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Boundary layer of Hsieh's equation with conservative nonlinearity

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In this paper, we consider an initial-boundary value problem of Hsieh's equation with conservative nonlinearity. The global unique solvability in the framework of Sobolev is established. In particular, one of our main motivations is to investigate the boundary layer (BL) effect and the convergence rates as the diffusion parameter β goes zero. It is shown that the BL-thickness is of the order $O(\beta^\gamma)$ with $0 < \gamma < \frac{1}{2}$. We need to point out that, different from the previous work on nonconservative form of Hsieh's equations, the conservative nonlinearity $(\psi^\beta\theta^\beta)_x$ implies that new nonlinear term $\psi_x^\beta\theta^\beta$ needs to be handled. It is important that more regularities on the solution to the limit problem are required to obtain the convergence rates and BL-thickness. It is more difficult for initial-boundary problem due to the lack of boundary conditions (especially, higher-order derivatives) prevents us from applying the integration by part to derive the energy estimates directly. Thus it is more complicated than the case of nonconservative form. Consequently more subtle mathematical analysis needs to be introduced to overcome the difficulties.

Keywords: Hsieh's equations; conservative nonlinearity; boundary layer; BL-thickness; convergence rates

2020 *Mathematics Subject Classification:* Primary: 35K45
Secondary: 35B40, 35B45

1. Introduction

The boundary-layer theory has been developed by Ludwig Prandtl in 1904 (see [17]). Although this theory is now more than 110 years old, it is nowadays still being applied in industry and research, because many important fields of fluid mechanics (i.e. aeronautics, ship hydrodynamics, automobile aerodynamics) refer to flows at high Reynolds numbers. Mathematical analysis on the boundary layer (BL) theory has been extensively studied in different contexts. In particular, when parabolic equations with small viscosity are applied as perturbations, the question of boundary layer problem also arises in the theory of hyperbolic systems. Gisclon and Serre in [8] developed a method to detect the boundary layer effect for a viscous perturbation of some class of quasi-linear hyperbolic systems in one space dimension, which was also generalized to the multi-dimensional case by Grenier and Gues in [**9**].

In this paper, we consider a boundary layer problem between two horizontal parallel plates. Such kinds of boundary layer problems were also studied in [**6**, **12**–**14**, **26**]. Precisely speaking, we consider the following initial-boundary value problem of Hsieh's equation with conservative nonlinearity related to Lorenz system on the strip $[0,\,1]\times[0,\,\infty)$

$$\begin{cases}\psi_t^\beta=-(\sigma-\alpha)\psi^\beta-\sigma\theta_x^\beta+\alpha\psi_{xx}^\beta, & 0<x<1,\ t>0,\\ \theta_t^\beta=-(1-\beta)\theta^\beta+\nu\psi_x^\beta+\left(\psi^\beta\theta^\beta\right)_x+\beta\theta_{xx}^\beta, & 0<x<1,\ t>0\end{cases}\tag{1.1}$$

with initial data

$$\left(\psi^\beta,\theta^\beta\right)(x,0)=(\psi_0,\theta_0)(x),\quad 0\leqslant x\leqslant 1\tag{1.2}$$

and the Dirichlet boundary conditions

$$\left(\psi^\beta,\theta^\beta\right)(0,t)=\left(\psi^\beta,\theta^\beta\right)(1,t)=(0,0),\quad t\geqslant 0,\tag{1.3}$$

which implies

$$\left(\psi_t^\beta,\theta_t^\beta\right)(0,t)=\left(\psi_t^\beta,\theta_t^\beta\right)(1,t)=(0,0),\quad t\geqslant 0.\tag{1.4}$$

Here both ψ^β and θ^β are unknown. The parameters α, β, σ and ν are all positive constants satisfying the relation $\alpha<\sigma$ and $0<\beta<1$. We can refer to [**11**, **21**] for the physical background of the system (1.1).

We expect to prove that as the diffusion parameter $\beta\to 0^+$, the solution sequences $\left\{(\psi^\beta,\,\theta^\beta)\right\}$ of the initial-boundary value problem (1.1)–(1.3) with $\nu=o(\sqrt{\beta})$ converge to the solution $(\psi^0,\,\theta^0)$ of the following formal limit problem (1.5)–(1.7) (at least formally; this will be made precisely below.)

$$\begin{cases}\psi_t^0=-(\sigma-\alpha)\psi^0-\sigma\theta_x^0+\alpha\psi_{xx}^0,\\ \theta_t^0=-\theta^0+(\psi^0\theta^0)_x, \quad 0<x<1,\ t>0\end{cases}\tag{1.5}$$

with initial data

$$(\psi^0,\theta^0)(x,0)=(\psi_0(x),\theta_0(x)),\quad 0\leqslant x\leqslant 1\tag{1.6}$$

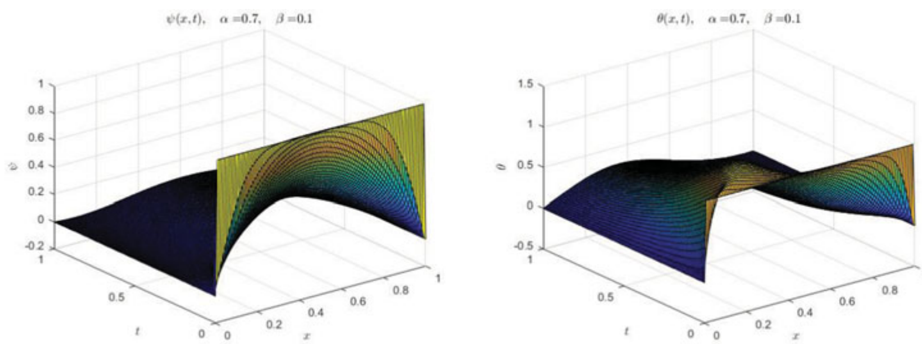
and the boundary conditions

$$\psi^0(0,t)=\psi^0(1,t)=0,\quad t\geqslant 0.\tag{1.7}$$

Note that one can get the following additional boundary conditions from (1.5) and (1.7):

$$\begin{cases}\psi_t^0(0,t)=\psi_t^0(1,t)=0,\\ \psi_{tt}^0(0,t)=\psi_{tt}^0(1,t)=0,\\ (\sigma\theta_x^0-\alpha\psi_{xx}^0)(0,t)=(\sigma\theta_x^0-\alpha\psi_{xx}^0)(1,t)=0,\quad t\geqslant 0,\end{cases}\tag{1.8}$$

which will be frequently used to handle the boundary term later.



conservative form of $\psi(x, t)$ conservative form of $\theta(x, t)$

Figure 1. Conservative form of $\psi(x,\, t)$ and conservative form of $\theta(x,\, t)$.

In addition, the initial data $(\psi_0,\, \theta_0)$ satisfies the compatibility conditions:

$$(\psi_0, \theta_0)\,(0) = (\psi_0, \theta_0)\,(1) = (0, 0). \tag{1.9}$$

The nonconservative form of the system (1.1) with $\alpha = 0$ was originally proposed by Hsieh in [**11**] to observe the nonlinear interaction between ellipticity and dissipation. Both conservative form and nonconservative form of the system (1.1) were studied in Tang's Ph.D. thesis [**21**] to understand chaos phenomenon. The nonconservative form corresponding to the system (1.1) reads as follows:

$$\begin{cases} \psi_t^\beta = -(\sigma - \alpha)\psi^\beta - \sigma\theta_x^\beta + \alpha\psi_{xx}^\beta, \\ \theta_t^\beta = -(1 - \beta)\theta^\beta + \nu\psi_x^\beta + 2\psi^\beta\theta_x^\beta + \beta\theta_{xx}^\beta. \end{cases}$$

Numerical experiments demonstrated and found drastically different behaviour between conservative form and nonconservative form of Hsieh's equations. One of our motivations is that it will become clear how the behaviour of conservative form, sometimes consistent with the behaviour of nonconservative form and sometimes utterly different, can be explained. Boundary layer theory studied in [**19**] for nonconservative form continues to be considered for conservative form in present paper (figures 1 and 2).

For latter presentation, we state function spaces and the notations as follows.

Notation 1.1. *Throughout this paper, we denote positive constants independent of* β *by* C. *And the character 'C' may differ in different places.* $L^2 = L^2([0,\, 1])$ *and* $L^\infty = L^\infty([0,\, 1])$ *denote the usual* L^p *space on* $[0,\, 1]$ *with its norm* $\|f\|_{L^2([0,\,1])} = \|f\| = (\int_0^1 |f(x)|^2 \,\mathrm{d}x)^{1/2}$ *and* $\|f\|_{L^\infty} = \sup\limits_{x\in[0,1]} |f(x)|$. $H^l([0,\, 1])$ *denotes the usual* l*-th order Sobolev space with its norm* $\|f\|_{H^l([0,1])} = \|f\|_l = (\sum_{i=0}^l \|\partial_x^i f\|^2)^{1/2}$. *For simplicity,* $\|f(\cdot,\, t)\|_{L^2}$, $\|f(\cdot,\, t)\|_{L^\infty}$ *and* $\|f(\cdot,\, t)\|_l$ *are denoted by* $\|f(t)\|$, $\|f(t)\|_{L^\infty}$ *and* $\|f(t)\|_l$ *respectively.*

In order to state the main results, let us describe the definition of BL-thickness, which is borrowed from [**6**, **20**].

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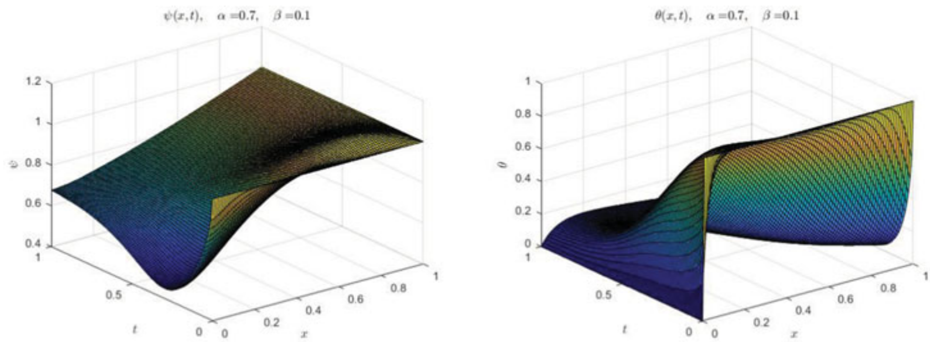

nonconservative form of $\psi(x,t)$ nonconservative form of $\theta(x,t)$

Figure 2. Nonconservative form of $\psi(x,\,t)$ and nonconservative form of $\theta(x,\,t)$.

Definition 1.2 (BL-thickness). *A function $\delta(\beta)$ is called a BL-thickness for the problem* (1.1)–(1.3) *with vanishing diffusion β, if $\delta(\beta)\downarrow 0$ as $\beta\downarrow 0$, and*

$$\lim_{\beta\to 0}\left\|\psi^\beta-\psi^0\right\|_{L^\infty(0,T;L^\infty[0,1])}=0, \tag{1.10}$$

$$\lim_{\beta\to 0}\left\|\theta^\beta-\theta^0\right\|_{L^\infty(0,T;L^\infty[\delta,1-\delta])}=0, \tag{1.11}$$

$$\liminf_{\beta\to 0}\left\|\theta^\beta-\theta^0\right\|_{L^\infty(0,T;L^\infty[0,1])}>0, \tag{1.12}$$

where $0<\delta=\delta(\beta)<1$, and $(\psi^\beta,\,\theta^\beta)$ (rep. $(\psi^0,\,\theta^0)$) is the solution to the problem (1.1)–(1.3) *(resp. to the limit problem* (1.5)–(1.7)*).*

Clearly, this definition does not determine the BL-thickness uniquely, since any function $\delta_*(\beta)$ with $\delta_*(\beta)\downarrow 0$ as $\beta\downarrow 0$ satisfying the inequality $\delta_*(\beta)\geqslant\delta(\beta)$ is also a BL-thickness if $\delta(\beta)$ is a BL-thickness.

The main results can be stated as follows.

Theorem 1.3. *Assume that the initial data $(\psi_0,\,\theta_0)\in H^1$ and $\|\psi_0\|_1+\|\theta_0\|_1$ be sufficiently small. Then we have*

(i) *There exist a unique solution $(\psi^\beta,\,\theta^\beta)$ to the initial-boundary value problem* (1.1)–(1.3) *satisfying*

$$\psi^\beta\in L^\infty\left(0,T;H^1\right)\bigcap L^2\left(0,T;H^2\right),\quad \theta^\beta\in L^\infty\left(0,T;H^1\right)\bigcap L^2\left(0,T;H^1\right),$$
$$\sqrt{\beta}\theta_x^\beta\in L^2\left(0,T;H^1\right).$$

(ii) *Further assume the initial data be imposed on more regularity $(\psi_0,\,\theta_0)\in H^2$. Then more regularities on solution $(\psi^\beta,\,\theta^\beta)$ to the initial-boundary value problem* (1.1)–(1.3) *are obtained to satisfy*

$$\psi^\beta\in L^\infty\left(0,T;H^2\right)\bigcap L^2\left(0,T;H^2\right),\quad \theta^\beta\in L^\infty\left(0,T;H^1\right)\bigcap L^2\left(0,T;H^1\right),$$
$$\psi_t^\beta\in L^\infty\left(0,T;L^2\right)\bigcap L^2\left(0,T;H^1\right),\quad \theta_t^\beta\in L^\infty\left(0,T;L^2\right)\bigcap L^2\left(0,T;L^2\right),$$
$$\sqrt{\beta}\theta_x^\beta\in L^2\left(0,T;H^1\right),\quad \sqrt{\beta}\theta_{xt}^\beta\in L^2\left(0,T;L^2\right),\ \beta\theta_{xx}^\beta\in L^\infty\left(0,T;L^2\right).$$

Here the norms are all uniform in β.

