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Well-posedness of Third Order Differential Equations in Hölder Continuous Function Spaces

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Abstract. In this paper, by using operator-valued $\dot{C}^\alpha$-Fourier multiplier results on vector-valued Hölder continuous function spaces, we give a characterization of the C^α-well-posedness for the third order differential equations $au'''(t) + u''(t) = Au(t) + Bu'(t) + f(t)$, $(t \in \mathbb{R})$, where A, B are closed linear operators on a Banach space X such that $D(A) \subset D(B)$, $a \in \mathbb{C}$ and $0 < \alpha < 1$.

1 Introduction

The well-posedness of third order differential equations has been investigated by many researchers, since these differential equations describe a large number of models arising from natural phenomena, such as flexible space structures with internal damping; see [3, 4, 7, 11, 12] for more information and references therein. For example, Poblete and Pozo studied the existence and uniqueness of strong solutions for the abstract third order equation

$$(1.1)\qquad \begin{cases} \alpha u'''(t) + u''(t) = \beta Au(t) + \gamma Bu'(t) + f(t), & (t \in [0, 2\pi]), \\ u(0) = u(2\pi), u'(0) = u'(2\pi), u''(0) = u''(2\pi), \end{cases}$$

where A and B are closed linear operators defined on a Banach space X with $D(A) \cap D(B) \neq \{0\}$, the constants $\alpha, \beta, \gamma \in \mathbb{R}^+$, and f belongs to either Lebesgue–Bochner spaces $L^p(\mathbb{T}; X)$, periodic Besov spaces $B^s_{p,q}(\mathbb{T}; X)$, or periodic Triebel–Lizorkin spaces $F^s_{p,q}(\mathbb{T}; X)$ [12]. They gave necessary and sufficient conditions for (1.1) to be L^p-well-posed (respectively $B^s_{p,q}$-well-posed and $F^s_{p,q}$-well-posed) by using operator-valued Fourier multipliers.

On the other hand, the well-posedness of differential equations in Hölder continuous function spaces have been extensively studied. See [5, 6, 10, 13, 14] for more

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information and references therein. The purpose of this paper is to study the well-posedness of the following third order differential equations on the real line:

$$(P) \qquad au'''(t) + u''(t) = Au(t) + Bu'(t) + f(t), \quad t \in \mathbb{R}$$

on Hölder continuous function spaces $C^\alpha(\mathbb{R}; X)$, where A and B are closed linear operators on a complex Banach space X such that $D(A) \subset D(B)$, $a \in \mathbb{C}$ and $0 < \alpha < 1$ are fixed scalars.

We say that (P) is C^α-well-posed if, for every $f \in C^\alpha(\mathbb{R}; X)$, there exists a unique $u \in C^\alpha\big(\mathbb{R}; D(A)\big) \cap C^{3+\alpha}(\mathbb{R}; X)$, such that $u' \in C^\alpha\big(\mathbb{R}; D(B)\big)$ and (P) is satisfied for all $t \in \mathbb{R}$. Here we consider $D(A)$ and $D(B)$ as Banach spaces equipped with the graph norms, while $C^{3+\alpha}(\mathbb{R}; X)$ is the space of all C^3-functions $u: \mathbb{R} \to X$ satisfying $u', u'', u''' \in C^\alpha(\mathbb{R}; X)$. Using known operator-valued $\dot{C}^\alpha$-Fourier multiplier results obtained by Arendt, Batty and Bu [1], we completely characterize the C^α-well-posedness of (P): when $0 < \alpha < 1$, then (P) is C^α-well-posed if and only if $i\mathbb{R} \subset \rho(P)$ and

$$\sup_{s \in \mathbb{R}} \| s^3 [ias^3 + s^2 + A + isB]^{-1} \| < \infty,$$
$$\sup_{s \in \mathbb{R}} \| sB [ias^3 + s^2 + A + isB]^{-1} \| < \infty$$

(see Theorem 3.1 below), where $\rho(P)$ is the resolvent set defined by the problem (P) (see the precise definition in the third section). Since the above estimations do not depend on the space parameter $0 < \alpha < 1$, we deduce that when (P) is C^α-well-posed for some $0 < \alpha < 1$, then it is C^α-well-posed for all $0 < \alpha < 1$.

It is remarkable that our characterization of the C^α-well-posedness of (P) does not depend on the geometry of the underlying Banach space X and the involved closed operator A does not need to generate a semigroup on X. Our result may be regarded as generalizations of the previous known results in the simpler cases when $a = 0$ and/or $B = 0$ [1].

This paper is organized as follows: in the second section, we give some preliminaries concerning $\dot{C}^\alpha$-Fourier multipliers and Carleman transform for functions of subexponential growth. In Section 3, we present our main result which gives a necessary and sufficient condition for the problem (P) to be C^α-well-posed. In the last section, we give a concrete example to which our abstract result may be applied.

