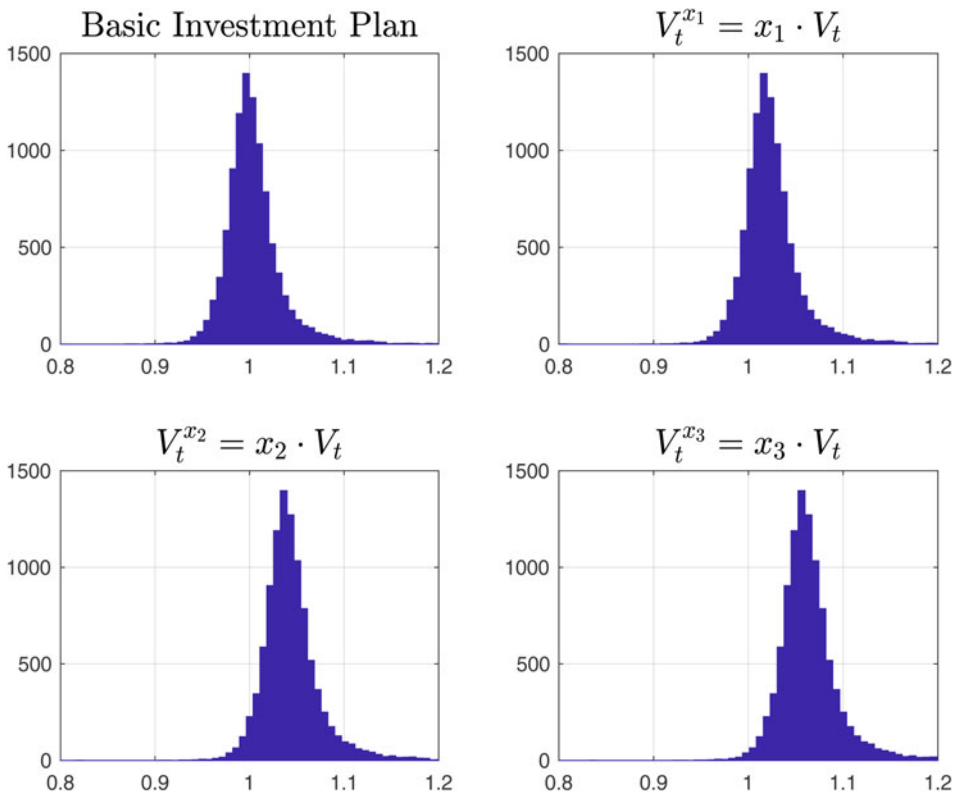

Figure 14. Histogram of the final values of the overfunded plans: The four panels capture $V_T^x = xV_T$ for the basic investment plan (top left) with no overfunding ($x = 1$) and the three overfunded plans, with $x_1 = 1.02$ (top right), $x_2 = 1.04$ (bottom left), and $x_3 = 1.06$ (bottom right).

Table 2. Summary statistics for the period January 1920 to August 2020 of the basic investment plan with no overfunding ($x = 1$), and for the three overfunded plans, with $x_1 = 1.02$, $x_2 = 1.04$, and $x_3 = 1.06$

Overfunding weight	Mean	Median	25%	75%	99.9%	Ratio	ES
$x = 1$	1.0059	0.9996	0.9869	1.0155	1.4841	0.5072	0.0083
$x_1 = 1.02$	1.0260	1.0197	1.0067	1.0358	1.5138	0.1561	0.0021
$x_2 = 1.04$	1.0461	1.0397	1.0264	1.0561	1.5435	0.0356	0.0005
$x_3 = 1.06$	1.0662	1.0597	1.0462	1.0764	1.5731	0.0087	0.0002

All values are computed for the final value of the investment plan, defined as $V_T^x = xV_T$.

as the investment plan covers more than 60 years, a time diversification effect is evident, as demonstrated by the lower failure rate documented for the 100-year analysis presented in Table 2. Overall, the number of positive outcomes increases with the length of the time window. However, this is only true for plans longer than 60 years.[6]

4.1 Model robustness

Given the set-up presented in this paper, an investor is confronted with some basic problems, that is: what happens to the performances of the model in the presence of some model uncertainty? What

[6]For ease of space, the graphical outputs of the respective cited simulations have been omitted and are available upon request.

