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Generic properties of periodic reflecting rays

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Abstract It is shown that for generic domains D in $\mathbb{R}^n$, $n \geq 2$, every periodic billiard trajectory in D passes only once through each of its reflection points, and any two different periodic billiard trajectories in D have no common reflection point

1 *Introduction*

Given a compact domain D in $\mathbb{R}^n$, $n \geq 2$, with smooth boundary $X = \partial D$, we consider periodic billiard trajectories in D, and call them periodic reflecting rays on X Generic properties of smooth compact $(n-1)$-dimensional submanifolds X of $\mathbb{R}^n$ concerning periodic reflecting rays on them were established by Lazutkin [**5**] and Petkov and Stojanov [**9**], [**10**], [**11**] (see also [**7**] and [**8**]) In [**5**] Lazutkin proved an analogue of Kupka-Smale theorem for billiards in strictly convex domains in $\mathbb{R}^2$ Studying inverse spectral problems in connection with the so-called Poisson relation for manifolds with boundary (*cf* [**2**], [**4**], [**6**]), V Petkov and the present author proved that generically most X in $\mathbb{R}^n$ have the following properties

(i) every two different periodic reflecting rays on X have rationally independent lengths ([**9**]),

(ii) (for $n = 2$) there do not exist generalized periodic geodesics on X ([**9**], [**10**]),

(iii) for every $s \geq 2$ there are at most a finite number of periodic reflecting rays on X having exactly s reflection points ([**11**]),

(iv) there do not exist periodic reflecting rays on X having segments tangent to X ([**10**], [**11**]),

(v) the spectrum of the Poincaré map related to every periodic reflecting ray on X does not contain roots of unity ([**11**])

In [**9**], [**10**] we had to overcome among others the following two difficulties

(A) in general a periodic reflecting ray could pass two or more times (in different directions) through some of its reflection points (see figure 1),

(B) some different periodic reflecting rays could have common reflection points (see figure 2)

In this paper we prove that for generic X in $\mathbb{R}^n$ the phenomena (A) and (B) cannot occur The latter means that for generic X given $x \in X$, there exist at most two directions $v \neq 0$ in $\mathbb{R}^n$ which are symmetric with respect to the normal N_x to X at x (and of course may coincide with N_x) such that starting from x in the direction v and reflecting on X satisfying the usual law of reflection, we get a periodic reflecting ray on X Note that, according to (iii), for generic X there are at most a countable number of such $x \in X$ for which there exist directions v with the above properties

The proofs of our results use the technique of [**9**], [**10**] which is based on the multi-jet transversality theorem One of these results (see theorem A in § 2) has been already used to prove the main theorem in [**11**], and both theorems A and B established below could be used to simplify the proofs in [**9**], [**10**] and to get other properties of generic X in $\mathbb{R}^n$

Thanks are due to the referee for his remarks and suggestions Especially, he pointed out that theorem B (the proof of which in the first version of the paper was different from the present one) can be established using the same idea as those in the proof of theorem A

2 *Definitions and main results*

Let X be a smooth compact $(n-1)$-dimensional submanifold of $\mathbb{R}^n$, $n \geq 2$, and $C^\infty(X, \mathbb{R}^n)$ be the set of all smooth maps $X \to \mathbb{R}^n$ endowed with the *Whitney C^∞ topology* (cf ch II of [**1**] or 2 1 of [**3**]) Denote by $C^\infty_{\mathrm{emb}}(X, \mathbb{R}^n)$ the subspace of $C^\infty(X, \mathbb{R}^n)$ consisting of all smooth immersive embeddings $X \to \mathbb{R}^n$ As is well known, $C^\infty(X, \mathbb{R}^n)$ is a Baire space and $C^\infty_{\mathrm{emb}}(X, \mathbb{R}^n)$ is open in it (cf loc cit), therefore the latter is also a Baire space

By a *periodic reflecting ray* on X we mean a closed curve γ formed by a finite number of straightline segments $l_1, l_2, \quad, l_k$, where $l_i = [x_i, x_{i+1}] = \{z \in \mathbb{R}^n \quad z = px_i + (1-p)x_{i+1}, 0 \leq p \leq 1\}$, $x_i \in X$ $(i = 1, 2, \quad, k)$, $x_{k+1} = x_1$, such that any open segment $\mathring{l}_i = (x_i, x_{i+1})$ does not intersect transversally X, and for every $i = 1, \quad, k$, l_i and l_{i+1} make equal acute angles with one of the normal vectors N_{i+1} to X at x_{i+1} (for convenience we set $l_{k+1} = l_1$), and l_i, l_{i+1} and N_{i+1} lie in a common two-dimensional plane The points $x_1, x_2, \quad x_k$ are called *reflection points* of γ