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Theorem 1.4. *Assume that the initial data* $(\psi_0, \theta_0) \in H^1$ *and* $\|\psi_0\|_1 + \|\theta_0\|_1$ *be sufficiently small. Then we have*

(i) *There exists a unique solution* (ψ^0, θ^0) *to the limit problem* (1.5)–(1.7) *satisfying*

$$\psi^0 \in L^\infty\left(0,T;H^1\right) \bigcap L^2\left(0,T;H^2\right),$$
$$\theta^0 \in L^\infty\left(0,T;H^1\right) \bigcap L^2\left(0,T;H^1\right).$$

(ii) *Further assume the initial data be imposed on more regularity* $\psi_0 \in H^2$. *Then more regularities on solution* (ψ^0, θ^0) *to the initial-boundary value problem* (1.5)–(1.7) *are obtained to satisfy*

$$\psi^0 \in L^\infty\left(0,T;H^2\right) \bigcap L^2\left(0,T;H^2\right), \quad \theta^0 \in L^\infty\left(0,T;H^1\right) \bigcap L^2\left(0,T;H^1\right)$$
$$\psi^0_t \in L^\infty\left(0,T;L^2\right) \bigcap L^2\left(0,T;H^1\right), \quad \theta^0_t \in L^\infty\left(0,T;L^2\right) \bigcap L^2\left(0,T;L^2\right).$$

(iii) *Further assume the initial data be imposed on more regularity* $(\psi_0, \theta_0) \in H^3$ *and* $\|\psi_0\|_2 + \|\theta_0\|_2$ *be sufficiently small. Then more regularities on solution* (ψ^0, θ^0) *to the initial-boundary value problem* (1.5)–(1.7) *are obtained to satisfy*

$$\psi^0 \in L^\infty\left(0,T;H^3\right) \bigcap L^2\left(0,T;H^3\right), \quad \theta^0 \in L^\infty\left(0,T;H^3\right) \bigcap L^2\left(0,T;H^3\right)$$
$$\psi^0_t \in L^\infty\left(0,T;L^2\right) \bigcap L^2\left(0,T;H^2\right), \quad \theta^0_t \in L^\infty\left(0,T;L^2\right) \bigcap L^2\left(0,T;H^1\right).$$

Theorem 1.5. *Under the same conditions of theorem* 1.4, *any function* $\delta(\beta)$ *satisfying the conditions* $\delta(\beta) \to 0$ *and* $\beta^{1/2}/\delta(\beta) \to 0$ *as* $\beta \to 0^+$, *is a BL-thickness such that*

$$\left\|\psi^\beta - \psi^0\right\|_{L^\infty(0,T;L^\infty[0,1])} \leqslant C\beta^{3/8},$$
$$\left\|\theta^\beta - \theta^0\right\|_{L^\infty(0,T;L^\infty[\delta,1-\delta])} \leqslant C(\beta^{1/2}/\delta(\beta))^{1/2},$$
$$\liminf_{\beta\to 0} \left\|\theta^\beta - \theta^0\right\|_{L^\infty(0,T;L^\infty[0,1])} > 0.$$

Consequently,

$$\lim_{\beta\to 0} \left\|\psi^\beta - \psi^0\right\|_{L^\infty(0,T;L^\infty[0,1])} = 0,$$
$$\lim_{\beta\to 0} \left\|\theta^\beta - \theta^0\right\|_{L^\infty(0,T;L^\infty[\delta,1-\delta])} = 0.$$

Remark 1.6. It is reasonable that our results show that boundary layer phenomenon only occur for θ^β, but not for ψ^β. The limit of ψ^β is really passed as the diffusion β goes zero.

We need to point out that, different from the previous work on nonconservative form of Hsieh's equations in [**19**], the conservative nonlinearity $(\psi^\beta\theta^\beta)_x$ implies that new nonlinear term $\psi^\beta_x\theta^\beta$ need to be handled. Part (iii) of theorem 1.4 will play important roles to obtain the convergence rates and boundary layer thickness. That is, more regularities on the solution to the limit problem are required. Generally speaking, it is more difficult for initial-boundary problem due to the lack of boundary conditions on higher-order derivatives. In addition, lack of boundary

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conditions on θ^0 prevents us from applying the integration by part to derive the energy estimates directly. Thus it is more complicated than the case of nonconservative form. Consequently more subtle mathematical analysis need to be introduced to overcome the difficulties.

We now review some related work to the problem studied in this paper. There have been several mathematical studies of various aspects of the system (1.1) or some slightly modified systems. In the case that all parameters are fixed constants, the reader is referred, for example, to [**1**, **3**–**5**, **10**, **15**, **18**, **22**–**25**, **27**].

An interesting problem mentioned as before is the zero diffusion limit, i.e. consider the limit problem of solution consequences when one or more of parameters vanishes for the corresponding Cauchy problem or initial-boundary value problem. Chen and Zhu in [**2**] considered the Cauchy problem of nonconservative form of Hsieh's equation

$$\begin{cases} \psi_t = -(\sigma - \alpha)\psi - \sigma\theta_x + \alpha\psi_{xx}, \\ \theta_t = -(1-\beta)\theta + \nu\psi_x + 2\psi\theta_x + \beta\theta_{xx} \end{cases} \tag{1.13}$$

with initial data

$$(\psi, \theta)(x, 0) = (\psi_0, \theta_0)(x) \to (\psi_\pm, \theta_\pm) \text{ as } x \to \pm\infty. \tag{1.14}$$

It was proved that the solution sequences $\{(\psi^\alpha,\, \theta^\alpha)\}$ of the Cauchy problem (1.13), (1.14) with $\sigma = 1$, $\alpha = \beta$ and $\nu < 0$ converge to the corresponding limit system with $\alpha = 0$ as $\alpha \to 0^+$. In [**18**], the global unique solvability on C^∞-solution to the Cauchy problem of equations (1.13) for the cases of $\alpha = \beta$ and $\alpha \neq \beta$ was established. Furthermore, the convergence rates as the diffusion parameter β goes zero is also obtained.

For the initial-boundary value problem, Ruan and Zhu in [**19**] considered equations (1.13) on the strip $[0,\, 1] \times [0,\, \infty)$ with the zero Dirichlet boundary conditions

$$(\psi, \theta)(0, t) = (\psi, \theta)(1, t) = (0, 0), \quad t \geqslant 0. \tag{1.15}$$

It was shown that the solution sequences $\left\{(\psi^\beta,\, \theta^\beta)\right\}$ of the initial-boundary value problem converge to the corresponding limit system with $\beta = 0$ as $\beta \to 0^+$ in the framework of Sobolev. The convergence rates and boundary layer thickness were also obtained. Similar result on the initial-boundary value problem of equations (1.13) with zero Dirichlet–Neumann boundary conditions was also obtained in [**16**].

The rest of this paper is arranged as follows. In § 2, a uniform priori estimates on the initial-boundary value problem (1.1)–(1.3) are derived. Then the global solvability and more regularities on the limit problem (1.5)–(1.7) are established in § 3. And in § 4, convergence rates and the BL-thickness as the diffusion parameter $\beta \to 0^+$ are obtained. Finally, we use a conclusion section to summarize the results of the paper in § 5.

2. A uniform priori estimates

In this section, we devote ourselves to the *a priori* estimates of the solution $(\psi^\beta(x,\, t),\, \theta^\beta(x,\, t))$ to the initial-boundary value problem (1.1)–(1.3) under the *a* *priori* assumption

$$N_1(T) = \sup_{0<t<T} \left\| \left(\psi^{\beta}, \theta^{\beta}\right)(t) \right\|_1^2 \leqslant \varepsilon_1^2, \tag{2.1}$$

which implies by Sobolev inequality

$$\left\| \left(\psi^{\beta}, \theta^{\beta}\right)(t) \right\|_{L^{\infty}} \leqslant C\varepsilon_1, \tag{2.2}$$

where ε_1 is a positive constant satisfying $0 < \varepsilon_1 \ll 1$, independent of β.

From now on we drop the superscript β for simplicity of notations. We will derive uniform-in-β estimates on $(\psi,\ \theta)$ in two lemmas.

LEMMA 2.1. *Under the same assumptions of theorem 1.3, the parameters σ, α, β and ν satisfy the relation $(\sigma+\nu)^2 < 4\alpha(1-\beta)$ with $\nu = o(\beta^{1/2})$, we have the following estimates*:

$$\begin{aligned} &\int_0^1 \left(\psi^2+\theta^2\right) \mathrm{d}x \\ &\quad + \int_0^t \int_0^1 \left[\psi^2+\theta^2+(\psi_x)^2+\beta\left(\theta_x\right)^2\right] \mathrm{d}x\mathrm{d}\tau \leqslant C\left\|(\psi_0,\theta_0)\right\|^2 \end{aligned} \tag{2.3}$$

and

$$\begin{aligned} &\int_0^1 \left[(\psi_x)^2+(\theta_x)^2\right] \mathrm{d}x \\ &\quad + \int_0^t \int_0^1 \left[(\psi_x)^2+(\theta_x)^2+(\psi_{xx})^2+\beta\left(\theta_{xx}\right)^2\right] \mathrm{d}x\mathrm{d}\tau \leqslant C\left\|(\psi_0,\theta_0)\right\|_1^2, \end{aligned} \tag{2.4}$$

where C is a positive constant independent of β.

Proof. First, we prove (2.3). Integrating equations $(1.1)_1 \times \psi + (1.1)_2 \times \theta$ over $(0,\ t) \times (0,\ 1)$ and using integration-by-part, the boundary condition (1.3), Cauchy–Schwarz inequality and (4.49), we obtain for any $\lambda > 0$

$$\begin{aligned} &\frac{1}{2}\int_0^1 \left(\psi^2+\theta^2\right) \mathrm{d}x + \int_0^t \int_0^1 \left[(\sigma-\alpha)\psi^2+(1-\beta)\theta^2\right] \mathrm{d}x\mathrm{d}\tau \\ &\quad + \alpha \int_0^t \int_0^1 (\psi_x)^2 \,\mathrm{d}x\mathrm{d}\tau + \beta \int_0^t \int_0^1 (\theta_x)^2 \,\mathrm{d}x\mathrm{d}\tau \\ &= \frac{1}{2}\int_0^1 \left(\psi_0^2+\theta_0^2\right) \mathrm{d}x + (\sigma+\nu)\int_0^t \int_0^1 \psi_x\theta \mathrm{d}x\mathrm{d}\tau + \frac{1}{2}\int_0^t \int_0^1 \psi_x\theta^2 \mathrm{d}x\mathrm{d}\tau \\ &\leqslant \frac{1}{2}\int_0^1 \left(\psi_0^2+\theta_0^2\right) \mathrm{d}x + \int_0^t \left(|\sigma+\nu| + \frac{1}{2}\left\|\theta(t)\right\|_{L^{\infty}}\right) \int_0^1 |\psi_x\theta| \,\mathrm{d}x\mathrm{d}\tau \\ &\leqslant \frac{1}{2}\int_0^1 \left(\psi_0^2+\theta_0^2\right) \mathrm{d}x + \lambda \int_0^t \int_0^1 (\psi_x)^2 \,\mathrm{d}x\mathrm{d}\tau \\ &\quad + \frac{(|\sigma+\nu|+C\varepsilon_1)^2}{4\lambda} \int_0^t \int_0^1 \theta^2 \mathrm{d}x\mathrm{d}\tau, \end{aligned} \tag{2.5}$$

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which implies (2.3) holds provided $\lambda > 0$ is chosen to satisfy

$$\alpha > \lambda, \quad 1 - \beta > \frac{\left(|\sigma + \nu| + C\varepsilon_1\right)^2}{4\lambda}. \tag{2.6}$$

In fact, $\lambda > 0$ can be chosen such as

$$\lambda = \frac{1}{2}\left\{\alpha + \frac{(\sigma+\nu)^2}{4(1-\beta)}\right\}. \tag{2.7}$$

This proves (2.3).