Table 3. Summary statistics for three 30-year periods of the basic investment plan with no overfunding ($x = 1$), and of the three overfunded plans with $x_1 = 1.02$, $x_2 = 1.04$, and $x_3 = 1.06$

Overfunding weight	Mean	Median	25%	75%	99.9%	Ratio	ES
Period 1920–1950							
$x = 1$	1.0052	0.9941	0.9751	1.0182	1.9618	0.5800	0.0165
$x_1 = 1.02$	1.0253	1.014	0.9946	1.0385	2.0011	0.3192	0.0078
$x_2 = 1.04$	1.0455	1.0339	1.0141	1.0589	2.0403	0.1207	0.0039
$x_3 = 1.06$	1.0656	1.0537	1.0336	1.0793	2.0795	0.0394	0.0025
Period 1950–1980							
$x = 1$	1.0028	0.9938	0.9779	1.0134	1.7822	0.5921	0.0133
$x_1 = 1.02$	1.0228	1.0137	0.9974	1.0337	1.8178	0.2831	0.0047
$x_2 = 1.04$	1.0429	1.0336	1.0170	1.0540	1.8535	0.0891	0.0015
$x_3 = 1.06$	1.0630	1.0535	1.0366	1.0742	1.8891	0.0258	0.0005
Period 1990–2020							
$x = 1$	1.0007	0.9974	0.9893	1.0050	1.3388	0.6087	0.0078
$x_1 = 1.02$	1.0208	1.0174	1.0091	1.0251	1.3656	0.1246	0.0018
$x_2 = 1.04$	1.0408	1.0374	1.0289	1.0452	1.3924	0.0307	0.0005
$x_3 = 1.06$	1.0608	1.0573	1.0487	1.0653	1.4192	0.0085	0.0002

All values are computed for the final value of the investment plan, defined as $V_T^x = xV_T$.

Table 4. Summary statistics for 10-, 20-, and 50-year periods of the basic investment plan with no overfunding ($x = 1$), and of the three overfunded plans with $x_1 = 1.02$, $x_2 = 1.04$, and $x_3 = 1.06$

Overfunding weight	Mean	Median	25%	75%	99.9%	Ratio	ES
Period 2010–2020							
$x = 1$	1.0004	0.9996	0.9987	1.0007	1.0749	0.6008	0.0008
$x_1 = 1.02$	1.0204	1.0196	1.0187	1.0208	1.0964	0.0006	1.29×10^{-5}
$x_2 = 1.04$	1.0404	1.0396	1.0387	1.0408	1.1179	0.0003	6.32×10^{-6}
$x_3 = 1.06$	1.0604	1.0596	1.0587	1.0608	1.1394	0.0003	6.78×10^{-7}
Period 2000–2020							
$x = 1$	1.0010	0.9998	0.9987	1.0006	1.0335	0.6152	0.0008
$x_1 = 1.02$	1.0201	1.0196	1.0187	1.0206	1.0542	0.0002	4.48×10^{-6}
$x_2 = 1.04$	1.0408	1.0396	1.0387	1.0406	1.0748	0.0001	1.02×10^{-6}
$x_3 = 1.06$	1.0612	1.0596	1.0586	1.0607	1.0955	0	0
Period 1970–2020							
$x = 1$	0.9828	0.9928	0.981	1.0061	1.4048	0.6507	0.0255
$x_1 = 1.02$	1.0025	1.0127	1.0006	1.0263	1.4329	0.2385	0.0171
$x_2 = 1.04$	1.0222	1.0326	1.0202	1.0464	1.4610	0.0508	0.0151
$x_3 = 1.06$	1.0418	1.0525	1.0399	1.0665	1.4891	0.0113	0.0148

All values are computed for the final value of the investment plan, defined as $V_T^x = xV_T$. The periods end in August 2020 and start in January 2010, January 2000, and January 1970, respectively.

happens if the model parameters are mis-estimated? For which time horizons is the model more robust? In this subsection we provide answers to these questions and show the robustness of the basic model and the overfunded models. Only results for underestimated parameters are presented because overestimated parameters improve convergence, at the cost of a higher initial investment.

