## Green's and allied theorems: a historical sketch.

By George A. Gibson, M.A.

[Abstract.]

The chief purpose of the paper was to indicate the rise of transformations of the type $\int\int\int\frac{dV}{dx}dxdydz = \int V\cos\alpha dS$ where the integral in the first member of the equation is taken throughout a closed surface, and that in the second member over the surface, $\alpha$ being the angle made with the axis of $x$ by the normal to the element $dS$ drawn outwards. It is on this transformation the analytical proof of Green's theorem depends, and it was shown to have been employed in various forms by Poisson, Duhamel, Gauss, and others, before Green's essay was generally known on the Continent. It may be observed that the essay was published at Nottingham in 1828, and seems to have been unknown to continental mathematicians till its reprint in *Crelle's Journal*, vols. 39 (1850), 44 (1852), and 47 (1854). The following references were given in the paper:—

Lagrange, in the *Mécanique Analytique* (2nd edition, 1811), Part I., sect. vii., arts. 29, 30, gives the transformation:—

$$\mathbf{S}\lambda'(\delta x'.dydz + \delta y'.dzdx + \delta z'.dxdy) = \mathbf{S}\lambda'(\cos\alpha'.\delta x' + \cos\beta'.\delta y' +$$

$\cos\gamma'.\delta z')ds^2$, $ds^2$ being an element of surface.

Laplace, in the *Supplément à la Théorie de l'Action Capillaire* (which forms a supplement to Liv. x., Part II., of the *Mécanique Céleste*, published 1806)—*Oeuvres Complètes*, 1880, t4. pp. 428–432 —transforms the integral

$$\int\int dxdy\left(\frac{d}{dx}\cdot\frac{p}{\sqrt{1+p^2+q^2}} + \frac{d}{dy}\frac{q}{\sqrt{1+p^2+q^2}}\right)$$

taken over the area of a section of a cylinder, whose generators are parallel to the axis of $z$, into the integral $\pm\int\frac{pdy - qdx}{R}$ taken along the boundary of the section, the $+$ sign holding for the part of the curve convex to the axis of $x$, the $-$ sign for that concave to the same axis.

Gauss gives a series of remarkable theorems, closely related to the transformation in question, in the introductory articles of his

"Theoria Attractionis corporum sphæroidicorum ellipticorum homogeneorum," *Comment. Soc. Reg. Gott.*, vol. ii. (1813), Werke, Bd. V. Great care is taken to make the proofs quite general. Some of these theorems will be found in Williamson's *Integral Calculus* (3rd edition), arts. 192, 193. Transformations of a like character occur in Gauss's Memoirs, "Principia Generalia Theoriæ Figuræ Fluidorum in Statu Aequilibrii," *Comment. Soc. Reg. Gott.*, vol. vii. (1830), Werke, Bd. V., and "Allgemeine Lehrsätze in Bez. auf die . . . Anziehungs-und Abstossungs-Kräfte," *Resultate des Magn. Vereins*, 1840, Werke, Bd. V. The proof of Poisson's equation, in §§ 9, 10 of the latter memoir, depends really on the transformation of a volume integral of the kind mentioned into the difference of a volume and of a surface integral, the manner of carrying out the transformation being quite special.

Poisson in various memoirs uses with considerable effect the transformation employed (later) by Green, and establishes in his earlier memoirs particular cases of Green's theorem. In his "Mémoire sur la Théorie du Magnetisme," *Mém. de l'Inst.*, t.v., 1826, read February 2, 1824, pp. 294-298, he proves the equation

$$\iiint k'\left(\alpha'\frac{d}{dx'}.\frac{1}{\rho}+\beta'\frac{d}{dy'}.\frac{1}{\rho}+\gamma'\frac{d}{dz'}\frac{1}{\rho}\right)dx'dy'dz'$$
$$=\iint k'(\alpha'\cos l'+\beta'\cos m'+\gamma'\cos n')\frac{d\omega'}{\rho}$$
$$-\iiint\left(\frac{d(\alpha'k')}{dx'}+\frac{d(\beta'k')}{dy'}+\frac{d(\gamma'k')}{dz'}\right)\frac{dx'dy'dz'}{\rho}$$

where $\rho^2=(x-x')^2+(y-y')^2+(z-z')^2$ and the volume integrals are taken throughout a closed surface and the double integrals over the surface. He considers the possibility of a line cutting the surface in more points than two, but makes no reference to Gauss's memoir of 1813. In § 18 of the same memoir he discusses the case in which $1/\rho$ becomes infinite within the limits of integration.