2 Preliminaries

Let X be a complex Banach space and $0 < \alpha < 1$. We denote by $C^\alpha(\mathbb{R}; X)$ the space of all X-valued functions u on $\mathbb{R}$ satisfying

$$\|u\|_\alpha := \sup_{s \neq t} \frac{\|u(s) - u(t)\|}{|s-t|^\alpha} < \infty.$$

Define

$$\|u\|_{C^\alpha(\mathbb{R};X)} := \|u(0)\| + \|u\|_\alpha.$$

It is easy to see that the space $C^\alpha(\mathbb{R}; X)$ equipped with norm $\|\cdot\|_{C^\alpha(\mathbb{R};X)}$ becomes a Banach space. The kernel of the seminorm $\|\cdot\|_\alpha$ on $C^\alpha(\mathbb{R}; X)$ is the space of all constant functions. The corresponding quotient space $\dot{C}^\alpha(\mathbb{R}; X)$ is also a Banach space under the quotient norm. We will identify a function $u \in C^\alpha(\mathbb{R}; X)$ with its equivalent class in $\dot{C}^\alpha(\mathbb{R}; X)$, that is, $\dot{u} := \{v \in C^\alpha(\mathbb{R}; X) : u - v \equiv \text{constant}\}$.

Let X, Y be Banach spaces. We denote by $\mathcal{L}(X, Y)$ the space of all bounded linear operators from X to Y. We will simply denote it by $\mathcal{L}(X)$ if $X = Y$.

We need the notion of operator-valued $\dot{C}^\alpha$-multipliers which has been studied in [1].

Definition 2.1 Let X, Y be complex Banach spaces, $m: \mathbb{R} \setminus \{0\} \to \mathcal{L}(X, Y)$ be continuous. m is said to be a $\dot{C}^\alpha$-Fourier multiplier if there exists a mapping $L: \dot{C}^\alpha(\mathbb{R}; X) \to \dot{C}^\alpha(\mathbb{R}; Y)$ such that

$$\int_{\mathbb{R}} \mathcal{F}\varphi(s)(Lf)(s)\, ds = \int_{\mathbb{R}} \mathcal{F}(\varphi m)(s) f(s)\, ds \tag{2.1}$$

for all $f \in C^\alpha(\mathbb{R}; X)$ and all $\varphi \in \mathcal{D}(\mathbb{R} \setminus \{0\})$, where $\mathcal{D}(\mathbb{R} \setminus \{0\})$ is the space of all C^∞-functions on $\mathbb{R} \setminus \{0\}$ with compact support contained in $\mathbb{R} \setminus \{0\}$, $\mathcal{F}$ is the Fourier transform given by

$$(\mathcal{F}h)(s) := \tilde{h}(s) := \int_{\mathbb{R}} h(t) e^{-ist}\, dt, \quad s \in \mathbb{R}$$

when $h \in L^1(\mathbb{R}; X)$.

Remark 2.1 By [1, Lemma 5.1], the right-hand side of (2.1) does not depend on the representative of $\dot{f}$ as

$$\int_{\mathbb{R}} \mathcal{F}(\varphi m)(s)\, ds = 2\pi(\varphi m)(0) = 0.$$

Moreover, the identity (2.1) defines $Lf \in C^\alpha(\mathbb{R}; X)$ uniquely up to an additive constant by [1, Lemma 5.1].

The following result gives a sufficient condition for a C^2-function to be a $\dot{C}^\alpha$-Fourier multiplier.

Theorem 2.1 (Arendt, Batty and Bu [1]) *Let X, Y be Banach spaces and $m: \mathbb{R} \setminus \{0\} \to \mathcal{L}(X, Y)$ be a C^2-function satisfying*

$$\sup_{s \neq 0} \big(\|m(s)\| + \|sm'(s)\| + \|s^2 m''(s)\| \big) < \infty.$$

Then m is a $\dot{C}^\alpha$-Fourier multiplier whenever $0 < \alpha < 1$.

Let $0 < \alpha < 1$. Then we denote by $C^{1+\alpha}(\mathbb{R}; X)$ the space of all X-valued functions u defined on $\mathbb{R}$, such that $u \in C^1(\mathbb{R}; X)$ and $u' \in C^\alpha(\mathbb{R}; X)$. The space $C^{1+\alpha}(\mathbb{R}; X)$ is equipped with the following norm

$$\|u\|_{C^{1+\alpha}(\mathbb{R};X)} := \|u(0)\| + \|u'\|_{C^\alpha(\mathbb{R};X)},$$

and it is a Banach space. It follows from [1, Lemma 6.2] that if $u, v \in C^\alpha(\mathbb{R}; X)$, then $u \in C^{1+\alpha}(\mathbb{R}; X)$ and $u' = v + x$ for some $x \in X$ if and only if

$$\int_{\mathbb{R}} \mathcal{F}(\mathrm{id} \cdot \varphi)(s) u(s)\, ds = \int_{\mathbb{R}} (\mathcal{F}\varphi)(s) v(s)\, ds$$

whenever $\varphi \in \mathcal{D}(\mathbb{R} \setminus \{0\})$, where $\mathrm{id}(s) := is$ when $s \in \mathbb{R}$.