As a first test, we consider the case faced by an investor who estimates the model parameters with an error. In such a case the investor builds the portfolio and hedges it with the net growth rate $\eta = 0.048$, which turns out to be different to those we use for the simulation of the testing data. The simulated testing data are generated with a misspecified version of the Euler scheme presented by equation (6):

$$S_{t_{i+1}} - S_{t_i} = \alpha^* e^{\eta^* t_i}(t_{i+1} - t_i) + \sqrt{S_{t_i} \alpha^* e^{\eta^* t_i}}(W_{t_{i+1}} - W_{t_i}). \tag{9}$$

where $\alpha^\star$ and $\eta^\star$ are the perturbed versions of the normalization factor and the net growth rate, respectively.

To better disentangle the role of the misspecification of parameters we modify only one parameter at a time, and we perturb it by a multiple $n = 1, 2, \ldots, n < \infty$ of their estimated standard error. We start with a misspecified normalization parameter $\alpha^* = \alpha - (\varepsilon^\alpha \cdot n)$, where ε^α is the standard error of the normalization factor α and n the number of standard errors considered. This first test shows an extreme model robustness for the 100 years 6 percent overfunded plan, as it achieves the target for $n = 7$. Interestingly, the model becomes more robust as we shorten the time horizons. With a time horizon of 30 years, which is a more realistic time horizon for an investor who saves for her pension, the 6 percent overfunded plan achieves its target even for $n = 10$ (while the basic plan is 5% off target).

Next, we keep α fixed and misspecify the net growth rate $\eta^* = \eta - (\varepsilon^\eta \cdot n)$, where ε^η is the standard error of the growth factor η. Empirical results show that, with a one standard error perturbation, the 6 percent overfunded plan still achieves the target at both 100 and 30 years, while the basic plan is 7 percent off target. Interestingly, the 30 years plan is the one that shows more robustness, as with $n = 2$ the overfunded plan is just 1.5 percent off target.

With an additional robustness test we check what happens if one misspecifies the parameters α and η in both the portfolio construction and hedging. Such a test requires the mis-estimation of

$$V_t = 1 - \exp\left\{-\frac{2\eta^* S_t}{\alpha^*(\exp\{\eta^* T\} - \exp\{\eta^* t\})}\right\} \tag{10}$$

and the relative passages necessary to build and hedge the portfolio.

Parameters, $\alpha^* = \alpha - (\alpha \cdot y)$ and $\eta^* = \eta - (\eta \cdot y)$ are once more the original parameters, this time perturbed by a percentage y. Once more, the model shows a strong robustness to the normalization parameter α, which still achieves the target with a misspecification of the parameter of $y = -5\%$. For the growth rate, the robustness of the 6 percent overfunded model remains very high up to $y = -1\%$, and it decreases as y increases. Remarkably, and once more, the 30 years plan shows more model robustness than those for most other time horizons.

4.2 The impact of mean-reversion and risk-premium for the overfunding model

Among the tests performed in the previous section we analyzed what happens to the overfunding model in the presence of parameters mis-estimation. Among others, we have tested the robustness of the model in the presence of a contaminated growth factor. In this section, we dig even further into the role of the risk-premium and we shed light on the role of the mean-reversion on the final performance of the overfunding model. We add some robustness simulations where we first eliminate the mean-reversion from our simulations, then we simulate our model with the MMM but only half of the risk-premium, and finally we mix these two effects, thus producing a simulation without mean-reversion and half of the risk-premium.

In particular, we examine the effect of making wrong assumptions on the stochastic process generating our simulated returns. Hedge ratios are computed under the same parameter values we used previously, as if the process generating the data had not changed, but the new simulation data are generated with (i) half of the risk-premium, (ii) from a geometric Brownian motion and (iii) with one half of the risk-premium applied to a Brownian motion in the third panel. In other words, we are hedging incorrectly to reflect our ignorance of the true process. We focus our analysis on finite horizons only, always controlling the risk through overfunding.

For the first test, we simply set $\eta = 0.048/2$, which is half of the value used throughout the paper and that we obtained with the numerical approach presented in Appendix A for the period from January 1920 until August 2020. Results of the simulated outcomes are presented in Table 5.

As it emerges from Table 5, the proposed approach remains robust with respect to the effect of a much lower risk-premium, which could be mitigated through a modest overfunding.