Next, we turn to prove (2.4). Integrating equations $(1.1)_1 \times (-\psi_{xx}) + (1.1)_2 \times (-\theta_{xx})$ over $(0, t) \times (0, 1)$, using integration-by-parts and the boundary conditions (1.3) and (1.4), we arrive at

$$\begin{aligned} &\frac{1}{2}\int_0^1 \left[(\psi_x)^2 + (\theta_x)^2\right] \mathrm{d}x + \int_0^t\int_0^1 \left[(\sigma-\alpha)(\psi_x)^2 + (1-\beta)(\theta_x)^2\right] \mathrm{d}x\mathrm{d}\tau \\ &\quad + \alpha\int_0^t\int_0^1 (\psi_{xx})^2 \mathrm{d}x\mathrm{d}\tau + \beta\int_0^t\int_0^1 (\theta_{xx})^2 \mathrm{d}x\mathrm{d}\tau \\ &= \frac{1}{2}\int_0^1 \left(\psi_{0x}^2 + \theta_{0x}^2\right) \mathrm{d}x + \sigma\int_0^t\int_0^1 \psi_{xx}\theta_x \mathrm{d}x\mathrm{d}\tau - \nu\int_0^t\int_0^1 \psi_x\theta_{xx}\mathrm{d}x\mathrm{d}\tau \\ &\quad + \frac{3}{2}\int_0^t\int_0^1 \psi_x(\theta_x)^2 \mathrm{d}x\mathrm{d}\tau + \int_0^t\int_0^1 \psi_{xx}\theta\theta_x\mathrm{d}x\mathrm{d}\tau \\ &= \frac{1}{2}\int_0^1 \left(\psi_{0x}^2 + \theta_{0x}^2\right) \mathrm{d}x + \sum_{i=1}^4 I_i. \end{aligned} \tag{2.8}$$

Now we estimate I_1–I_4 term by term as follows.

One has by using the Cauchy inequality

$$I_1 \leqslant \lambda\int_0^t\int_0^1 (\psi_{xx})^2 \mathrm{d}x\mathrm{d}\tau + \frac{\sigma^2}{4\lambda}\int_0^t\int_0^1 (\theta_x)^2 \mathrm{d}x\mathrm{d}\tau \tag{2.9}$$

and

$$I_2 \leqslant \frac{\nu^2}{2\beta}\int_0^t\int_0^1 (\psi_x)^2 \mathrm{d}x\mathrm{d}\tau + \frac{\beta}{2}\int_0^t\int_0^1 (\theta_{xx})^2 \mathrm{d}x\mathrm{d}\tau. \tag{2.10}$$

The constant λ in (2.9) is chosen the same as one in (2.6).

For suitably small $\lambda_1 > 0$, we have by using Cauchy–Schwarz inequality, Sobolev inequality and the *a priori* assumption (2.1) and (4.49)

$$\begin{aligned} I_3 &\leqslant C\int_0^t\int_0^1 (\psi_x\theta_x)^2 \mathrm{d}x\mathrm{d}\tau + \lambda_1\int_0^t\int_0^1 (\theta_x)^2 \mathrm{d}x\mathrm{d}\tau \\ &\leqslant C\int_0^t \|\psi_x(\tau)\|_{L^\infty}^2 \int_0^1 (\theta_x)^2 \mathrm{d}x\mathrm{d}\tau + \lambda_1\int_0^t\int_0^1 (\theta_x)^2 \mathrm{d}x\mathrm{d}\tau \\ &\leqslant C\varepsilon_1^2\int_0^t\int_0^1 \left[(\psi_x)^2 + (\psi_{xx})^2\right] \mathrm{d}x\mathrm{d}\tau + \lambda_1\int_0^t\int_0^1 (\theta_x)^2 \mathrm{d}x\mathrm{d}\tau \end{aligned} \tag{2.11}$$

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and

$$\begin{aligned} I_4 &\leqslant \int_0^t \int_0^1 |\psi_{xx}\theta\theta_x| \,\mathrm{d}x\mathrm{d}\tau \\ &\leqslant \|\theta\|_{L^\infty} \int_0^t \int_0^1 |\psi_{xx}\theta_x| \,\mathrm{d}x\mathrm{d}\tau \\ &\leqslant C\varepsilon_1 \int_0^t \int_0^1 \left[(\psi_{xx})^2 + (\theta_x)^2\right] \mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{2.12}$$

Substituting (2.9)–(2.12) into (2.8), and using (2.3) with the smallness of ε_1 and λ_1, we obtain (2.4). This completes the proof of lemma 2.1. □

Now, we can show that the *a priori* assumption (2.1) can be closed. Since, under this *a priori* assumption (2.1), we deduced that (2.3) and (2.4) hold provided ε_1 is sufficiently small. Therefore the assumption (2.1) is always true provided $\|(\psi_0, \theta_0)\|_1$ is sufficiently small.

In order to obtain the boundary layer thickness and the convergence rates in next section, it is required to derive the desired estimates on $\|(\psi_{xx}, \theta_{xx})(t)\|$ in the following lemma.

Lemma 2.2. *Under the same assumptions of theorem 1.3, we have the following estimates*:

$$\begin{aligned} &\int_0^1 \left[(\psi_t)^2 + (\theta_t)^2\right] \mathrm{d}x \\ &\quad + \int_0^t \int_0^1 \left[(\psi_t)^2 + (\theta_t)^2 + (\psi_{xt})^2 + \beta\,(\theta_{xt})^2\right] \mathrm{d}x\mathrm{d}\tau \leqslant C\,\|(\psi_0, \theta_0)\|_2^2 \end{aligned} \tag{2.13}$$

and

$$\int_0^1 \left[(\psi_{xx})^2 + (\beta\theta_{xx})^2\right] \mathrm{d}x \leqslant C\,\|(\psi_0, \theta_0)\|_2^2, \tag{2.14}$$

where C is a positive constant independent of β.

Proof. Differentiating (1.1) with respect to t, we get

$$\begin{cases} \psi_{tt} = -(\sigma-\alpha)\psi_t - \sigma\theta_{xt} + \alpha\psi_{xxt}, \\ \theta_{tt} = -(1-\beta)\theta_t + \nu\psi_{xt} + (\psi\theta)_{xt} + \beta\theta_{xxt}. \end{cases} \tag{2.15}$$

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Integrating equations $(2.15)_1 \times \psi_t + (2.15)_2 \times \theta_t$ over $(0, t) \times (0, 1)$, using integration-by-parts and the boundary conditions (1.4), we arrive at

$$\begin{aligned} &\frac{1}{2}\int_0^1 \left[(\psi_t)^2 + (\theta_t)^2\right] \mathrm{d}x \\ &\quad + \int_0^t \int_0^1 \left[(\sigma - \alpha)(\psi_t)^2 + (1-\beta)(\theta_t)^2\right] \mathrm{d}x\mathrm{d}\tau \\ &\quad + \alpha \int_0^t \int_0^1 (\psi_{xt})^2 \mathrm{d}x\mathrm{d}\tau + \beta \int_0^t \int_0^1 (\theta_{xt})^2 \mathrm{d}x\mathrm{d}\tau \\ &= \frac{1}{2}\int_0^1 \left[(\psi_t)^2 + (\theta_t)^2\right]_{t=0} \mathrm{d}x + \sum_{i=5}^{8} I_i, \end{aligned} \tag{2.16}$$

where

$$\begin{cases} I_5 = (\sigma+\nu) \displaystyle\int_0^t \int_0^1 \psi_{xt}\theta_t \mathrm{d}x\mathrm{d}\tau, \\ I_6 = \displaystyle\int_0^t \int_0^1 \psi_t\theta_x\theta_t \mathrm{d}x\mathrm{d}\tau, \\ I_7 = \dfrac{1}{2}\displaystyle\int_0^t \int_0^1 \psi_x (\theta_t)^2 \mathrm{d}x\mathrm{d}\tau, \\ I_8 = \displaystyle\int_0^t \int_0^1 \psi_{xt}\theta\theta_t \mathrm{d}x\mathrm{d}\tau. \end{cases} \tag{2.17}$$

I_5–I_8 are estimated term by term as follows.

By using Cauchy–Schwarz inequality, Sobolev inequality and the *a priori* assumption (2.1) and (4.49), we have

$$I_5 \leqslant \lambda \int_0^t \int_0^1 (\psi_{xt})^2 \mathrm{d}x\mathrm{d}\tau + \frac{(\sigma+\nu)^2}{4\lambda}\int_0^t \int_0^1 (\theta_t)^2 \mathrm{d}x\mathrm{d}\tau, \tag{2.18}$$

$$\begin{aligned} I_6 &\leqslant \frac{1}{2}\int_0^t \int_0^1 (\psi_t\theta_x)^2 \mathrm{d}x\mathrm{d}\tau + \frac{1}{2}\int_0^t \int_0^1 (\theta_t)^2 \mathrm{d}x\mathrm{d}\tau \\ &\leqslant C\varepsilon_1^2 \int_0^t \int_0^1 \left[(\psi_t)^2 + (\psi_{xt})^2\right] \mathrm{d}x\mathrm{d}\tau + \frac{1}{2}\int_0^t \int_0^1 (\theta_t)^2 \mathrm{d}x\mathrm{d}\tau, \end{aligned} \tag{2.19}$$

$$\begin{aligned} I_7 &\leqslant \frac{1}{4}\int_0^t \int_0^1 (\psi_x\theta_t)^2 \mathrm{d}x\mathrm{d}\tau + \frac{1}{4}\int_0^t \int_0^1 (\theta_t)^2 \mathrm{d}x\mathrm{d}\tau \\ &\leqslant C\int_0^t \int_0^1 \left[(\psi_x)^2 + (\psi_{xx})^2\right] \mathrm{d}x \int_0^1 (\theta_t)^2 \mathrm{d}x\mathrm{d}\tau + \frac{1}{4}\int_0^t \int_0^1 (\theta_t)^2 \mathrm{d}x\mathrm{d}\tau \end{aligned} \tag{2.20}$$

and

$$\begin{aligned} I_8 &\leqslant \|\theta\|_{L^\infty} \int_0^t \int_0^1 |\psi_{xt}\theta_t| \mathrm{d}x\mathrm{d}\tau \\ &\leqslant C\varepsilon_1 \int_0^t \int_0^1 (\psi_{xt})^2 \mathrm{d}x\mathrm{d}\tau + C\varepsilon_1 \int_0^t \int_0^1 (\theta_t)^2 \mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{2.21}$$

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Substituting (2.18)–(2.21) into (2.16), we have from (1.1)

$$\begin{aligned}
&\int_0^1 \left[(\psi_t)^2 + (\theta_t)^2\right] \mathrm{d}x + \int_0^t \int_0^1 \left[(\psi_t)^2 + (\theta_t)^2\right] \mathrm{d}x\mathrm{d}\tau \\
&\quad + \int_0^t \int_0^1 (\psi_{xt})^2 \,\mathrm{d}x\mathrm{d}\tau + \beta \int_0^t \int_0^1 (\theta_{xt})^2 \,\mathrm{d}x\mathrm{d}\tau \\
&\leqslant C \int_0^1 \left[(\psi_0)^2 + (\theta_0)^2 + (\psi_{0x})^2 + (\theta_{0x})^2 + (\psi_{0xx})^2 + (\theta_{0xx})^2\right] \mathrm{d}x \\
&\quad + C \int_0^t \int_0^1 \left[(\psi_x)^2 + (\psi_{xx})^2\right] \mathrm{d}x \int_0^1 (\theta_t)^2 \,\mathrm{d}x\mathrm{d}\tau,
\end{aligned} \tag{2.22}$$

which implies

$$\begin{aligned}
\int_0^1 (\theta_t)^2 \,\mathrm{d}x \leqslant{}& C \int_0^1 \left[(\psi_0)^2 + (\theta_0)^2 + (\psi_{0x})^2 + (\theta_{0x})^2 + (\psi_{0xx})^2 + (\theta_{0xx})^2\right] \mathrm{d}x \\
&+ C \int_0^t \int_0^1 \left[(\psi_x)^2 + (\psi_{xx})^2\right] \mathrm{d}x \int_0^1 (\theta_t)^2 \,\mathrm{d}x\mathrm{d}\tau.
\end{aligned}$$

Using Gronwall's inequality and (2.4) in lemma 2.1, one has

$$\begin{aligned}
\int_0^1 (\theta_t)^2 \,\mathrm{d}x \leqslant{}& C \int_0^1 \left[(\psi_0)^2 + (\theta_0)^2 + (\psi_{0x})^2 + (\theta_{0x})^2 + (\psi_{0xx})^2 + (\theta_{0xx})^2\right] \mathrm{d}x \\
&\times \exp\left\{C \int_0^t \int_0^1 \left[(\psi_x)^2 + (\psi_{xx})^2\right] \mathrm{d}x\mathrm{d}\tau\right\} \\
\leqslant{}& C \int_0^1 \left[(\psi_0)^2 + (\theta_0)^2 + (\psi_{0x})^2 + (\theta_{0x})^2 + (\psi_{0xx})^2 + (\theta_{0xx})^2\right] \mathrm{d}x \\
&\times \exp\left\{C \|(\psi_0, \theta_0)\|_1^2\right\} \\
\leqslant{}& C.
\end{aligned}$$

Substituting the above inequality into (2.22) and using (2.4) in lemma 2.1 once again

$$\begin{aligned}
&\int_0^1 \left[(\psi_t)^2 + (\theta_t)^2\right] \mathrm{d}x + \int_0^t \int_0^1 \left[(\psi_t)^2 + (\theta_t)^2\right] \mathrm{d}x\mathrm{d}\tau \\
&\quad + \int_0^t \int_0^1 (\psi_{xt})^2 \,\mathrm{d}x\mathrm{d}\tau + \beta \int_0^t \int_0^1 (\theta_{xt})^2 \,\mathrm{d}x\mathrm{d}\tau \\
&\leqslant C \int_0^1 \left[(\psi_0)^2 + (\theta_0)^2 + (\psi_{0x})^2 + (\theta_{0x})^2 + (\psi_{0xx})^2 + (\theta_{0xx})^2\right] \mathrm{d}x \\
&\quad + C \int_0^t \int_0^1 \left[(\psi_x)^2 + (\psi_{xx})^2\right] \mathrm{d}x\mathrm{d}\tau
\end{aligned}$$

$$\leqslant C\int_0^1 \left[(\psi_0)^2+(\theta_0)^2+(\psi_{0x})^2+(\theta_{0x})^2+(\psi_{0xx})^2+(\theta_{0xx})^2\right]\mathrm{d}x$$
$$+C\|(\psi_0,\theta_0)\|_1^2,$$

which implies (2.13).