Another important example from Poisson is furnished by § 89 of the sixth chapter of his *Théorie de la Chaleur* (Paris, 1835; a date later than that of the publication of Green's essay and of the memoir of Duhamel referred to below). He arrives at the following equation:—

$$\frac{d}{dt}.\iiint cu\mathrm{P}dxdydz$$

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$$=\int\int\int \mathrm{P}\left\{\frac{d}{dx}\left(k\frac{du}{dx}\right)+\frac{d}{dy}\left(k\frac{du}{dy}\right)+\frac{d}{dz}\left(k\frac{du}{dz}\right)\right\}dxdydz$$

$$=\int\int \mathrm{P}k\left(\frac{du}{dx}\cos\alpha+\frac{du}{dy}\cos\beta+\frac{du}{dz}\cos\gamma\right)d\mathrm{S}$$

$$-\int\int uk\left(\frac{d\mathrm{P}}{dx}\cos\alpha+\frac{d\mathrm{P}}{dy}\cos\beta+\frac{d\mathrm{P}}{dz}\cos\gamma\right)d\mathrm{S}$$

$$+\int\int\int u\left\{\frac{d}{dx}\left(k\frac{d\mathrm{P}}{dx}\right)+\frac{d}{dy}\left(k\frac{d\mathrm{P}}{dy}\right)+\frac{d}{dz}\left(k\frac{d\mathrm{P}}{dz}\right)\right\}dxdydz$$

The last equation is in form the extension of Green's theorem given in Thomson and Tait's *Natural Philosophy*, vol. i., p. 168.

Another interesting example from Poisson occurs in a "Mémoire sur l'Équilibre des Fluides," *Mém. de l'Inst.*, t.ix, 1830, read Nov. 24, 1828, pp. 22-24.

*Ostrogradsky*, in a memoir with the title "Note sur la Théorie de la Chaleur," *Mém. de l'Acad. de St Petersbourg*, 6s. t.i (1831), read Nov. 5, 1828, p. 129, establishes the theorem

$$\int\left(\frac{dp}{dx}+\frac{dq}{dy}+\frac{dr}{dz}\right)\omega=\int\left(\mathrm{P}\cos\lambda+\mathrm{Q}\cos\mu+\mathrm{R}\cos\nu\right)s$$

where $\omega$ is a volume-element, $s$ a surface-element, and P, Q, R the values of $p$, $q$, $r$ at $s$.

*Duhamel*, in Note 1. appended to a memoir on the "Theory of Heat," published in the *Journ. Ecol. Polyt.*, t.xiv, cah. 22 (1833), gives, at pp. 67-71, a proof of the equation

$$\int \mathrm{U}'\left(\frac{d^2\mathrm{U}}{dx^2}+\frac{d^2\mathrm{U}}{dy^2}+\frac{d^2\mathrm{U}}{dz^2}\right)dxdydz$$

$$=\int \mathrm{U}\left(\frac{d^2\mathrm{U}'}{dx^2}+\frac{d^2\mathrm{U}'}{dy^2}+\frac{d^2\mathrm{U}'}{dz^2}\right)dxdydz$$

$$+\int \mathrm{U}\left(l\frac{d\mathrm{U}'}{dx}+m\frac{d\mathrm{U}}{dy}+n\frac{d\mathrm{U}}{dz}\right)d\mathrm{S}$$

$$-\int \mathrm{U}\left(l\frac{d\mathrm{U}'}{dx}+m\frac{d\mathrm{U}'}{dy}+n\frac{d\mathrm{U}'}{dz}\right)d\mathrm{S}$$

$l$, $m$, $n$, being the direction cosines of the outward normal at $d$S.

*Lamé*, in the same volume, p. 204 and p. 231, gives investigations of a similar character.

*Sir W. Thomson*, in Article XII. of his "Electrostatics and Magnetism" (reprinted from the *Camb. Math. Jour.*, Nov., 1842, and Feb. 1843), furnishes examples of the analysis with which the paper dealt, and which, he says, was suggested to him by the analysis used by Poisson in the article of his *Théorie de la Chaleur*, quoted above.

Questions of priority are usually somewhat difficult to answer; but while it seems clear that the theorem generally quoted as Green's was given independently of Green, yet the importance which he rightly attached to it, and the splendid use to which he put it, amply justify us in keeping to the customary mode of citation.

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## Some new Properties of the Triangle.

By J. S. MACKAY, M.A., LL.D.

[The substance of this paper will be included in Dr Mackay's paper on the triangle in the first volume of the *Proceedings*, now about to be published.]

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R. E. ALLARDICE, Esq., M.A., Vice-President, in the Chair.

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## A special case of three-bar motion.

By Professor STEGGALL.

The questions involved in the consideration of three-bar motion have attracted a good deal of attention (*Proceedings of Mathematical Society of London* passim, and elsewhere); but I am not aware of any complete account of the figures that can be derived from such a motion. The present paper gives a complete list of all the different kinds of curve that are obtained by a tracing point at the middle of the middle bar, the two outer bars being equal.

It may be advisable to briefly obtain the general equation to the curve traced by any point on the middle bar, without any condition of equality in the lengths of the other two.

Let $2a$ be the distance of the fixed centres, $b$, $2c$, $d$ the lengths of the three bars in order, $h$ the distance of the tracing point from the middle of the middle bar measured from the bar $b$, $\theta$, $\phi$, $\psi$ the