To remove the mean-reversion and generate the simulated returns from a geometric Brownian motion, we first need to estimate the parameters entering the Black–Scholes dynamics. We first estimate

Table 5. Summary statistics for the period January 1990 to August 2020 of the basic investment plan with no overfunding ($x = 1$), and for the three overfunded plans, with $x_1 = 1.02$, $x_2 = 1.04$, and $x_3 = 1.06$

Overfunding weight	Mean	Median	25%	75%	99.9%	Ratio	ES
$x = 1$	0.99986	0.9999	0.9987	1.0003	1.1052	0.6215	0.0017259
$x_1 = 1.02$	1.0199	1.0199	1.0187	1.0203	1.1273	0.0141	0.0001498
$x_2 = 1.04$	1.0399	1.0399	1.0387	1.0403	1.1494	0.0023	0.0000207
$x_3 = 1.06$	1.0599	1.0599	1.0587	1.0603	1.1716	0.0003	0.0000056

All values are computed for the final value of the investment plan, defined as $V_T^x = xV_T$ and applying a growth factor equal to $\eta = 0.048/2$.

Table 6. Summary statistics for the period January 1990 to August 2020 of the basic investment plan with no overfunding ($x = 1$), and for the three overfunded plans, with $x_1 = 1.02$, $x_2 = 1.04$, and $x_3 = 1.06$

Overfunding weight	Mean	Median	25%	75%	99.9%	Ratio	ES
$x = 1$	0.9878	1.0015	1.001	1.022	1.0738	0.1392	0.023985
$x_1 = 1.02$	1.0076	1.0215	1.019	1.0425	1.0953	0.1192	0.021888
$x_2 = 1.04$	1.0274	1.0416	1.039	1.0629	1.1167	0.1063	0.020116
$x_3 = 1.06$	1.0471	1.0616	1.058	1.0834	1.1382	0.0921	0.018601

All values are computed for the final value of the investment plan, defined as $V_T^x = xV_T$ and generated by a Black–Scholes dynamics.

Table 7. Summary statistics for the period January 1990 to August 2020 of the basic investment plan with no overfunding ($x = 1$), and for the three overfunded plans, with $x_1 = 1.02$, $x_2 = 1.04$, and $x_3 = 1.06$

Overfunding weight	Mean	Median	25%	75%	99.9%	Ratio	ES
$x = 1$	0.97496	1.0001	1.001	1.0074	1.0425	0.1769	0.029981
$x_1 = 1.02$	0.99446	1.0201	1.018	1.0276	1.0633	0.1471	0.027379
$x_2 = 1.04$	1.014	1.0401	1.037	1.0477	1.0842	0.1265	0.025247
$x_3 = 1.06$	1.0335	1.0601	1.059	1.0679	1.1051	0.1121	0.023442

All values are computed for the final value of the investment plan, defined as $V_T^x = xV_T$ and generated by a Black–Scholes dynamics and $\eta = 0.048/2$.

the volatility of the MMM realized log-returns at 10-year periods. The value of 17 percent for σ is aligned with the literature concerning the annual volatility of the S&P 500. It is theoretically grounded, as the average multi-yearly volatility of returns of the MMM decreases over time because of long-term mean-reversion from 17 percent at 10-year period to 11 percent at 30-year period. This quantity is used as a volatility input (σ) to simulate returns through a geometric Brownian motion:

$$S_{t_{i+1}} = S_{t_i} exp^{\left[\left(\mu - \frac{\sigma^2}{2}\right)(t_{i+1} - t_i) + (W_{t_{i+1}} - W_{t_i})\right]}. \tag{11}$$

with a drift parameter $\mu = \eta + \sigma^2/2$, $\eta = 0.048$ and W denoting the stochastic differential of the Brownian motion. The portfolio analysis is then performed as if the simulated returns would come from an MMM dynamics. Results are depicted in Table 6.

The effect of removing the mean-reversion from the simulated returns has a minor impact on all variables, and the overall strategy remains robust and an investor can still achieve its target by slightly increasing the overfunding.

Finally, we repeat the same exercise, with a reduced risk-premium, so that $\eta = 0.048/2$ and $\mu = \eta + \sigma^2/2$. Results are depicted in Table 7.

As from Table 7 the impact of the mean-reversion and the risk-premium are not of major importance, as the overall strategy remains robust.

The intuition behind the above results is that mean-reversion is hard to detect on individual stock paths of finite length. Therefore, it is not surprising that its effects on the proposed strategies are modest at a 30-year time horizons. Overall, we confirm that the role of mean-reversion is potentially very important in the very long run (Pastor and Stambaugh (2012)), but much less on the thirty year horizon on which we focus our empirical analysis. It is also worth noticing that our asset allocation

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tends to reduce stock allocation in later years, reducing the effects of long-term uncertainty in stock parameters. Likewise, also the size risk-premium has just a minor effect on the proposed overfunding scheme, and both the mean-reversion and the size risk-premium are not driving the performances of the overfunding scheme and for more risk averse investors, their effect can be alleviated through a minimal extra overfunding.