In a similar way, $C^{2+\alpha}(\mathbb{R};X)$ is the space of all X-valued functions u defined on $\mathbb{R}$ such that $u \in C^2(\mathbb{R};X)$ and $u', u'' \in C^\alpha(\mathbb{R};X)$. $C^{3+\alpha}(\mathbb{R};X)$ is the space of all X-valued functions u defined on $\mathbb{R}$ such that $u \in C^3(\mathbb{R};X)$ and u', u'', $u''' \in C^\alpha(\mathbb{R};X)$. $C^{3+\alpha}(\mathbb{R};X)$ is also a Banach space equipped with the norm

$$\|u\|_{C^{3+\alpha}(\mathbb{R};X)} := \|u(0)\| + \|u'(0)\| + \|u''(0)\| + \|u'''\|_{C^\alpha(\mathbb{R};X)}.$$

Let $u \in L^1_{\mathrm{loc}}(\mathbb{R};X)$. We say that u is of subexponential growth if for all $\epsilon > 0$

$$\int_{-\infty}^{\infty} e^{-\epsilon|t|}\|u(t)\|\,dt < \infty.$$

For such function u, we define its Carleman transform on $\mathbb{C} \setminus i\mathbb{R}$ by

$$\hat{u}(\lambda) := \begin{cases} \int_0^\infty e^{-\lambda t}u(t)\,dt, & \operatorname{Re}\lambda > 0, \\ -\int_0^\infty e^{\lambda t}u(-t)\,dt, & \operatorname{Re}\lambda < 0 \end{cases}$$

[2, (4.25), p. 292]. A point $\eta \in \mathbb{R}$ is called a regular point of u if its Carleman transform has a holomorphic extension to a neighborhood of $i\eta$. The Carleman spectrum of u is given by $\mathrm{sp}_C(u) := \{\eta \in \mathbb{R} : \eta \text{ is not regular}\}$. It is well known that $u = 0$ if and only if the $\mathrm{sp}_C(u) = \phi$.

3 The C^α-Well-Posedness of (P)

Let X be a complex Banach space, let $A\colon D(A) \to X$ and $B\colon D(B) \to X$ be closed linear operators on X satisfying $D(A) \subset D(B)$ and let $a \in \mathbb{C}$, $0 < \alpha < 1$. We consider the C^α-well-posedness of the third order differential equations:

$$(P) \qquad au'''(t) + u''(t) = Au(t) + Bu'(t) + f(t), \quad t \in \mathbb{R},$$

on Hölder continuous function spaces $C^\alpha(\mathbb{R};X)$.

Definition 3.1 We say that (P) is C^α-well-posed, if for all $f \in C^\alpha(\mathbb{R};X)$, there exists a unique $u \in C^\alpha\big(\mathbb{R};D(A)\big) \cap C^{3+\alpha}(\mathbb{R};X)$, such that $u' \in C^\alpha\big(\mathbb{R};D(B)\big)$ and (P) is satisfied for all $t \in \mathbb{R}$, here $D(A)$ and $D(B)$ are equipped with the graph norms, so that they become Banach spaces.

We define the resolvent set for the problem (P) by

$$\rho(P) := \{z \in \mathbb{C} : az^3 + z^2 - A - zB : D(A) \to X \text{ is a bijection and } [az^3 + z^2 - A - zB]^{-1} \in \mathcal{L}(X)\}.$$

Let $z \in \rho(P)$. Then $[az^3 + z^2 - A - zB]^{-1} \in \mathcal{L}(X)$ is a bijection from X onto $D(A)$ by definition. This implies that $A[az^3 + z^2 - A - zB]^{-1}$ and $B[az^3 + z^2 - A - zB]^{-1} \in \mathcal{L}(X)$ by the closed graph theorem and the closedness of A and B. In particular, $[az^3 + z^2 - A - zB]^{-1} \in \mathcal{L}\big(X, D(A)\big) \cap \mathcal{L}\big(X, D(B)\big)$. Here again we consider $D(A)$ and $D(B)$ as Banach spaces equipped with the graph norms.

The following results give a necessary and sufficient condition for (P) to be C^α-well-posed.

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Theorem 3.1 *Let X be a complex Banach space, $a \in \mathbb{C}$, $0 < \alpha < 1$ and let A, B be closed linear operators on X satisfying $D(A) \subset D(B)$. Then (P) is C^α-well-posed if and only if $i\mathbb{R} \subset \rho(P)$ and*

$$\sup_{s\in\mathbb{R}} \|s^3[ias^3 + s^2 + A + isB]^{-1}\| < \infty,$$
$$\sup_{s\in\mathbb{R}} \|sB[ias^3 + s^2 + A + isB]^{-1}\| < \infty.$$

Proof Assume that $i\mathbb{R} \subset \rho(P)$ and $\sup_{s\in\mathbb{R}} \|s^3[ias^3 + s^2 + A + isB]^{-1}\| < \infty$, $\sup_{s\in\mathbb{R}} \|sB[ias^3 + s^2 + A + isB]^{-1}\| < \infty$. Then $A: D(A) \to X$ is invertible and its inverse $A^{-1} \in \mathcal{L}(X)$ as $0 \in \rho(P)$, by assumption. Let

$$m(s) := [ias^3 + s^2 + A + isB]^{-1}, \quad p(s) := s^3 m(s), \quad q(s) := sBm(s)$$

when $s \in \mathbb{R}$. It is easy to verify that m, p and q are $\mathcal{L}(X)$-valued C^∞-functions on $\mathbb{R}$. We have

$$\sup_{s\in\mathbb{R}} \|p(s)\| < \infty, \quad \sup_{s\in\mathbb{R}} \|q(s)\| < \infty, \tag{3.1}$$

by assumption. The identity $(ias^3 + s^2)m(s) + Am(s) + isBm(s) = I_X$ and the fact that the inverse of $A: D(A) \to X$ is in $\mathcal{L}(X)$ implies that

$$\sup_{s\in\mathbb{R}} \|Am(s)\| < \infty, \quad \sup_{s\in\mathbb{R}} \|m(s)\| < \infty. \tag{3.2}$$