It directly follows from (1.1)

$$\begin{cases}(\psi_{xx})^2=\dfrac{1}{\alpha^2}\left[\psi_t+(\sigma-\alpha)\psi+\sigma\theta_x\right]^2,\\ (\beta\theta_{xx})^2=\left[\theta_t+(1-\beta)\theta-\nu\psi_x-\psi\theta_x-\psi_x\theta\right]^2.\end{cases}\tag{2.23}$$

Integrating (2.23) over (0,1) and using Cauchy–Schwarz inequality, (4.49), (2.13) with lemma 2.1, we obtain

$$\begin{aligned}\int_0^1 &\left[(\psi_{xx})^2+(\beta\theta_{xx})^2\right]\mathrm{d}x\\ &=\frac{1}{\alpha^2}\int_0^1\left[\psi_t+(\sigma-\alpha)\psi+\sigma\theta_x\right]^2\mathrm{d}x\\ &\quad+\int_0^1\left[\theta_t+(1-\beta)\theta-\nu\psi_x-\psi\theta_x-\psi_x\theta\right]^2\mathrm{d}x\\ &\leqslant C\int_0^1\left[(\psi_t)^2+\psi^2+(\theta_x)^2+(\theta_t)^2+\theta^2+(\psi_x)^2\right]\mathrm{d}x\\ &\leqslant C\left\|(\psi_0,\theta_0)\right\|_2^2.\end{aligned}\tag{2.24}$$

This completes the proof of lemma 2.2. □

3. More regularities on the limit problem

In this section, we will establish the *a priori* estimates of the solution $(\psi^0,\,\theta^0)$ to initial-boundary value problem (1.5)–(1.7). In particular, the more regularities on the solutions will be obtained provided the initial data is more regular. This will play an important role in proving boundary layer thickness and convergence rates in next section. It is required on the *a priori* assumption

$$N_2(T)=\sup_{0<t<T}\left\|\left(\psi^0,\theta^0\right)(t)\right\|_1^2\leqslant\varepsilon_2^2,\tag{3.1}$$

which implies by Sobolev inequality

$$\left\|\left(\psi^0,\theta^0\right)(t)\right\|_{L^\infty}\leqslant C\varepsilon_2,\tag{3.2}$$

where $0<\varepsilon_2\ll 1$. From now on we drop the superscript 0 for simplicity of notations and denote $(\psi,\,\theta)$ instead of $(\psi^0,\,\theta^0)$.

Lemma 3.1. *Assume that the initial data satisfy the conditions*: $(\psi_0,\,\theta_0)\in H^1$ *and* $\|\psi_0\|_1+\|\theta_0\|_1$ *is sufficiently small. The parameters* α *and* σ *satisfy the relation* $\frac{\sigma^2}{4}<\alpha<\sigma$.

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(i) *Then there exists a unique solutions* $(\psi,\, \theta)$ *to the initial-boundary value problem* (1.5)–(1.7) *satisfying*

$$\begin{aligned} &\int_0^1 \left(\psi^2+\theta^2\right) \mathrm{d}x \\ &\quad + \int_0^t \int_0^1 \left[\psi^2+\theta^2+(\psi_x)^2\right] \mathrm{d}x\mathrm{d}\tau \leqslant C\|(\psi_0,\theta_0)\|^2 \end{aligned} \tag{3.3}$$

and

$$\begin{aligned} &\int_0^1 \left[(\psi_x)^2+(\theta_x)^2\right] \mathrm{d}x \\ &\quad + \int_0^t \int_0^1 \left[(\psi_x)^2+(\theta_x)^2+(\psi_{xx})^2\right] \mathrm{d}x\mathrm{d}\tau \leqslant C\|(\psi_{0x},\theta_{0x})\|^2. \end{aligned} \tag{3.4}$$

(ii) *Furthermore assume that* $\psi_0 \in H^2$, *we have*:

$$\begin{aligned} &\int_0^1 \left[(\psi_t)^2+(\theta_t)^2+(\psi_{xx})^2\right] \mathrm{d}x \\ &\quad + \int_0^t \int_0^1 \left[(\psi_t)^2+(\theta_t)^2+(\psi_{xt})^2\right] \mathrm{d}x\mathrm{d}\tau \leqslant C\left(\|\psi_0\|_2^2+\|\theta_0\|_1^2\right). \end{aligned} \tag{3.5}$$

(iii) *Furthermore assume that* $(\psi_0,\, \theta_0) \in H^3$ *and* $\|\psi_0\|_2+\|\theta_0\|_2$ *is sufficiently small, more regularity on the solution* $(\psi,\, \theta)$ *is obtained as follows:*

$$\int_0^1 (\theta_{xx})^2 \mathrm{d}x + \int_0^t \int_0^1 \left[(\theta_{xx})^2+(\psi_{xxx})^2\right] \mathrm{d}x\mathrm{d}\tau \leqslant C\left(\|\psi_0\|_2^2+\|\theta_0\|_1^2\right), \tag{3.6}$$

$$\int_0^t \int_0^1 (\theta_{xt})^2 \mathrm{d}x\mathrm{d}\tau \leqslant C\left(\|\psi_0\|_2^2+\|\theta_0\|_1^2\right), \tag{3.7}$$

$$\int_0^t \int_0^1 (\psi_{xxt})^2 \mathrm{d}x\mathrm{d}\tau \leqslant C\left(\|\psi_0\|_3^2+\|\theta_0\|_2^2\right) \tag{3.8}$$

and

$$\int_0^1 (\theta_{xxx})^2 \mathrm{d}x + \int_0^t \int_0^1 (\theta_{xxx})^2 \mathrm{d}x\mathrm{d}\tau \leqslant C\left(\|\psi_0\|_3^2+\|\theta_0\|_3^2\right). \tag{3.9}$$

Proof. Proof of (3.3).

Integrating the resulting equations $(1.5)_1 \times \psi + (1.5)_2 \times \theta$ over $(0,\, t) \times (0,\, 1)$, using integration-by-parts and the boundary conditions (1.7), Cauchy–Schwarz inequality and (3.1), we obtain for $\lambda > 0$ taken in lemma 2.1

$$\begin{aligned} &\frac{1}{2}\int_0^1 \left(\psi^2+\theta^2\right) \mathrm{d}x + \int_0^t \int_0^1 \left[(\sigma-\alpha)\psi^2+\theta^2\right] \mathrm{d}x\mathrm{d}\tau \\ &\quad + \alpha \int_0^t \int_0^1 (\psi_x)^2 \mathrm{d}x\mathrm{d}\tau \end{aligned}$$

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$$
\begin{aligned}
&= \frac{1}{2}\int_0^1 \left(\psi_0^2+\theta_0^2\right) \mathrm{d}x + \sigma\int_0^t\int_0^1 \psi_x\theta \mathrm{d}x\mathrm{d}\tau + \frac{1}{2}\int_0^t\int_0^1 \psi_x\theta^2 \mathrm{d}x\mathrm{d}\tau \qquad (3.10)\\
&\leqslant \frac{1}{2}\int_0^1 \left(\psi_0^2+\theta_0^2\right) \mathrm{d}x + \int_0^t \left(\sigma + \frac{1}{2}\left\|\theta(t)\right\|_{L^\infty}\right)\int_0^1 \left|\psi_x\theta\right| \mathrm{d}x\mathrm{d}\tau\\
&\leqslant \frac{1}{2}\int_0^1 \left(\psi_0^2+\theta_0^2\right) \mathrm{d}x + \lambda\int_0^t\int_0^1 \left(\psi_x\right)^2 \mathrm{d}x\mathrm{d}\tau\\
&\quad + \frac{(\sigma+C\varepsilon_2)^2}{4\lambda}\int_0^t\int_0^1 \theta^2 \mathrm{d}x\mathrm{d}\tau. \qquad (3.1)
\end{aligned}
$$

Then we will deduce

$$
\begin{aligned}
&\int_0^1 \left(\psi^2+\theta^2\right) \mathrm{d}x\\
&\quad + \int_0^t\int_0^1 \left[\psi^2+\theta^2+\left(\psi_x\right)^2\right] \mathrm{d}x\mathrm{d}\tau \leqslant C\left\|(\psi_0,\theta_0)\right\|^2.
\end{aligned} \qquad (3.11)
$$

Proof of (3.4). Differentiating (1.5) with respect to x, we get

$$
\begin{cases}
\psi_{xt} = -(\sigma-\alpha)\psi_x - \sigma\theta_{xx} + \alpha\psi_{xxx},\\
\theta_{xt} = -\theta_x + \left(\psi\theta\right)_{xx}.
\end{cases} \qquad (3.12)
$$

Integrating equation $(3.12)_1\times\psi_x + (3.12)_2\times\theta_x$ over $(0,\,t)\times(0,\,1)$, using integration-by-parts and the boundary conditions (1.7) and (1.8), we arrive at

$$
\begin{aligned}
&\frac{1}{2}\int_0^1 \left[\left(\psi_x\right)^2+\left(\theta_x\right)^2\right] \mathrm{d}x + \int_0^t\int_0^1 \left[\left(\sigma-\alpha\right)\left(\psi_x\right)^2+\left(\theta_x\right)^2\right] \mathrm{d}x\mathrm{d}\tau\\
&\quad + \alpha\int_0^t\int_0^1 \left(\psi_{xx}\right)^2 \mathrm{d}x\mathrm{d}\tau \qquad (3.13)\\
&= \frac{1}{2}\int_0^1 \left(\psi_{0x}^2+\theta_{0x}^2\right) \mathrm{d}x + J_1 + J_2 + J_3,
\end{aligned}
$$

where

$$
\begin{cases}
J_1 = \sigma\int_0^t\int_0^1 \psi_{xx}\theta_x \mathrm{d}x\mathrm{d}\tau,\\
J_2 = \frac{3}{2}\int_0^t\int_0^1 \psi_x\left(\theta_x\right)^2 \mathrm{d}x\mathrm{d}\tau,\\
J_3 = \int_0^t\int_0^1 \psi_{xx}\theta\theta_x \mathrm{d}x\mathrm{d}\tau.
\end{cases} \qquad (3.14)
$$

We have by using Cauchy–Schwarz inequality, Sobolev inequality and the *a priori* assumptions (3.1)–(3.2)

$$
J_1 \leqslant \lambda\int_0^t\int_0^1 \left(\psi_{xx}\right)^2 \mathrm{d}x\mathrm{d}\tau + \frac{\sigma^2}{4\lambda}\int_0^t\int_0^1 \left(\theta_x\right)^2 \mathrm{d}x\mathrm{d}\tau, \qquad (3.15)
$$

$$\begin{aligned} J_2 &\leqslant \frac{3}{8\lambda_1}\int_0^t\int_0^1 (\psi_x\theta_x)^2\,\mathrm{d}x\mathrm{d}\tau + \frac{3\lambda_1}{2}\int_0^t\int_0^1 (\theta_x)^2\,\mathrm{d}x\mathrm{d}\tau \\ &\leqslant C\varepsilon_2^2\int_0^t\int_0^1 \left[(\psi_x)^2 + (\psi_{xx})^2\right]\mathrm{d}x\mathrm{d}\tau + \frac{3\lambda_1}{2}\int_0^t\int_0^1 (\theta_x)^2\,\mathrm{d}x\mathrm{d}\tau \end{aligned} \tag{3.16}$$

and

$$J_3 \leqslant C\varepsilon_2\int_0^t\int_0^1 \left[(\psi_{xx})^2 + (\theta_x)^2\right]\mathrm{d}x\mathrm{d}\tau. \tag{3.17}$$

Substituting (3.15)–(3.17) into (3.13), and using the smallness of ε_2 and λ_1, we deduce (3.4). This completes the proof of lemma 3.1(i).

Now, by the similar argument to those in § 2, we can show that the *a priori* assumption (3.1) is closed.