5. Conclusion

The strategy we discussed promises to achieve the long-term objectives of traditional pension and target date investment policies at a fraction of their traditional theoretical initial cost. Its practical implementation requires an assessment of its long-term reliability and an evaluation of their safety margin. To fulfill these prudential requirements, the proposed overfunded scheme allows for the choice and the evaluation of the desired safety level. Our results show that an overfunding of 6 percent achieves the desired final value more than 99 percent of the time, with an expected shortfall not greater than 0.2 percent of the targeted value. To corroborate the robustness of our model, similar results are achieved when changing the market scenario by adding market uncertainty, model misspecification, lower risk-premium, and removing the mean-reversion. Therefore, appropriate overfunding appears to allow for the safe use of this strategy in the investment industry. With a wider use of this methodology, a similar gain in the efficiency of long-term risk management for pensions and other long-term investments may become achievable.

Our aim is the introduction of a simple framework for the management of shortfall risk in TDFs. Additional capital or policy changes may be required to improve the robustness of the proposed scheme in order to achieve the desired outcomes with even higher confidence. Where appropriate, remedial measures must be introduced to address potential pitfalls. The effects of randomness in volatility, price jumps, interest rates (when cash is targeted as the payout), and the equity risk-premium on the performance of the proposed policy should also be analyzed. Particular attention should be given to the additional funding requirements under a variety of stress conditions. Finally, the effect of a large-scale implementation on the overall stability of financial markets should be evaluated. As such, although we provide an initial basic overfunded approach to implement a pension scheme, many questions remain for further investigation.

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A. The minimal market model (MMM)

As defined in the main text, we model the stock index dynamics as:

$$dS_t = \alpha_t dt + \sqrt{S_t \alpha_t} dW_t, \tag{12}$$

where $\alpha_t = S_t \theta_t^2 = \alpha \exp(\eta \cdot t)$, with θ_t denoting the volatility. Here, $\alpha > 0$ is a normalization parameter, $\eta > 0$ is the (net) growth rate of S_t, and $W = \{W_t \geq 0, t\}$ is the driving canonical Brownian motion.

This is the MMM introduced by Platen (2001) and described in Platen and Heath (2010), where S_t models the growth optimal portfolio as a time-transformed squared Bessel process of dimension four. This yields the ergodic Cox–Ingersoll–Ross process $Y_t = S_t/\alpha_t$ when normalized, which is linearly mean-reverting. The index conditioned on previous values follows a noncentral chi-square distribution with four degrees of freedom with respect to market time, which depends on the value of the index and keeps the index going up and down but never reaches zero.

Note that the volatility $\theta_t = \sqrt{\alpha_t/S_t}$ with respect to market time is equal to $1/\sqrt{Y_t}$, which generates the well-known leverage effect. Thus, $S_t = Y_t\alpha_t$ is modeled as the product of the mean-reverting process Y_t with stationary density, and an exponential function α_t of market time. It is important that we realistically capture with these dynamics the leverage effect and the mean-reversion, which are both typical for the long-term dynamics of a stock index.

The mean-reversion is clearly visible for the logarithm of the stock index shown in Figure 2 and it is difficult to imagine a reason for not assuming that this mean-reversion with respect to market time will continue as long as the market exists. Of course, the mean-reversion is not too important when modeling over shorter time periods. The key property that makes the proposed approach working under the model is the strict supermartingale property of the risk-free security when denominated in units of the stock index.

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B. Parameter estimation under the MMM

In this appendix we present the maximum likelihood estimation approach used to estimate the MMM parameters in the paper. Specifically, the goal is to find the parameters α and θ of the SDE presented in equation (12) that maximize the occurrence of the observations under the hypothesis that the MMM holds true. To achieve it we first need to define the transition density function of the discounted stock index (see Revuz and Yor (1999)):

$$p_S(t, x_t, T, x_T) = \frac{1}{2(\varphi_T - \varphi_t)}\sqrt{\frac{x_T}{x_t}}\exp\left(-\frac{x_t + x_T}{2(\varphi_T - \varphi_t)}\right)I_t\left(\frac{\sqrt{x_T x_t}}{\varphi_T - \varphi_t}\right) \tag{13}$$

where $I_\upsilon(z) = \sum_{m=0}^{\infty}((1/2z)^{2m+\upsilon}/m!\Gamma(m+1+\upsilon))$ is the modified Bessel function of the first kind with index υ and $\varphi_t = 1/4\eta(\alpha(\exp(\eta t) - 1))$ is the quadratic variation of $\sqrt{S}$.