We have

$$m'(s) = -m(s)[3ias^2 + 2s + iB]m(s),$$

and

$$m''(s) = 2m(s)[3ias^2 + 2s + iB]m(s)[3ias^2 + 2s + iB]m(s) - m(s)[6ias + 2]m(s)$$

when $s \in \mathbb{R}$. It follows that

$$\sup_{s\in\mathbb{R}} \|sm'(s)\| < \infty, \quad \sup_{s\in\mathbb{R}} \|s^2 m''(s)\| < \infty,$$
$$\sup_{s\in\mathbb{R}} \|sAm'(s)\| < \infty, \quad \sup_{s\in\mathbb{R}} \|s^2 Am''(s)\| < \infty,$$
$$\sup_{s\in\mathbb{R}} \|sBm'(s)\| < \infty, \quad \sup_{s\in\mathbb{R}} \|s^2 Bm''(s)\| < \infty$$

by (3.1) and (3.2). Here we have used the fact that

$$\sup_{s\in\mathbb{R}} \|Bm(s)\| < \infty, \quad \sup_{s\in\mathbb{R}} \|s^2 m(s)\| < \infty, \tag{3.3}$$

which are easy consequences of the uniform boundedness of p, q and the continuity. Therefore, considering $m: \mathbb{R} \to \mathcal{L}\big(X, D(A)\big)$ or $m: \mathbb{R} \to \mathcal{L}\big(X, D(B)\big)$, m is a $\dot{C}^\alpha$-Fourier multiplier by Theorem 2.1. On the other hand, we have

$$p'(s) = 3s^2 m(s) - s^3 m(s)[3ias^2 + 2s + iB]m(s),$$

and

$$p''(s) = 6sm(s) - 6s^2m(s)[3ias^2 + 2s + iB]m(s) - s^3m(s)[6ias + 2]m(s) \\ + 2s^3m(s)[3ias^2 + 2s + iB]m(s)[3ias^2 + 2s + iB]m(s)$$

by (3.1), (3.2) and (3.3). It follows that

$$\sup_{s\in\mathbb{R}} \|sp'(s)\| < \infty, \quad \sup_{s\in\mathbb{R}} \|s^2p''(s)\| < \infty.$$

Consequently p is a $\dot{C}^\alpha$-Fourier multiplier by Theorem 2.1.

For h, we have

$$q'(s) = Bm(s) - sBm(s)[3ias^2 + 2s + iB]m(s),$$

and

$$q''(s) = -2Bm(s)[3ias^2 + 2s + iB]m(s) - sBm(s)[6ias + 2]m(s) \\ + 2sBm(s)[3ias^2 + 2s + iB]m(s)[3ias^2 + 2s + iB]m(s).$$

It follows that

$$\sup_{s\in\mathbb{R}} \|sq'(s)\| < \infty, \quad \sup_{s\in\mathbb{R}} \|s^2q''(s)\| < \infty$$

by (3.1), (3.2) and (3.3). Hence q is also a $\dot{C}^\alpha$-Fourier multiplier by Theorem 2.1.

Let $k(s) = sm(s)$ and $l(s) = s^2m(s)$. In a similar way, we show by using (3.1), (3.2) and (3.3) that

$$\sup_{s\in\mathbb{R}} \|sk'(s)\| < \infty, \quad \sup_{s\in\mathbb{R}} \|s^2k''(s)\| < \infty,$$
$$\sup_{s\in\mathbb{R}} \|sBk'(s)\| < \infty, \quad \sup_{s\in\mathbb{R}} \|s^2Bk''(s)\| < \infty,$$
$$\sup_{s\in\mathbb{R}} \|sl'(s)\| < \infty, \quad \sup_{s\in\mathbb{R}} \|s^2l''(s)\| < \infty.$$

Therefore l is a $\dot{C}^\alpha$-Fourier multiplier, and considering $k:\mathbb{R} \to \big(\mathcal{X}, D(B)\big)$, k is also a $\dot{C}^\alpha$-Fourier multiplier by Theorem 2.1.

Let $f \in C^\alpha(\mathbb{R}; X)$ be fixed. Then there exist $u_1 \in C^\alpha\big(\mathbb{R}; D(A)\big) \cap C^\alpha\big(\mathbb{R}; D(B)\big)$ and $u_2 \in C^\alpha\big(\mathbb{R}; D(B)\big)$, $u_3, u_4, u_5 \in C^\alpha(\mathbb{R}; X)$, such that

$$\int_{\mathbb{R}} (\mathcal{F}\phi_1)(s)u_1(s)\,ds = -\int_{\mathbb{R}} \mathcal{F}(\phi_1 m)(s)f(s)\,ds, \tag{3.4}$$

$$\int_{\mathbb{R}} (\mathcal{F}\phi_2)(s)u_2(s)\,ds = -i\int_{\mathbb{R}} \mathcal{F}(\phi_2 k)(s)f(s)\,ds, \tag{3.5}$$

$$\int_{\mathbb{R}} (\mathcal{F}\phi_3)(s)u_3(s)\,ds = -i\int_{\mathbb{R}} \mathcal{F}(\phi_3 q)(s)f(s)\,ds,$$

$$\int_{\mathbb{R}} (\mathcal{F}\phi_4)(s)u_4(s)\,ds = \int_{\mathbb{R}} \mathcal{F}(\phi_4 l)(s)f(s)\,ds, \tag{3.6}$$