Proof of (3.5). Differentiating $(1.5)_1$ with respect to t, we get

$$\psi_{tt} = -(\sigma-\alpha)\psi_t - \sigma\theta_{xt} + \alpha\psi_{xxt}. \tag{3.18}$$

Integrating equation (3.18) $\times\, \psi_t + (1.5)_2 \times \theta_t$ over $(0,\, t)\times(0,\, 1)$, using integration-by-parts, the boundary conditions (1.8) and equation $(1.5)_1$, we have

$$\begin{aligned} &\frac{1}{2}\int_0^1 (\psi_t)^2\,\mathrm{d}x + \int_0^t\int_0^1 \left[(\sigma-\alpha)(\psi_t)^2 + \alpha(\psi_{xt})^2 + (\theta_t)^2\right]\mathrm{d}x\mathrm{d}\tau \\ &\quad= \frac{1}{2}\int_0^1 (\psi_t)^2(x,0)\mathrm{d}x + \frac{1}{2}\int_0^1 \left[(\theta_0)^2 - \theta^2\right]\mathrm{d}x + \sigma\int_0^t\int_0^1 \psi_{xt}\theta_t\mathrm{d}x\mathrm{d}\tau \\ &\qquad+ \int_0^t\int_0^1 \psi\theta_x\theta_t\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_0^1 \psi_x\theta\theta_t\mathrm{d}x\mathrm{d}\tau + \sigma\int_0^t\int_0^1 \psi_x\theta_t\mathrm{d}x\mathrm{d}\tau \\ &\quad\leqslant C\int_0^1 \left(\psi_0^2 + \theta_0^2 + \theta_{0x}^2 + \psi_{0xx}^2\right)\mathrm{d}x + \lambda\int_0^t\int_0^1 (\psi_{xt})^2\,\mathrm{d}x\mathrm{d}\tau \\ &\qquad+ \frac{\sigma^2}{4\lambda}\int_0^t\int_0^1 (\theta_t)^2\,\mathrm{d}x\mathrm{d}\tau + C\varepsilon_2\int_0^t\int_0^1 (\theta_t)^2\,\mathrm{d}x\mathrm{d}\tau \\ &\qquad+ C\int_0^t\int_0^1 \left[(\psi_x)^2 + (\theta_x)^2\right]\mathrm{d}x\mathrm{d}\tau, \end{aligned} \tag{3.19}$$

which implies due to the smallness of ε_2 and λ_1

$$\begin{aligned} &\int_0^1 (\psi_t)^2\,\mathrm{d}x + \int_0^t\int_0^1 \left[(\psi_t)^2 + (\psi_{xt})^2 + (\theta_t)^2\right]\mathrm{d}x\mathrm{d}\tau \\ &\quad\leqslant C\left(\|\psi_0\|_2^2 + \|\theta_0\|_1^2\right). \end{aligned} \tag{3.20}$$

From $(1.5)_2$, we have

$$(\theta_t)^2 = \left(-\theta + \psi_x\theta + \psi\theta_x\right)^2. \tag{3.21}$$

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Integrating (3.21) over (0, 1), we obtain by Cauchy–Schwarz inequality and (3.2)–(3.4)

$$\begin{aligned}\int_0^1 (\theta_t)^2\,\mathrm{d}x &\leqslant C\int_0^1 \left[\theta^2+(\psi\theta_x)^2+(\psi_x\theta)^2\right]\mathrm{d}x\\ &\leqslant C\int_0^1 \theta^2\mathrm{d}x + C\varepsilon_2^2\int_0^1 (\theta_x)^2\,\mathrm{d}x + C\varepsilon_2^2\int_0^1 (\psi_x)^2\,\mathrm{d}x\\ &\leqslant C\int_0^1 \left(\psi_0^2+\theta_0^2+\psi_{0x}^2+\theta_{0x}^2\right)\mathrm{d}x.\end{aligned} \tag{3.22}$$

From $(1.5)_1$, (3.3) and (3.4), one easily gets by Cauchy inequality

$$\begin{aligned}\int_0^1 (\psi_{xx})^2\,\mathrm{d}x &\leqslant C\int_0^1 \left[(\psi_t)^2+\psi^2+(\theta_x)^2\right]\mathrm{d}x\\ &\leqslant C\int_0^1 \left(\psi_0^2+\theta_0^2+\psi_{0x}^2+\theta_{0x}^2+\psi_{0xx}^2\right)\mathrm{d}x.\end{aligned} \tag{3.23}$$

(3.22) and (3.23) imply (3.5). This completes the proof of lemma 3.1(ii).

Next, we prove (iii) of lemma 3.1 under the *a priori* assumption

$$N_3(T)=\sup_{0<t<T}\|(\psi,\theta)(t)\|_2^2\leqslant\varepsilon_3^2, \tag{3.24}$$

which implies by Sobolev inequality

$$\|\psi(t)\|_{W^{1,\infty}}+\|\theta(t)\|_{W^{1,\infty}}\leqslant C\varepsilon_3, \tag{3.25}$$

where $0<\varepsilon_3\ll 1$.

Proof of (3.6). Differentiating $(1.5)_2$ with respect to x twice, we get

$$\theta_{xxt}=-\theta_{xx}+(\psi\theta_x)_{xx}+(\psi_x\theta)_{xx}\,. \tag{3.26}$$

Integrating equation (3.26) $\times\,\theta_{xx}$ over $(0,\,t)\times(0,\,1)$, using integration-by-parts and the boundary conditions (1.7), we arrive at

$$\begin{aligned}&\frac{1}{2}\int_0^1 (\theta_{xx})^2\,\mathrm{d}x+\int_0^t\int_0^1(\theta_{xx})^2\,\mathrm{d}x\mathrm{d}\tau\\ &\quad=\frac{1}{2}\int_0^1(\theta_{0xx})^2\,\mathrm{d}x+3\int_0^t\int_0^1\psi_{xx}\theta_x\theta_{xx}\mathrm{d}x\mathrm{d}\tau+\frac{5}{2}\int_0^t\int_0^1\psi_x(\theta_{xx})^2\,\mathrm{d}x\mathrm{d}\tau\\ &\qquad+\int_0^t\int_0^1\psi_{xxx}\theta\theta_{xx}\mathrm{d}x\mathrm{d}\tau\\ &\quad=\frac{1}{2}\int_0^1(\theta_{0xx})^2\,\mathrm{d}x+J_4+J_5+J_6.\end{aligned} \tag{3.27}$$

Here J_4–J_6 are estimated term by term as follows.

One has from Cauchy–Schwarz inequality and (3.25)

$$J_4 \leqslant C\varepsilon_3 \int_0^t \int_0^1 \left[(\psi_{xx})^2 + (\theta_{xx})^2\right] \mathrm{d}x\mathrm{d}\tau \tag{3.28}$$

and

$$J_5 \leqslant C\varepsilon_3 \int_0^t \int_0^1 (\theta_{xx})^2 \,\mathrm{d}x\mathrm{d}\tau. \tag{3.29}$$

In order to estimate J_6–J_7, let's differentiate $(1.5)_1$ with respect to x

$$\alpha\psi_{xxx} = \psi_{xt} + (\sigma - \alpha)\psi_x + \sigma\theta_{xx}. \tag{3.30}$$

Substituting (3.30) into J_6, we get by (3.2) and Cauchy–Schwarz inequality

$$\begin{aligned} J_6 &= \frac{1}{\alpha}\int_0^t \int_0^1 \theta\theta_{xx}\left[\psi_{xt} + (\sigma-\alpha)\psi_x + \sigma\theta_{xx}\right] \mathrm{d}x\mathrm{d}\tau \\ &\leqslant \frac{1}{\alpha}\int_0^t \|\theta(\tau)\|_{L^\infty} \int_0^1 \left|\theta_{xx}\left[\psi_{xt} + (\sigma-\alpha)\psi_x + \sigma\theta_{xx}\right]\right| \mathrm{d}x\mathrm{d}\tau \\ &\leqslant C\varepsilon_2 \int_0^t \int_0^1 \left[(\theta_{xx})^2 + (\psi_{xt})^2 + (\psi_x)^2\right] \mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{3.31}$$

Collecting the estimates (3.27)–(3.31) and using (3.4) with (3.5), we derive

$$\int_0^1 (\theta_{xx})^2 \,\mathrm{d}x + \int_0^t \int_0^1 (\theta_{xx})^2 \,\mathrm{d}x\mathrm{d}\tau \leqslant C\left(\|\psi_0\|_2^2 + \|\theta_0\|_1^2\right), \tag{3.32}$$

Finally, integrating (3.30) over $(0,\,t) \times (0,\,1)$ and using Cauchy inequality, we get

$$\int_0^t \int_0^1 (\psi_{xxx})^2 \,\mathrm{d}x\mathrm{d}\tau \leqslant C \int_0^t \int_0^1 \left[(\psi_{xt})^2 + (\psi_x)^2 + (\theta_{xx})^2\right] \mathrm{d}x\mathrm{d}\tau. \tag{3.33}$$

(3.4), (3.5), (3.32) with (3.33) imply (3.6).

Now, we can show that the *a priori* assumption (3.24) can be closed. Since, under this *a priori* assumption (3.24), we have deduced that (3.5) and (3.6) hold provided ε_3 is sufficiently small. Therefore the assumption (3.24) is always true provided $\psi_0 \in H^2$, $\theta_0 \in H^1$ and $\|\psi_0\|_2 + \|\theta_0\|_1$ is sufficiently small.

Next, let's continue to the proof of the rest estimates one by one under the additional regularity on initial data.

Proof of (3.7). Differentiating $(1.5)_2$ with respect to x, we have

$$\begin{aligned} \theta_{xt} &= -\theta_x + (\psi\theta)_{xx} \\ &= -\theta_x + \psi_{xx}\theta + 2\psi_x\theta_x + \psi\theta_{xx}. \end{aligned} \tag{3.34}$$

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Integrating (3.34) over $(0, t) \times (0, 1)$ and using the Cauchy inequality, we get

$$\begin{aligned} &\int_0^t \int_0^1 (\theta_{xt})^2 \,\mathrm{d}x\mathrm{d}\tau \\ &\quad \leqslant C \int_0^t \int_0^1 \left[(\theta_x)^2 + (\psi_{xx}\theta)^2 + (\psi_x\theta_x)^2 + (\psi\theta_{xx})^2 + (\psi_{xx})^2\right] \mathrm{d}x\mathrm{d}\tau \\ &\quad \leqslant C \int_0^t \int_0^1 \left[(\theta_x)^2 + (\psi_{xx})^2 + (\psi_x)^2 + (\theta_{xx})^2\right] \mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{3.35}$$

(3.4), (3.6) and (3.35) imply (3.7).

Proof of (3.8). Integrating equation (3.18) $\times\, \psi_{xxt}$ over $(0, t) \times (0, 1)$, and using integration-by-parts with the boundary conditions (1.8), we arrive at

$$\begin{aligned} &\alpha \int_0^1 \int_0^t (\psi_{xxt})^2 \mathrm{d}x\mathrm{d}\tau + (\sigma - \alpha) \int_0^1 \int_0^t (\psi_{xt})^2 \mathrm{d}x\mathrm{d}\tau + \frac{1}{2} \int_0^1 (\psi_{xt})^2 \mathrm{d}x \\ &\quad = \frac{1}{2} \int_0^1 (\psi_{xt})^2|_{t=0} \mathrm{d}x + \sigma \int_0^1 \int_0^t \theta_{xt}\psi_{xxt} \mathrm{d}x\mathrm{d}\tau \\ &\quad \leqslant \frac{1}{2} \int_0^1 (\psi_{xt})^2|_{t=0} \mathrm{d}x + \frac{\alpha}{2} \int_0^1 \int_0^t (\psi_{xxt})^2 \mathrm{d}x\mathrm{d}\tau + \frac{\sigma^2}{2\alpha} \int_0^1 \int_0^t (\theta_{xt})^2 \mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{3.36}$$

From (3.30), we have

$$\psi_{xt}\Big|_{t=0} = -(\sigma - \alpha)\psi_{0x} - \sigma\theta_{0xx} + \alpha\psi_{0xxx}, \tag{3.37}$$

which implies

$$\begin{aligned} \int_0^1 \left(\psi_{xt}\Big|_{t=0}\right)^2 \mathrm{d}x &\leqslant C \int_0^1 \left[(\psi_{0x})^2 + (\theta_{0xx})^2 + (\psi_{0xxx})^2\right] \mathrm{d}x \\ &\leqslant C \left(\|\psi_{0x}\|_2^2 + \|\theta_{0x}\|_1^2\right). \end{aligned} \tag{3.38}$$

Combining (3.36) and (3.38), we derive (3.8).