Given a finite time series $t_0 < t_1 < \cdots < t_n < \infty$ of observations of the discounted index $S_{t_0}, S_{t_1}, \ldots, S_{t_n}$ and using the above defined transition density function, we maximize the logarithm of the likelihood function:

$$l(\alpha, \eta) = \tag{14}$$

$$= \sum_{i=1}^{n}\log\left(\frac{1}{2(\varphi_{t_i} - \varphi_{t_i-1})}\sqrt{\frac{S_{t_i}}{S_{t_i-1}}}\exp\left(-\frac{S_{t_i-1} + S_{t_i}}{2(\varphi_{t_i} - \varphi_{t_i-1})}\right)I_1\left(\frac{\sqrt{S_{t_i}S_{t_i-1}}}{\varphi_{t_i} - \varphi_{t_i-1}}\right)\right) \tag{15}$$

$$= \sum_{i=1}^{n}\left\{\log\left(\frac{1}{2(\varphi_{t_i} - \varphi_{t_i-1})}\right) + \frac{1}{2}\log\left(\frac{S_{t_i}}{S_{t_i-1}}\right) + \left(-\frac{S_{t_i-1} + S_{t_i}}{2(\varphi_{t_i} - \varphi_{t_i-1})}\right) + \log\left(I_1\left(\frac{\sqrt{S_{t_i}S_{t_i-1}}}{\varphi_{t_i} - \varphi_{t_i-1}}\right)\right)\right\}. \tag{16}$$

To initialize the computation we need to find the initial estimates of both α and θ, which can be found by equating the theoretical quadratic variation of $\sqrt{S}$:

$$\langle\sqrt{S}\rangle_{t_j} = \frac{1}{4\eta}\alpha(\exp(\eta t_j) - 1) \tag{17}$$

with its empirical counterpart

$$\langle\sqrt{S}\rangle_{t_j} \approx \sum_{\sigma+j}^{i=1}\left(\sqrt{S_{t_i}} - \sqrt{S_{t_{i-1}}}\right)^2. \tag{18}$$

For $t = t_k$ and $t = t_{2k}$, where $k = \lfloor +;\ n/2\rfloor$, the initial estimates of the parameters are defined as:

$$\alpha_0 = \langle\sqrt{S}\rangle_{t_k}\frac{4\eta}{\exp(\eta t_k) - 1} \tag{19}$$

and

$$\eta_0 = \log\frac{\langle\sqrt{S}\rangle_{t_{2k}}/\langle\sqrt{S}\rangle_{t_k} - 1}{t_k - t_0}. \tag{20}$$

The first iteration to find the logarithm of the likelihood function is computed using the above defined initial estimates at points $\alpha_0 + i\delta\alpha$ and $\eta_0 + j\delta\eta$ for $i, j = -2, -1, 0, 1, 2$, $\delta\alpha = \alpha_0/4$ and $\delta\eta = \eta_0/4$ and fitting the quadratic form

$$Q(x) = x^T Ax - 2b^T x + c \tag{21}$$

where $x = (\alpha/\eta)$ is a 2-by-1 vector that contains the parameters of interest, A is a 2-by-2 negative definite matrix, b is a 2-by-1 vector and c a scalar. After the algorithm is initialized, the subsequent values (α_1 and η_1) are results of the matrix expression $A^{-1}b$, which corresponds to the maximum value of the quadratic form. The algorithm stops when it finds the maximum likelihood estimates of both α and η, obtained by applying the Newton-Raphson root-finding method to the first order partial derivatives of the log-likelihood function.

Finally, we obtain the standard errors of the estimated parameters from the Cramer-Rao inequality for the covariance matrix, as this provides the lower bound for the variance of an unbiased estimator of a parameter as the number of observations increases,

$$VAR(\alpha, \eta) \geq -\frac{1}{\nabla^2 l(\alpha, \eta)}. \tag{22}$$

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