$$\int_{\mathbb{R}} (\mathcal{F}\phi_5)(s)u_5(s)\,ds = i\int_{\mathbb{R}} \mathcal{F}(\phi_5 p)(s)f(s)\,ds \tag{3.7}$$

for all $\phi_i \in \mathcal{D}(\mathbb{R} \setminus \{0\})$. Since $u_1 \in C^\alpha\big(\mathbb{R}; D(A)\big)$ and $u_2 \in C^\alpha\big(\mathbb{R}; D(B)\big)$, we have $Au_1, Bu_2 \in C^\alpha(\mathbb{R}; X)$. It follows from (3.4) and (3.5) that

$$\int_{\mathbb{R}} (\mathcal{F}\phi_1)(s) Au_1(s)\, ds = -\int_{\mathbb{R}} \mathcal{F}(\phi_1 Am)(s) f(s)\, ds, \tag{3.8}$$

$$\int_{\mathbb{R}} (\mathcal{F}\phi_2)(s) Bu_2(s)\, ds = -i\int_{\mathbb{R}} \mathcal{F}(\phi_2 q)(s) f(s)\, ds \tag{3.9}$$

when $\phi_1, \phi_2 \in \mathcal{D}(\mathbb{R} \setminus \{0\})$ by the closedness of A and B. Choosing $\phi_1 = \mathrm{id}\cdot\phi_2$ in (3.4), where $\mathrm{id}(s) := is$ when $s \in \mathbb{R}$, we obtain from (3.5) that

$$\int_{\mathbb{R}} \mathcal{F}(\mathrm{id}\cdot\phi_2)(s) u_1(s)\, ds = \int_{\mathbb{R}} (\mathcal{F}\phi_2)(s) u_2(s)\, ds$$

whenever $\phi_2 \in \mathcal{D}(\mathbb{R} \setminus \{0\})$. Thus $u_1 \in C^{1+\alpha}\big(\mathbb{R}; D(B)\big)$ and $u_1' = u_2 + y_1$ for some $y_1 \in D(B)$ by [1, Lemma 6.2]. Here we have used the facts that $u_1, u_2 \in C^\alpha\big(\mathbb{R}; D(B)\big)$.

Similarly choosing $\phi_2 = \mathrm{id}\cdot\phi_4$ in (3.5), we deduce that $u_2 \in C^{1+\alpha}(\mathbb{R}; X)$ and $u_2' = u_4 + y_2$ for some $y_2 \in X$ by [1, Lemma 6.2] and (3.6). Taking $\phi_4 = \mathrm{id}\cdot\phi_5$ in (3.6), we deduce that $u_4 \in C^{1+\alpha}(\mathbb{R}; X)$ and $u_4' = u_5 + y_3$ for some $y_3 \in X$ by [1, Lemma 6.2] and (3.7). Thus $u_1 \in C^{3+\alpha}(\mathbb{R}; X)$ and $u_1''' = u_5 + y_3$.

Now the identity

$$ias^3 m(s) + s^2 m(s) = -Am(s) - isBm(s) + I_X, \quad s \in \mathbb{R}$$

implies that

$$\begin{aligned}\int_{\mathbb{R}} \mathcal{F}(iap)(s) f(s)\, ds &+ \int_{\mathbb{R}} (\mathcal{F}l)(s) f(s)\, ds \\ &= -\int_{\mathbb{R}} \mathcal{F}(Am)(s) f(s)\, ds - \int_{\mathbb{R}} \mathcal{F}(iq)(s) f(s)\, ds + \int_{\mathbb{R}} f(s)(\mathcal{F}\phi)(s)\, ds.\end{aligned}$$

Hence we have

$$\int_{\mathbb{R}} (au_5 + u_4 - Au_1 - Bu_2 - f)(s)\mathcal{F}(\phi)(s)\, ds = 0,$$

for all $\phi \in \mathcal{D}(\mathbb{R} \setminus \{0\})$ by (3.8) and (3.9), or equivalently,

$$au_1''' + u_1'' = Au_1 + Bu_1' + f + y$$

for some $y \in X$ by [1, Lemma 5.1].

Let A^{-1} be the inverse of $A\colon D(A) \to X$ and let $x = A^{-1}y$. Then $x \in D(A)$ and $u = u_1 + x$ solves (P). This shows the existence.

To show the uniqueness, we let $u \in C^\alpha\big(\mathbb{R}; D(A)\big) \cap C^{3+\alpha}(\mathbb{R}; X)$ be such that $u' \in C^\alpha\big(\mathbb{R}; D(B)\big)$ and

$$au'''(t) + u''(t) = Au(t) + Bu'(t)$$

when $t \in \mathbb{R}$. Taking the Carleman transform $\hat{u}$ of u, we have $\hat{u}(\lambda) \in D(A) \cap D(B)$ and

$$\widehat{Au}(\lambda) = A\hat{u}(\lambda), \quad \widehat{Bu'}(\lambda) = \lambda B\hat{u}(\lambda) - Bu(0),$$
$$\widehat{u''}(\lambda) = \lambda^2\hat{u}(\lambda) - \lambda u(0) - u'(0),$$
$$\widehat{u'''}(\lambda) = \lambda^3\hat{u}(\lambda) - \lambda^2 u(0) - \lambda u'(0) - u''(0),$$

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for all $\lambda \in \mathbb{C}\backslash i\mathbb{R}$ by [2]. It follows that

$$\begin{aligned}&[a\lambda^3 + \lambda^2 - A - \lambda B]\hat{u}(\lambda)\\ &\qquad = (a\lambda^2 + \lambda)u(0) + (a\lambda + 1)u'(0) + au''(0) - Bu(0),\end{aligned}$$

when $\lambda \in \mathbb{C}\backslash i\mathbb{R}$. This implies that the Carleman spectrum $\mathrm{sp}_C(u)$ of u is empty as $i\mathbb{R} \subset \rho(P)$ by assumption. Therefore $u = 0$ by [2, Theorem 4.8.2]. Hence (P) is C^α-well-posed.