Proof of (3.9). Differentiating $(1.5)_2$ with respect to x three times, we get

$$\theta_{xxxt} = -\theta_{xxx} + (\psi\theta_x)_{xxx} + (\psi_x\theta)_{xxx}\,. \tag{3.39}$$

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Integrating equation (3.39) $\times\ \theta_{xxx}$ over $(0,\, t)\times(0,\, 1)$, using integration-by-parts with the boundary conditions (1.7), we arrive at

$$\begin{aligned}
&\frac{1}{2}\int_0^1 (\theta_{xxx})^2\,\mathrm{d}x + \int_0^t\int_0^1 (\theta_{xxx})^2\,\mathrm{d}x\mathrm{d}\tau \\
&= \frac{1}{2}\int_0^1 (\theta_{0xxx})^2\,\mathrm{d}x + 4\int_0^t\int_0^1 \psi_{xxx}\theta_x\theta_{xxx}\mathrm{d}x\mathrm{d}\tau \\
&\quad + 6\int_0^t\int_0^1 \psi_{xx}\theta_{xx}\theta_{xxx}\mathrm{d}x\mathrm{d}\tau + \frac{7}{2}\int_0^t\int_0^1 \psi_x\,(\theta_{xxx})^2\,\mathrm{d}x\mathrm{d}\tau \\
&\quad + \int_0^t\int_0^1 \psi_{xxxx}\theta\theta_{xxx}\mathrm{d}x\mathrm{d}\tau \\
&= \frac{1}{2}\int_0^1 (\theta_{0xxx})^2\,\mathrm{d}x + \sum_{i=7}^{10} J_i.
\end{aligned} \tag{3.40}$$

Now we estimate J_7–J_{10} term by term as follows.

It is obvious to get

$$J_7 \leqslant C\varepsilon_3\int_0^t\int_0^1 \left[(\psi_{xxx})^2 + (\theta_{xxx})^2\right]\mathrm{d}x\mathrm{d}\tau \tag{3.41}$$

and

$$J_9 \leqslant C\varepsilon_3\int_0^t\int_0^1 (\theta_{xxx})^2\,\mathrm{d}x\mathrm{d}\tau. \tag{3.42}$$

We have from Cauchy–Schwarz inequality, Sobolev inequality and (3.25)

$$\begin{aligned}
J_8 &\leqslant C\int_0^t\int_0^1 (\psi_{xx})^2\,(\theta_{xx})^2\,\mathrm{d}x\mathrm{d}\tau + \lambda_1\int_0^t\int_0^1 (\theta_{xxx})^2\,\mathrm{d}x\mathrm{d}\tau \\
&\leqslant C\varepsilon_3^2\int_0^t\int_0^1 \left[(\psi_{xx})^2 + (\psi_{xxx})^2\right]\mathrm{d}x\mathrm{d}\tau + \lambda_1\int_0^t\int_0^1 (\theta_{xxx})^2\,\mathrm{d}x\mathrm{d}\tau.
\end{aligned} \tag{3.43}$$

In order to estimate J_{10}, let's differentiate $(1.5)_1$ with respect to x twice

$$\alpha\psi_{xxxx} = \psi_{xxt} + (\sigma-\alpha)\psi_{xx} + \sigma\theta_{xxx}. \tag{3.44}$$

Substituting (3.44) into J_{10}, using (3.2) and Cauchy–Schwarz inequality, we have

$$\begin{aligned}
J_{10} &= \int_0^t\int_0^1 \psi_{xxxx}\theta\theta_{xxx}\mathrm{d}x\mathrm{d}\tau \\
&= \frac{1}{\alpha}\int_0^t\int_0^1 \theta\theta_{xxx}\left[\psi_{xxt} + (\sigma-\alpha)\psi_{xx} + \sigma\theta_{xxx}\right]\mathrm{d}x\mathrm{d}\tau \\
&\leqslant C\varepsilon_2\int_0^t\int_0^1 \left[(\theta_{xxx})^2 + (\psi_{xxt})^2 + (\psi_{xx})^2\right]\mathrm{d}x\mathrm{d}\tau.
\end{aligned} \tag{3.45}$$

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Substituting the estimates on J_7–J_{10} into (3.40), and using (3.4), (3.6), (3.8), we obtain

$$\begin{aligned} &\frac{1}{2}\int_0^1 (\theta_{xxx})^2\,\mathrm{d}x + \int_0^t\int_0^1 (\theta_{xxx})^2\,\mathrm{d}x\mathrm{d}\tau \\ &\quad \leqslant \frac{1}{2}\int_0^1 (\theta_{0xxx})^2\,\mathrm{d}x + C\left(\|\psi_0\|_3^2 + \|\theta_0\|_2^2\right), \end{aligned} \tag{3.46}$$

which implies (3.9).

This completes the proof of lemma 3.1. □

By the derived *a priori* estimates and the local existence of the solutions which can be proved by the slightly standard iteration method, we can get the global existence of the solutions to (1.1)–(1.3) and (1.5)–(1.7) by extending the local solution to the time $t = +\infty$. This completes the proof of theorem 1.3.

For readers' convenience, we give the proof of the local existence. For the shortness, we take the initial-boundary value problem (1.5)–(1.7) as an example to sketch the main idea of the proof. In fact, we construct the approximate solution sequences $(\psi_{n+1}^0,\, \theta_{n+1}^0)$, $n \geqslant 0$ by induction. Precisely, suppose that the n-th order approximate solution $(\psi_n^0,\, \theta_n^0)$, $n \geqslant 0$ is obtained for some time $0 < T_n \leqslant T$, we define $(\psi_{n+1}^0,\, \theta_{n+1}^0)$ by solving the following linear initial-boundary value problem, i.e. the iteration scheme

$$\begin{cases} (\psi_{n+1}^0)_t = -(\sigma-\alpha)\psi_{n+1}^0 - \sigma(\theta_n^0)_x + \alpha(\psi_{n+1}^0)_{xx}, \\ (\theta_{n+1}^0)_t = -\theta_{n+1}^0 + \psi_n^0(\theta_{n+1}^0)_x + \theta_n^0(\psi_{n+1}^0)_x, \quad 0 < x < 1, \quad t > 0, \\ (\psi_{n+1}^0, \theta_{n+1}^0)(x,0) = (\psi_0(x), \theta_0(x)), \quad 0 \leqslant x \leqslant 1, \\ \psi_{n+1}^0(0,t) = \psi_{n+1}^0(1,t) = 0, \quad t \geqslant 0. \end{cases}$$

Then, the existence of solutions to the above linearized problem is shown in a time interval $t \in [0,\, t_{n+1}]$ with $0 < T_{n+1} \leqslant T_n$. The rest is to derive the uniform-in-n estimates of $(\psi_{n+1}^0,\, \theta_{n+1}^0)$, which guarantee that the life-span T_{n+1} of the approximate solution $(\psi_{n+1}^0,\, \theta_{n+1}^0)$ has a strictly positive lower bound as n goes infinity. Finally, the local existence of the nonlinear problem (1.5)–(1.7) follows from the fixed point theorem.

4. Convergence rates and BL-thickness

In this section, we go back to use the symbol $(\psi^\beta,\, \theta^\beta)$ and $(\psi^0,\, \theta^0)$ to denote the solution to the initial-boundary value problems (1.1)–(1.3) and (1.5)–(1.7) respectively. Convergence rates of the vanishing diffusion viscosity and the BL-thickness will be obtained. That is, we will give the proof of theorem 1.5, and it suffices to show the following two lemmas.

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LEMMA 4.1 (Convergence rates). *Under the same assumptions of theorem 1.5, we have the following estimates:*

$$\int_0^1 \left[\left(\psi^\beta-\psi^0\right)^2+\left(\theta^\beta-\theta^0\right)^2\right]\mathrm{d}x + \int_0^t\int_0^1 \left[\left(\psi^\beta-\psi^0\right)^2+\left(\theta^\beta-\theta^0\right)^2\right]\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_0^1 \left(\psi^\beta-\psi^0\right)_x^2\mathrm{d}x\mathrm{d}\tau \leqslant C\beta \tag{4.1}$$

and

$$\int_0^1 \left(\psi^\beta-\psi^0\right)_x^2\mathrm{d}x + \int_0^t\int_0^1 \left(\psi^\beta-\psi^0\right)_t^2\mathrm{d}x\mathrm{d}\tau \leqslant C\beta^{1/2}, \tag{4.2}$$

where C is a positive constant, independent of β.

Proof. Set

$$u^\beta=\psi^\beta-\psi^0, \quad v^\beta=\theta^\beta-\theta^0. \tag{4.3}$$

Then we deduce from (1.1)–(1.3) and (1.5)–(1.7) that (u^β, v^β) satisfy the following initial-boundary value problem:

$$\begin{cases} u_t^\beta=-(\sigma-\alpha)u^\beta-\sigma v_x^\beta+\alpha u_{xx}^\beta, \quad 0<x<1,\ t>0, \\ v_t^\beta=-(1-\beta)v^\beta+\nu u_x^\beta+\psi^\beta v_x^\beta+u^\beta\theta_x^0+\beta\left(\theta^0+\theta_{xx}^\beta\right)+\nu\psi_x^0 \\ \qquad +\theta^\beta u_x^\beta+\psi_x^0 v^\beta,\ 0<x<1,\ t>0 \end{cases} \tag{4.4}$$

with initial data

$$\left(u^\beta, v^\beta\right)(x,0)=(0,0), \quad 0\leqslant x\leqslant 1 \tag{4.5}$$

and the boundary conditions

$$u^\beta(0,t)=u^\beta(1,t)=0, \quad t\geqslant 0, \tag{4.6}$$

which implies

$$u_t^\beta(0,t)=u_t^\beta(1,t)=0, \quad t\geqslant 0. \tag{4.7}$$

Part I. The proof of (4.1).

Integrating equation $(4.4)_1 \times u^\beta + (4.4)_2 \times v^\beta$ over $(0,\, t) \times (0,\, 1)$, and using the boundary conditions (1.3) and (4.5), (4.6) we arrive at

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left[\left(u^\beta\right)^2 + \left(v^\beta\right)^2\right] \mathrm{d}x \\
&\quad + \int_0^t\int_0^1 \left[(\sigma-\alpha)\left(u^\beta\right)^2 + (1-\beta)\left(v^\beta\right)^2 + \alpha\left(u^\beta_x\right)^2\right] \mathrm{d}x\mathrm{d}\tau \\
&= (\sigma+\nu)\int_0^t\int_0^1 u^\beta_x v^\beta \mathrm{d}x\mathrm{d}\tau - \frac{1}{2}\int_0^t\int_0^1 \psi^\beta_x \left(v^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau \\
&\quad + \int_0^t\int_0^1 u^\beta v^\beta \theta^0_x \mathrm{d}x\mathrm{d}\tau + \int_0^t\int_0^1 \left(\beta\theta^0 + \beta\theta^\beta_{xx} + \nu\psi^0_x\right) v^\beta \mathrm{d}x\mathrm{d}\tau \\
&\quad + \int_0^t\int_0^1 \theta^\beta u^\beta_x v^\beta \mathrm{d}x\mathrm{d}\tau + \int_0^t\int_0^1 \psi^0_x (v^\beta)^2 \mathrm{d}x\mathrm{d}\tau \\
&= \sum_{i=1}^6 K_i.
\end{aligned} \tag{4.8}$$

We can estimate K_1–K_6 term by term as follows by using Cauchy–Schwarz inequality, Sobolev inequality and the *a priori* assumptions (4.49), (3.24) and (3.25):

$$K_1 \leqslant \lambda\int_0^t\int_0^1 \left(u^\beta_x\right)^2 \mathrm{d}x\mathrm{d}\tau + \frac{(\sigma+\nu)^2}{4\lambda}\int_0^t\int_0^1 \left(v^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau, \tag{4.9}$$

$$\begin{aligned}
K_2 &\leqslant \lambda_1\int_0^t\int_0^1 \left(v^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau + \frac{1}{16\lambda_1}\int_0^t\int_0^1 \left(\psi^\beta_x v^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau \\
&\leqslant \lambda_1\int_0^t\int_0^1 \left(v^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau \\
&\qquad + C\int_0^t \left(\left\|\psi^\beta_x(\tau)\right\|^2 + \left\|\psi^\beta_{xx}(\tau)\right\|^2\right)\int_0^1 \left(v^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau,
\end{aligned} \tag{4.10}$$

$$K_3 \leqslant C\varepsilon_3\int_0^t\int_0^1 \left(u^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau + C\varepsilon_3\int_0^t\int_0^1 \left(v^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau, \tag{4.11}$$

$$\begin{aligned}
K_4 &\leqslant C\beta^2\int_0^t\int_0^1 \left[\left(\theta^0\right)^2 + \left(\theta^\beta_{xx}\right)^2\right] \mathrm{d}x\mathrm{d}\tau + 2\lambda_1\int_0^t\int_0^1 \left(v^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau \\
&\quad + C\nu^2\int_0^t\int_0^1 \left(\psi^0_x\right)^2 \mathrm{d}x\mathrm{d}\tau \\
&\leqslant C\beta + 2\lambda_1\int_0^t\int_0^1 \left(v^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau,
\end{aligned} \tag{4.12}$$

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$$K_5 \leqslant C\varepsilon_1 \int_0^t \int_0^1 \left[\left(u_x^\beta\right)^2 + \left(v^\beta\right)^2\right] \mathrm{d}x\mathrm{d}\tau \tag{4.13}$$

and

$$K_6 \leqslant C\varepsilon_3 \int_0^t \int_0^1 \left(v^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau. \tag{4.14}$$

Plugging (4.9)–(4.14) into (4.8), and using lemmas 2.1, 3.1 and the smallness of ε_1, ε_3 and λ_1, we obtain

$$\begin{aligned}
&\int_0^1 \left[\left(u^\beta\right)^2 + \left(v^\beta\right)^2\right] \mathrm{d}x + \int_0^t \int_0^1 \left[\left(u^\beta\right)^2 + \left(v^\beta\right)^2 + \left(u_x^\beta\right)^2\right] \mathrm{d}x\mathrm{d}\tau \\
&\quad \leqslant C\beta + C\int_0^t \left(\left\|\psi_x^\beta(\tau)\right\|^2 + \left\|\psi_{xx}^\beta(\tau)\right\|^2\right) \int_0^1 \left(v^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau.
\end{aligned} \tag{4.15}$$

Therefore, Gronwall's inequality and lemma 2.2 yield (4.1).

Part II. The proof of (4.2).

Integrating equation $(4.4)_1 \times u_t^\beta$ over $(0, t) \times (0, 1)$, using integration-by-parts and the boundary conditions (4.7), we arrive at

$$\begin{aligned}
&\frac{1}{2}\int_0^1 \left[(\sigma - \alpha)\left(u^\beta\right)^2 + \alpha\left(u_x^\beta\right)^2\right] \mathrm{d}x + \int_0^t \int_0^1 \left(u_t^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau \\
&\quad = \sigma \int_0^t \int_0^1 u_{xt}^\beta v^\beta \mathrm{d}x\mathrm{d}\tau \\
&\quad \leqslant C\left(\int_0^t \int_0^1 \left(u_{xt}^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau\right)^{1/2} \left(\int_0^t \int_0^1 \left(v^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau\right)^{1/2}.
\end{aligned} \tag{4.16}$$

From (2.13), (3.5) and (4.3), one has

$$\int_0^t \int_0^1 \left(u_{xt}^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau \leqslant C \left\|(\psi_0, \theta_0)\right\|_2^2, \tag{4.17}$$

which together with (4.1) implies (4.2). This completes the proof of lemma 4.1. □

The following lemma will greatly contribute to the boundary layer thickness.

LEMMA 4.2. *Under the same assumptions of theorem 1.5, we have the following estimates*

$$\int_\delta^{1-\delta} \left|\left(\theta^\beta - \theta^0\right)_x\right| \mathrm{d}x \leqslant C\delta^{-1/2}\beta^{1/4}, \tag{4.18}$$

where C is a positive constant independent of β.

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Proof. Differentiating $(4.4)_2$, we have

$$\begin{aligned} v^\beta_{xt} &= -(1-\beta)v^\beta_x + \left(\psi^\beta v^\beta_x\right)_x + \left(u^\beta\theta^0_x\right)_x \\ &\quad + \beta v^\beta_{xxx} + \left(\theta^\beta u^\beta_x\right)_x + \left(\psi^0_x v^\beta\right)_x \\ &\quad + \beta\left(\theta^0_x + \theta^0_{xxx}\right) + \nu\psi^\beta_{xx}. \end{aligned} \tag{4.19}$$

Denote $z = v^\beta_x$, then we deduce that

$$\begin{aligned} z_t &= -(1-\beta)z + \left(\psi^\beta z\right)_x + \left(u^\beta\theta^0_x\right)_x + \beta z_{xx} \\ &\quad + (\theta^\beta u^\beta_x)_x + (\psi^0_x v^\beta)_x + \beta\left(\theta^0_x + \theta^0_{xxx}\right) + \nu\psi^\beta_{xx}. \end{aligned} \tag{4.20}$$

As in [6, 12], introduce the functions $\phi_\varepsilon(z) = \sqrt{z^2+\varepsilon^2}$ and

$$\xi_\delta(x) = \begin{cases} x, & 0 \leqslant x \leqslant \delta, \\ \delta, & \delta < x < 1-\delta, \\ 1-x, & 1-\delta \leqslant x \leqslant 1. \end{cases} \tag{4.21}$$

Notice $\phi_\varepsilon(z)$ and $\xi_\delta(x)$ respectively satisfy the following properties

$$\begin{cases} \text{(i)} & \phi_\varepsilon(z) \geqslant 0, \quad \lim\limits_{\varepsilon\to 0}\phi_\varepsilon(z) = |z|, \\ \text{(ii)} & |\phi'_\varepsilon(z)| \leqslant 1, \quad \lim\limits_{\varepsilon\to 0}\phi'_\varepsilon(z)z = |z|, \\ \text{(iii)} & \phi''_\varepsilon(z) \geqslant 0, \quad \phi''_\varepsilon(z)z^2 \leqslant \varepsilon \end{cases} \tag{4.22}$$

and

$$\begin{cases} 0 \leqslant \xi_\delta(x) \leqslant \delta \leqslant 1, \quad 0 \leqslant |\xi'_\delta(x)| \leqslant 1, \xi_\delta(0) = \xi_\delta(1) = 0, \\ |\xi_\delta(x)\phi'_\varepsilon(z)| \leqslant 1, \quad |\xi'_\delta(x)\phi'_\varepsilon(z)| \leqslant 1, \ |\xi'_\delta(x)| \leqslant \delta. \end{cases} \tag{4.23}$$

Integrating equation (4.20) $\times\, \xi_\delta(x)\phi'_\varepsilon(z)$ over $(0,\, t) \times (0,\, 1)$, we obtain

$$\begin{aligned} &\int_0^t\int_0^1 \xi_\delta(x)\phi'_\varepsilon(z)z_t\mathrm{d}x\mathrm{d}\tau \\ &= -(1-\beta)\int_0^t\int_0^1 \xi_\delta(x)\phi'_\varepsilon(z)z\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_0^1 \xi_\delta(x)\phi'_\varepsilon(z)\left(\psi^\beta z\right)_x\mathrm{d}x\mathrm{d}\tau \\ &\quad + \int_0^t\int_0^1 \xi_\delta(x)\phi'_\varepsilon(z)\left(u^\beta\theta^0_x\right)_x\mathrm{d}x\mathrm{d}\tau + \beta\int_0^t\int_0^1 \xi_\delta(x)\phi'_\varepsilon(z)z_{xx}\mathrm{d}x\mathrm{d}\tau \\ &\quad + \beta\int_0^t\int_0^1 \xi_\delta(x)\phi'_\varepsilon(z)\left(\theta^0_x + \theta^0_{xxx}\right)\mathrm{d}x\mathrm{d}\tau \\ &\quad + \int_0^t\int_0^1 \xi_\delta(x)\phi'_\varepsilon(z)(\theta^\beta u^\beta_x)_x\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_0^1 \xi_\delta(x)\phi'_\varepsilon(z)(\psi^0_x v^\beta)_x\mathrm{d}x\mathrm{d}\tau \\ &\quad + \nu\int_0^t\int_0^1 \xi_\delta(x)\phi'_\varepsilon(z)\psi^\beta_{xx}\mathrm{d}x\mathrm{d}\tau \\ &= \sum_{i=7}^{14} K_i. \end{aligned} \tag{4.24}$$

Next, we estimate each term in (4.24) one by one.

First, using the initial data (4.5) and (4.22)(ii), we get

$$\int_0^t \int_0^1 \xi_\delta(x)\phi'_\varepsilon(z)z_t \mathrm{d}x\mathrm{d}\tau = \int_0^1 \xi_\delta(x)\phi_\varepsilon(z)\mathrm{d}x - \int_0^1 \varepsilon\xi_\delta(x)\mathrm{d}x \tag{4.25}$$

and

$$\lim_{\varepsilon\to 0} K_7 \leqslant C\int_0^t \int_0^1 \xi_\delta(x)|z|\mathrm{d}x\mathrm{d}\tau. \tag{4.26}$$

Next, integrating by parts, we have

$$\begin{aligned} K_8 &= -\int_0^t \int_0^1 \xi'_\delta(x)\phi'_\varepsilon(z)\psi^\beta z\mathrm{d}x\mathrm{d}\tau - \int_0^t \int_0^1 \xi_\delta(x)\phi''_\varepsilon(z)\psi^\beta z z_x\mathrm{d}x\mathrm{d}\tau \\ &= K_8^1 + K_8^2 \end{aligned} \tag{4.27}$$

and

$$\begin{aligned} K_{10} &= -\beta\int_0^t \int_0^1 \xi'_\delta(x)\phi'_\varepsilon(z)z_x\mathrm{d}x\mathrm{d}\tau - \beta\int_0^t \int_0^1 \xi_\delta(x)\phi''_\varepsilon(z)z_x^2\mathrm{d}x\mathrm{d}\tau \\ &= K_{10}^1 + K_{10}^2. \end{aligned} \tag{4.28}$$

Using the property of $\xi_\delta(x)$, we can rewrite K_8^1 as

$$K_8^1 = -\int_0^t \int_0^\delta \phi'_\varepsilon(z)\psi^\beta z\mathrm{d}x\mathrm{d}\tau + \int_0^t \int_{1-\delta}^1 \phi'_\varepsilon(z)\psi^\beta z\mathrm{d}x\mathrm{d}\tau. \tag{4.29}$$

In addition, it is easy to see that

$$\begin{cases} \left|\psi^\beta(x,t)\right| \leqslant \int_0^x \left|\psi_y^\beta(y,t)\right|\mathrm{d}y \leqslant Cx \leqslant C\xi_\delta(x) \quad \text{for any } x\in[0,\delta], \\ \left|\psi^\beta(x,t)\right| \leqslant \int_x^1 \left|\psi_y^\beta(y,t)\right|\mathrm{d}y \\ \qquad\qquad\quad \leqslant C(1-x) \leqslant C\xi_\delta(x) \text{ for any } x\in[1-\delta,1]. \end{cases} \tag{4.30}$$

From (4.29), (4.30) and (4.22), we obtain

$$\begin{aligned} K_8^1 &\leqslant C\int_0^t \int_0^\delta \xi_\delta(x)\left|\phi'_\varepsilon(z)z\right|\mathrm{d}x\mathrm{d}\tau + C\int_0^t \int_{1-\delta}^1 \xi_\delta(x)\left|\phi'_\varepsilon(z)z\right|\mathrm{d}x\mathrm{d}\tau \\ &\leqslant C\int_0^t \int_0^1 \xi_\delta(x)\left|\phi'_\varepsilon(z)z\right|\mathrm{d}x\mathrm{d}\tau \end{aligned} \tag{4.31}$$

and

$$\lim_{\varepsilon\to 0} K_8^1 \leqslant C\int_0^t \int_0^1 \xi_\delta(x)|z|\mathrm{d}x\mathrm{d}\tau. \tag{4.32}$$

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By Hölder inequality, it follows from (4.21), (4.22), lemmas 2.1 and 3.1

$$\begin{aligned} K_{10}^1 &= -\beta \int_0^t \int_0^\delta \phi_\varepsilon'(z) z_x \mathrm{d}x\mathrm{d}\tau + \beta \int_0^t \int_{1-\delta}^1 \phi_\varepsilon'(z) z_x \mathrm{d}x\mathrm{d}\tau \\ &\leqslant C\beta \int_0^t \int_0^\delta |z_x| \,\mathrm{d}x\mathrm{d}\tau + C\beta \int_0^t \int_{1-\delta}^1 |z_x| \,\mathrm{d}x\mathrm{d}\tau \\ &\leqslant C\beta\delta^{1/2} \left(\int_0^t \int_0^\delta \left|\theta_{xx}^\beta - \theta_{xx}^0\right|^2 \mathrm{d}x\mathrm{d}\tau \right)^{1/2} \\ &\quad + C\beta\delta^{1/2} \left(\int_0^t \int_{1-\delta}^1 \left|\theta_{xx}^\beta - \theta_{xx}^0\right|^2 \mathrm{d}x\mathrm{d}\tau \right)^{1/2} \\ &\leqslant C\beta^{1/2}\delta^{1/2}. \end{aligned} \tag{4.33}$$

By the key property (4.22)(iii) of $\phi_\varepsilon(z)$ and Cauchy inequality, K_8^2 and K_{10}^2 can be bounded as follows:

$$\begin{aligned} &K_8^2 + K_{10}^2 \\ &\leqslant \frac{\beta}{2} \int_0^t \int_0^1 \xi_\delta(x) \phi_\varepsilon''(z) z_x^2 \mathrm{d}x\mathrm{d}\tau + \frac{1}{2\beta} \int_0^t \int_0^1 \xi_\delta(x) \phi_\varepsilon''(z) (\psi^\beta z)^2 \mathrm{d}x\mathrm{d}\tau \\ &\quad - \beta \int_0^t \int_0^1 \xi_\delta(x) \phi_\varepsilon''(z) z_x^2 \mathrm{d}x\mathrm{d}\tau \\ &= -\frac{\beta}{2} \int_0^t \int_0^1 \xi_\delta(x) \phi_\varepsilon''(z) z_x^2 \mathrm{d}x\mathrm{d}\tau + \frac{1}{2\beta} \int_0^t \int_0^1 \xi_\delta(x) \phi_\varepsilon''(z) (\psi^\beta z)^2 \mathrm{d}x\mathrm{d}\tau \\ &\leqslant \frac{1}{2\beta} \int_0^t \int_0^1 \xi_\delta(x) \phi_\varepsilon''(z) z^2 (\psi^\beta)^2 \mathrm{d}x\mathrm{d}\tau \to 0, \text{ as } \varepsilon \to 0. \end{aligned} \tag{4.34}$$