Conversely, assume that (P) is C^α-well-posed. Let $L: C^\alpha(\mathbb{R}; X) \to S(\mathbb{R}; X)$ be the solution operator which associates for each $f \in C^\alpha(\mathbb{R}; X)$, the unique solution $L(f)$ of (P), where $S(\mathbb{R}; X)$ is the solution space of (P) consisting of all $u \in C^\alpha\big(\mathbb{R}; D(A)\big) \cap C^{3+\alpha}(\mathbb{R}; X)$, such that $u' \in C^\alpha\big(\mathbb{R}; D(B)\big)$. $S(\mathbb{R}; X)$ equipped with the norm

$$\|u\|_{S(\mathbb{R};X)} := \|u\|_{C^\alpha(\mathbb{R};D(A))} + \|u\|_{C^{3+\alpha}(\mathbb{R};X)} + \|Bu'\|_{C^\alpha(\mathbb{R};X)}$$

is a Banach space. It is easy to show that L is linear and bounded by the closed graph theorem.

Let $s \in \mathbb{R}$ be fixed; we are going to show that $is, \in \rho(P)$. Let $x \in D(A)$ be such that $(-ias^3 - s^2)x = Ax + isBx$ and let $u = e_s \otimes x$, where $(e_s \otimes x)(t) := e^{ist}x$ when $t \in \mathbb{R}$. Then $u \in C^\alpha\big(\mathbb{R}; D(A)\big) \cap C^{3+\alpha}(\mathbb{R}; X)$, $u' \in C^\alpha\big(\mathbb{R}; D(B)\big)$ and

$$au'''(t) + u''(t) = Au(t) + Bu'(t)$$

for all $t \in \mathbb{R}$. This means that $u \in S(\mathbb{R}; X)$ and u solves (P) when taking $f = 0$. Hence $u = 0$ by the uniqueness of the solution of (P). Consequently $x = 0$. We have shown that $ias^3 + s^2 + A + isB$ is injective.

To show that $ias^3 + s^2 + A + isB$ is also surjective, we let $y \in X$ and consider $f = e_s \otimes y$. Then $f \in C^\alpha(\mathbb{R}; X)$. Let $u \in S(\mathbb{R}; X)$ be the unique solution of (P), *i.e.*,

$$au'''(t) + u''(t) = Au(t) + Bu'(t) + f(t)$$

for all $t \in \mathbb{R}$. For fixed $\xi \in \mathbb{R}$, we consider the function u_ξ given by $u_\xi(t) = u(t + \xi)$ when $t \in \mathbb{R}$. Then both functions u_ξ and $e^{i\xi s}u$ are in $S(\mathbb{R}; X)$ and solve the problem

$$av'''(t) + v''(t) = Av(t) + Bv'(t) + e^{i\xi s}f(t).$$

We deduce from the uniqueness that $u_\xi = e^{i\xi s}u$, that is, $u(t+\xi) = e^{i\xi s}u(t)$ for $t, \xi \in \mathbb{R}$. Let $x = u(0)$. Then $x \in D(A)$ and $u = e_s \otimes x$. Since u solves

$$au'''(t) + u''(t) = Au(t) + Bu'(t) + f(t),$$

we have $(-ias^3 - s^2)e_s \otimes x = e_s \otimes Ax + ise_s \otimes Bx + e_s \otimes y$. Letting $t = 0$, we obtain $(-ias^3 - s^2 - A - isB)x = y$. This shows that $ias^3 + s^2 + A + isB$ is surjective. Thus $ias^3 + s^2 + A + isB$ is a bijection from $D(A)$ onto X and $x = -(ias^3 + s^2 + A + isB)^{-1}y$. We have shown that

$$u = -e_s \otimes (ias^3 + s^2 + A + isB)^{-1}y.$$

Consequently

$$\begin{aligned}\|(ias^3 + s^2 + A + isB)^{-1}y\| &= \|u(0)\|\\ &\le \|L\|\,\|f\|_{C^\alpha(\mathbb{R};X)} = \|L\|\,(1 + \gamma_\alpha|s|^\alpha)\,\|y\|\end{aligned}$$

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for some constant $\gamma_\alpha > 0$ depending only on α [1, (3.1)]. Thus $(ias^3 + s^2 + A + isB)^{-1}$ is a bounded linear operator for every $s \neq 0$. That is, $is \in \rho(P)$ for all $s \in \mathbb{R} \smallsetminus \{0\}$.