Direct calculations with lemmas 2.1, 3.1, (4.22) and (4.23) show

$$\begin{aligned} K_9 &= \int_0^t \int_0^1 \xi_\delta(x) \phi_\varepsilon'(z) u_x^\beta \theta_x^0 \mathrm{d}x\mathrm{d}\tau + \int_0^t \int_0^1 \xi_\delta(x) \phi_\varepsilon'(z) u^\beta \theta_{xx}^0 \mathrm{d}x\mathrm{d}\tau \\ &\leqslant C\delta \left(\int_0^t \int_0^1 \left(u_x^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau \right)^{1/2} \left(\int_0^t \int_0^1 \left(\theta_x^0\right)^2 \mathrm{d}x\mathrm{d}\tau \right)^{1/2} \\ &\quad + C\delta \left(\int_0^t \int_0^1 \left(u^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau \right)^{1/2} \left(\int_0^t \int_0^1 \left(\theta_{xx}^0\right)^2 \mathrm{d}x\mathrm{d}\tau \right)^{1/2} \\ &\leqslant C\delta\beta^{1/2}. \end{aligned} \tag{4.35}$$

Using Hölder inequality, (4.22), (4.23) and lemmas 2.1, 3.1, we have

$$K_{11} \leqslant C\delta\beta \left\{ \int_0^t \int_0^1 \left[\left(\theta_x^0\right)^2 + \left(\theta_{xxx}^0\right)^2 \right] \mathrm{d}x\mathrm{d}\tau \right\}^{1/2} \leqslant C\delta\beta. \tag{4.36}$$

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Integrating by parts and using (4.23), we have

$$\begin{aligned} K_{12} &= \int_0^t \int_0^1 \xi_\delta(x)\phi'_\varepsilon(z)(\theta^\beta u_x^\beta)_x \mathrm{d}x\mathrm{d}\tau \\ &= \int_0^t \int_0^1 \phi'_\varepsilon(z)\xi_\delta(x)\theta_x^\beta u_x^\beta \mathrm{d}x\mathrm{d}\tau + \int_0^t \int_0^1 \phi'_\varepsilon(z)\xi_\delta(x)\theta^\beta u_{xx}^\beta \mathrm{d}x\mathrm{d}\tau \\ &= K_{12}^1 + K_{12}^2. \end{aligned} \tag{4.37}$$

Using (4.22), (4.23) and Hölder inequality, (2.3), (4.1) we get

$$\begin{aligned} K_{12}^1 &\leqslant \int_0^t \int_0^1 |\phi'_\varepsilon(z)\xi_\delta(x)\theta_x^\beta u_x^\beta| \mathrm{d}x\mathrm{d}\tau \\ &\leqslant C\delta \int_0^t \int_0^1 |\theta_x^\beta u_x^\beta| \mathrm{d}x\mathrm{d}\tau \\ &\leqslant C\delta \left(\int_0^t \int_0^1 (\theta_x^\beta)^2 \mathrm{d}x\mathrm{d}\tau \right)^{1/2} \left(\int_0^t \int_0^1 (u_x^\beta)^2 \mathrm{d}x\mathrm{d}\tau \right)^{1/2} \\ &\leqslant C\delta\beta^{1/2}. \end{aligned} \tag{4.38}$$

From $(4.4)_1$, we have

$$\begin{aligned} K_{12}^2 &= \frac{1}{\alpha} \int_0^t \int_0^1 \phi'_\varepsilon(z)\xi_\delta(x)\theta^\beta \left[u_t^\beta + (\sigma - \alpha)u^\beta + \sigma v_x^\beta \right] \mathrm{d}x\mathrm{d}\tau \\ &= \frac{1}{\alpha} \int_0^t \int_0^1 \phi'_\varepsilon(z)\xi_\delta(x)\theta^\beta u_t^\beta \mathrm{d}x\mathrm{d}\tau + \frac{\sigma - \alpha}{\alpha} \int_0^t \int_0^1 \phi'_\varepsilon(z)\xi_\delta(x)\theta^\beta u^\beta \mathrm{d}x\mathrm{d}\tau \\ &\quad + \frac{\sigma}{\alpha} \int_0^t \int_0^1 \phi'_\varepsilon(z)\xi_\delta(x)\theta^\beta v_x^\beta \mathrm{d}x\mathrm{d}\tau \\ &= K_{12}^{2,1} + K_{12}^{2,2} + K_{12}^{2,3}. \end{aligned} \tag{4.39}$$

Using (4.22), (4.23), Hölder inequality, (2.3) and (4.2), we have

$$\begin{aligned} K_{12}^{2,1} &\leqslant C\delta \int_0^t \int_0^1 \left| \theta^\beta u_t^\beta \right| \mathrm{d}x\mathrm{d}\tau \\ &\leqslant C\delta \left(\int_0^t \int_0^1 \left(\theta^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau \right)^{1/2} \left(\int_0^t \int_0^1 \left(u_t^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau \right)^{1/2} \\ &\leqslant C\delta\beta^{1/4}. \end{aligned} \tag{4.40}$$

Using (4.22), (4.23), Hölder inequality, (2.3) and (4.1), we have

$$\begin{aligned} K_{12}^{2,2} &\leqslant C\delta \int_0^t \int_0^1 \left| \theta^\beta u^\beta \right| \mathrm{d}x\mathrm{d}\tau \\ &\leqslant C\delta \left(\int_0^t \int_0^1 \left(\theta^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau \right)^{1/2} \left(\int_0^t \int_0^1 \left(u^\beta\right)^2 \mathrm{d}x\mathrm{d}\tau \right)^{1/2} \\ &\leqslant C\delta\beta^{1/2}. \end{aligned} \tag{4.41}$$

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From (4.22), we have

$$\begin{aligned} K_{12}^{2,3} &\leqslant C\int_0^t \left\|\theta^\beta(\tau)\right\|_{L^\infty}\int_0^1 \left|\xi_\delta(x)v_x^\beta\right|\mathrm{d}x\mathrm{d}\tau \\ &\leqslant C\int_0^t\int_0^1 \xi_\delta(x)|z|\mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{4.42}$$

Substituting (4.40)–(4.42) into (4.39), we get

$$K_{12}^2 \leqslant C\int_0^t\int_0^1 \xi_\delta(x)|z|\mathrm{d}x\mathrm{d}\tau + C\delta^{1/2}\beta^{1/4}, \tag{4.43}$$

which together with (4.37), (4.38) yields

$$K_{12} \leqslant C\int_0^t\int_0^1 \xi_\delta(x)|z|\mathrm{d}x\mathrm{d}\tau + C\delta^{1/2}\beta^{1/4}. \tag{4.44}$$

Direct calculation show

$$\begin{aligned} K_{13} &= \int_0^t\int_0^1 \xi_\delta(x)\phi_\varepsilon'(z)\psi_{xx}^0 v^\beta\mathrm{d}x\mathrm{d}\tau + \int_0^t\int_0^1 \xi_\delta(x)\phi_\varepsilon'(z)\psi_x^0 v_x^\beta\mathrm{d}x\mathrm{d}\tau \\ &= K_{13}^1 + K_{13}^2. \end{aligned} \tag{4.45}$$

From (4.22), (4.23), Hölder inequality, (4.1) and (3.4), we have

$$\begin{aligned} K_{13}^1 &\leqslant C\delta\int_0^t\int_0^1 \left|\psi_{xx}^0 v^\beta\right|\mathrm{d}x\mathrm{d}\tau \\ &\leqslant C\delta\left(\int_0^t\int_0^1 \left(v^\beta\right)^2\mathrm{d}x\mathrm{d}\tau\right)^{1/2}\left(\int_0^t\int_0^1 \left(\psi_{xx}^0\right)^2\mathrm{d}x\mathrm{d}\tau\right)^{1/2} \\ &\leqslant C\delta\beta^{1/2}. \end{aligned} \tag{4.46}$$

From (4.22), we have

$$\begin{aligned} K_{13}^2 &\leqslant C\int_0^t \left\|\psi_x^0(\tau)\right\|_{L^\infty}\int_0^1 \left|\xi_\delta(x)v_x^\beta\right|\mathrm{d}x\mathrm{d}\tau \\ &\leqslant C\int_0^t\int_0^1 \xi_\delta(x)|z|\mathrm{d}x\mathrm{d}\tau. \end{aligned} \tag{4.47}$$

which together with (4.45), (4.46) yields

$$K_{13} \leqslant C\int_0^t\int_0^1 \xi_\delta(x)|z|\mathrm{d}x\mathrm{d}\tau + C\delta^{1/2}\beta^{1/2}. \tag{4.48}$$

Using $\nu = o(\beta^{1/2})$, we have

$$K_{14} \leqslant C\nu\delta\int_0^t\int_0^1 \left(\psi_{xx}^\beta\right)^2\mathrm{d}x\mathrm{d}\tau \leqslant C\beta^{1/2}\delta. \tag{4.49}$$

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Collecting all estimates on K_7–K_{14}, and letting $\varepsilon \to 0$ in (4.24), we get

$$\int_0^1 \xi_\delta(x)|z|\mathrm{d}x \leqslant C\int_0^t\int_0^1 \xi_\delta(x)|z|\mathrm{d}x\mathrm{d}\tau + C\delta^{1/2}\beta^{1/4}. \tag{4.50}$$

By Gronwall's inequality, we obtain

$$\int_0^1 \xi_\delta(x)|z|\mathrm{d}x \leqslant C\delta^{1/2}\beta^{1/4}, \tag{4.51}$$

which imply (4.18). This completes the proof of lemma 4.2. □

Finally, based on lemmas 4.1–4.2, we can prove theorem 1.5.

Proof of theorem 1.5. First, using Hölder inequality, we have from lemma 4.1

$$\int_0^1 \left|\left(\theta^\beta - \theta^0\right)\right| \mathrm{d}x \leqslant C\left[\int_0^1 \left(\theta^\beta - \theta^0\right)^2 \mathrm{d}x\right]^{1/2} \leqslant C\beta^{1/2}. \tag{4.52}$$

Since $W^{1,1}([\delta,\, 1-\delta]) \hookrightarrow L^\infty([\delta,\, 1-\delta])$, we have from (4.52) and lemma 4.2

$$\begin{aligned} \left\|\left(\theta^\beta - \theta^0\right)(t)\right\|_{L^\infty([\delta,1-\delta])} &\leqslant \int_\delta^{1-\delta} \left|\left(\theta^\beta - \theta^0\right)\right| \mathrm{d}x + \int_\delta^{1-\delta} \left|\left(\theta^\beta - \theta^0\right)_x\right| \mathrm{d}x \\ &\leqslant C\beta^{1/2} + C\delta^{-1/2}\beta^{1/4} \\ &\leqslant C\delta^{-1/2}\beta^{1/4} \\ &= C(\beta^{1/2}/\delta(\beta))^{1/2} \to 0 \ \text{ as } \beta \to 0. \end{aligned} \tag{4.53}$$

(4.53) imply inequality (1.11).

In addition, using Sobolev inequality, we also have from (4.1)

$$\begin{aligned} \left\|\left(\psi^\beta - \psi^0\right)(t)\right\|_{L^\infty([0,1])} &\leqslant C\left\|\left(\psi^\beta - \psi^0\right)(t)\right\|_{L^2[0,1]}^{1/2} \left\|\left(\psi^\beta - \psi^0\right)_x(t)\right\|_{L^2[0,1]}^{1/2} \\ &\leqslant C\beta^{3/8} \to 0 \text{ as } \beta \to 0. \end{aligned} \tag{4.54}$$

(4.54) imply inequality (1.10). As in [**6**, **7**, **12**], we observe the inequality (1.12) holds. This completes the proof of theorem 1.5. □

5. Conclusion

In summary, three results are obtained in this paper:

- The global unique solvability of the initial-boundary value problem (1.1)–(1.3) of Hsieh's equation with conservative nonlinearity is established in the Sobolev framework presented in theorem 1.3.
- The global unique solvability and more regularities of the corresponding formal limit problem (1.5)–(1.7) is established in the Sobolev framework presented in theorem 1.4.

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- Convergence rates and the BL-thickness as the diffusion parameter $\beta \to 0^+$ are obtained and this result is stated in theorem 1.5.

We emphasize that the conservative nonlinearity is stronger than the nonconservative nonlinearity. Thus more regularities on the solution to the limit problem presented in part (iii) of theorem 1.4 are required so that the convergence rates and boundary layer thickness are obtained. However, generally speaking, it is more difficult for initial-boundary problem due to the lack of boundary conditions on higher-order derivatives. Thus it is more complicated than the case of nonconservative form.

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