On the other hand, we note that

$$\begin{aligned}\gamma_\alpha \|s\|^\alpha \, \|s^3(ias^3 + s^2 + A + isB)^{-1}y\| \\ = \|s^3 e_s \otimes (ias^3 + s^2 + A + isB)^{-1}y\|_\alpha \\ = \|u'''\|_\alpha \leq \|L\| \, \|f\|_{C^\alpha} == \|L\|(1 + \gamma_\alpha \|s\|^\alpha)\|y\|,\end{aligned}$$

by [1, (3.1)]. It follows that when $s \neq 0$,

$$\|s^3(ias^3 + s^2 + A + isB)^{-1}\| \leq \|L\|(1 + \gamma_\alpha^{-1}\|s\|^{-\alpha}).$$

Similarly using the inequality $\|Bu'\|_{C^\alpha} \leq \|L\| \, \|f\|_{C^\alpha}$, one obtains

$$\|sB(ias^3 + s^2 + A + isB)^{-1}\| \leq \|L\|(1 + \gamma_\alpha^{-1}\|s\|^{-\alpha}) \tag{3.15}$$

when $s \neq 0$.

When $s = 0$, f is the constant function y and the corresponding solution u is the constant function $-A^{-1}y$. Then the inequality $\|u\|_{C^\alpha} \leq \|L\| \, \|f\|_{C^\alpha}$ implies that

$$\|A^{-1}y\| \leq \|L\| \, \|y\|,$$

that is, $0 \in \rho(P)$. Hence we have $i\mathbb{R} \subset \rho(P)$. It is not hard to verify that $\rho(P)$ is an open subset of $\mathbb{C}$ and the functions defined on $\mathbb{R}$ by

$$s \to \|s^3(ias^3 + s^2 + A + isB)^{-1}\|, \quad s \to \|sB(ias^3 + s^2 + A + isB)^{-1}\|$$

are continuous. It follows that

$$\|s^3(ias^3 + s^2 + A + isB)^{-1}\| < \infty, \quad \|sB(ias^3 + s^2 + A + isB)^{-1}\| < \infty$$

by continuity, (3.14) and (3.15). This completes the proof. ∎

Since the necessary and sufficient condition for the problem (*P*) to be C^α-well-posed obtained in Theorem 3.1 does not depend on the space parameter $0 < \alpha < 1$, we have the following immediate corollary.

Corollary 3.1 *If the problem* (*P*) *is* C^α*-well-posed for some* $0 < \alpha < 1$, *then it is* C^α*-well-posed for all* $0 < \alpha < 1$.

4 Applications

In this section, we give an example where our abstract results may be applied. We recall that a closed densely defined operator A on a Banach space X is sectorial of angle $\beta \in (0, \pi)$ if $\sigma(A) \subset \overline{\Sigma_\beta}$ and for every $\beta' \in (\beta, \pi)$,

$$\sup_{z \in \mathbb{C} \smallsetminus \Sigma_{\beta'}} \|z(z - A)^{-1}\| < \infty,$$

where $\Sigma_\beta := \{z \in \mathbb{C} : |\arg(z)| < \beta\}$. For a sectorial operator A, we define the sectorial angle $\omega(A)$ by

$$\omega(A) := \inf\{\beta \in (0, \pi) : A \text{ is sectorial of angle } \beta\}.$$

For every $\beta \in (0, \pi)$, we put

$$H^\infty(\Sigma_\beta) := \Big\{ f : \Sigma_\beta \to \mathbb{C} \text{ holomorphic} : \|f\|_\infty := \sup_{z \in \Sigma_\beta} \|f(z)\| < \infty \Big\},$$

$$H_0^\infty(\Sigma_\beta) := \Big\{ f \in H^\infty(\Sigma_\beta) : \text{there exists } \epsilon > 0, \text{ such that } \sup_{z \in \Sigma_\beta} |f(z)\| \Big|\frac{1+z^2}{z}\Big|^\epsilon < \infty \Big\}.$$

If A is sectorial operator of angle $\beta \in (0, \pi)$, then

$$f(A) := \frac{1}{2\pi i} \int_{\partial \Sigma_{\beta'}} f(z)(z - A)^{-1}\, dz$$

defines a functional calculus from $H_0^\infty(\Sigma'_\beta)$ into $\mathcal{L}(X)$ for all $\beta' > \beta$. This functional calculus may be extended in a natural way in order to define the fractional powers A^ϵ for all $\epsilon > 0$. It is known that A^ϵ is still a sectorial operator and $D(A) \subset D(A^\epsilon)$ when $0 < \epsilon < 1$ [8].

We say that a sectorial operator A admits a bounded H^∞-functional calculus of angle $\beta \in \big[\omega(A), \pi\big)$, if the functional calculus on $H_0^\infty(\Sigma_{\beta'})$ defined above extends to a bounded linear operator on $H^\infty(\Sigma_{\beta'})$ for all $\beta' \in (\beta, \pi)$. The infimum of all such β is denoted by $\omega_H(A)$. When A admits a bounded H^∞-functional calculus of angle β, then there exists a constant $C_\beta \geq 0$, such that for all $f \in H^\infty(\Sigma_\beta)$, one has

$$\|f(A)\| \leq C_\beta \|f\|_\infty. \tag{4.1}$$

We refer to [8, 9] for the concepts of H^∞-functional calculus for sectorial operators.

Example 4.1 Let $a > 0$, $0 < \alpha < 1$ be fixed and X be a Banach space. We consider the following second order differential equation:

$$au'''(t) + u''(t) = Au(t) + \gamma A^{m/n} u'(t) + f(t), \quad t \in \mathbb{R}, \tag{P'}$$

where A is a sectorial operator on X admitting a bounded H^∞-functional calculus of angle β for some $\beta \in (0, \pi)$, m, n are given positive integers satisfying $0 < m/n < 2/3$, $i\mathbb{R} \subset \rho(P')$, and γ is a fixed scalar number. Then (P') is C^α-well-posed.

Proof It is clear that (P') is a special case of (P) when $B = \gamma A^{m/n}$. Let $s \in \mathbb{R}$, we consider the polynomial $g_s(z) := z^n + is\gamma z^m + s^2 + as^3$. Let z_s be one of the roots of $g_k(z) = 0$, that is,

$$z_s^n + is\gamma z_s^m + s^2 + as^3 = 0. \tag{4.1}$$

It follows from (4.1) that there exists no real sequence $(s_k)_{k \in \mathbb{Z}}$ converging to ∞ such that $(z_{s_k})_{k \in \mathbb{Z}}$ is bounded. This implies that $\lim_{s \to \infty} |z_s| = +\infty$. Let $y_s := \frac{z_s^n}{s^3}$. Then by (4.1) there exists no real sequence $(s_k)_{k \in \mathbb{Z}}$ converging to ∞ such that $(|y_{s_k}|)_{k \in \mathbb{Z}}$ converges to $+\infty$ as $0 < m/n < 2/3$ by assumption. Therefore there exists a constant $C \geq 0$, such that $|y_s| \leq C$ for all $s \in \mathbb{R}$. We deduce from (4.1) and the assumption $0 < m/n < 2/3$ that

$$\lim_{s \to \infty} y_s = -a. \tag{4.2}$$

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Let $z_{1,s}, z_{2,s}, \ldots, z_{n,s}$ be the roots of $g_s(z) = 0$. Then $g_s(z) = \prod_{j=1}^{n}(z - z_{j,s})$. Consider the function $h_s(z) = z + is\gamma z^{m/n} + s^2 + as^3$, where $z^{m/n} := e^{\frac{m}{n} log(z)}$ by using the main branch of the logarithm. We have $h_s(z) = \prod_{j=1}^{n}(z_{j,s} - z^{1/n})$. Then

$$\sup_{z\in\Sigma_\beta}\Big|\frac{s^3}{h_s(z)}\Big| = \sup_{z\in\Sigma_\beta}\Big|\frac{1}{\prod_{j=1}^{n}\big(\frac{z_{j,s}}{|s|^{3/n}} - \frac{z^{1/n}}{|s|^{3/n}}\big)}\Big| < \infty,$$
$$\sup_{z\in\Sigma_\beta}\Big|\frac{z}{h_s(z)}\Big| = \sup_{z\in\Sigma_\beta}\Big|\frac{1}{\prod_{j=1}^{n}\big(\frac{z_{j,k}}{z^{1/n}} - 1\big)}\Big| < \infty \tag{4.3}$$

when $|s|$ is big enough, by (4.2). an immediate consequence of (4.3) is

$$\sup_{z\in\Sigma_\beta}\Big|\frac{s^2}{h_s(z)}\Big| < \infty$$

when $|s|$ is big enough. This implies that

$$\sup_{z\in\Sigma_\beta}\Big|\frac{sz^{m/n}}{h_s(z)}\Big| < \infty$$

when $|s|$ is big enough, by the definition of h_s. Consequently the functions $f_{1,s}$ and $f_{2,s}$ defined by

$$f_{1,s}(z) = \frac{s^2}{z + is\gamma z^{m/n} + s^2 + as^3}, \quad f_{2,s}(z) = \frac{sz^{m/n}}{z + is\gamma z^{m/n} + s^2 + as^3}$$

belong to $H^\infty(\Sigma_\beta)$ when $|s|$ is big enough.

For all $s \in \mathbb{R}$, the operator $as^3 + s^2 + A + is\gamma A^{m/n}$ is a bijection from $D(A)$ onto X and $(as^3 + s^2 + A + is\gamma A^{m/n})^{-1} \in \mathcal{L}(X)$ as $i\mathbb{R} \subset \rho(P')$ by assumption. One can verify, using for example [8, Chapter 1], that

$$f_{1,s}(A) = s^2(as^3 + s^2 + A + is\gamma A^{m/n})^{-1},$$
$$f_{2,s}(A) = sA^{m/n}(as^3 + s^2 + A + is\gamma A^{m/n})^{-1}$$

when $|s|$ is big enough. The assumption that A admits a bounded H^∞-functional calculus of angle β implies that the sets

$$\{s^2(as^3 + s^2 + A + is\gamma A^{m/n})^{-1} : s \in \mathbb{R}\},$$
$$\{sA^{m/n}(as^3 + s^2 + A + is\gamma A^{m/n})^{-1} : s \in \mathbb{R}\}$$

are bounded by (4) and the assumption that $i\mathbb{R} \subset \rho(P')$. Consequently, ($P'$) is C^α-well-posed by Theorem 3.1. ∎

We do not know whether the result remains true if we replace the component m/n in (P') by arbitrary $0 < \epsilon < 2/3$.

Similar argument shows that when $a = 0$, then the result remains true when the assumption $0 < m/n < 2/3$ is replaced by $0 < m/n < 1/2$.

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