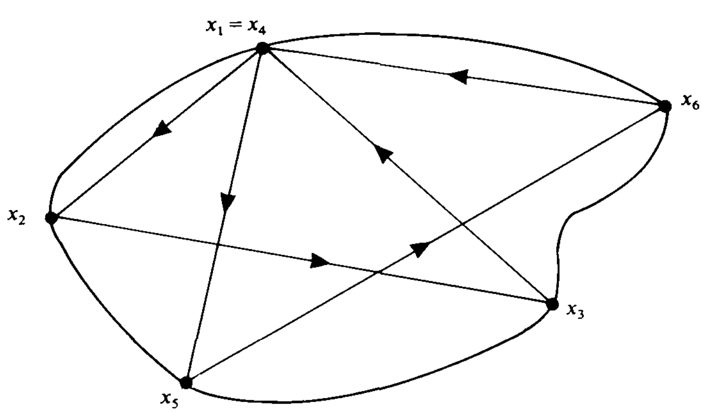

FIGURE 1

It may happen that some segments l_i of γ are tangent to X at some interior point z of l_i (see figure 3) Note especially that points like z_1 on figure 3 are not considered as reflection ones, while z_2 is a reflection point Clearly, a periodic reflecting ray on

X may lie 'inside X' as well as 'outside X' (see figure 4). Moreover, some of its reflection points may coincide (see figure 1). We mention also that X is not assumed to be connected.

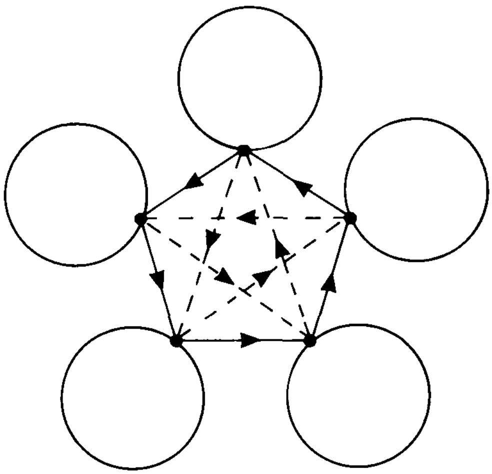

FIGURE 2

If there is a segment $l_i = [x_i, x_{i+1}]$ of γ which is orthogonal to X at x_i or x_{i+1}, then γ is called a *symmetric ray* ([**5**]). Otherwise, γ will be called a *non-symmetric* ray. In the first case either $k = 2$ or $k > 2$ and exactly two segments of γ are orthogonal to X at some of their end points (see figure 4).

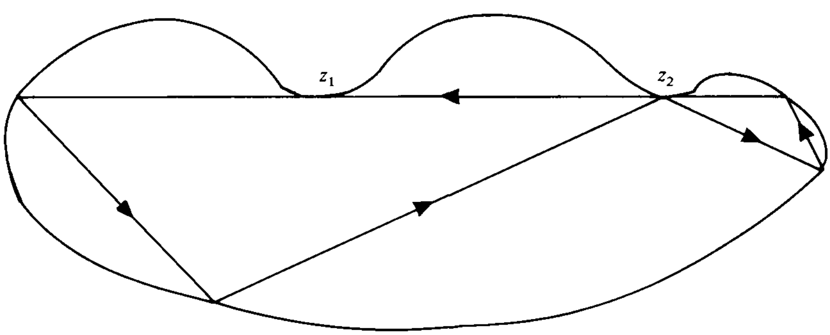

FIGURE 3

Given a periodic reflecting ray γ denote by m the number of all different segments of γ (thus for symmetric γ we should take only a half of the segments of γ), and let s be the number of all different reflection points of γ. The number $d(\gamma)$, defined by $d(\gamma) = m - s$ for non-symmetric γ and by $d(\gamma) = m + 1 - s$ for symmetric γ, will be called the *defect* of γ.

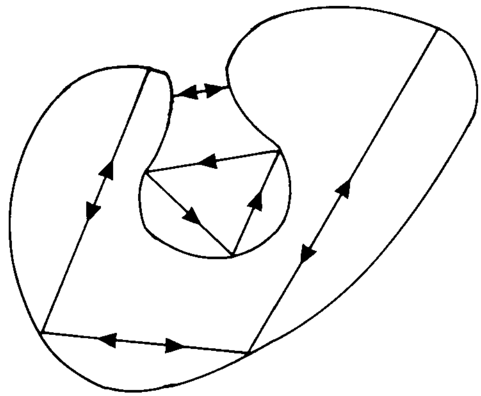

FIGURE 4

We are ready to state our results

THEOREM A *Let X be a smooth compact $(n-1)$-dimensional submanifold of $\mathbb{R}^n$, $n \geq 2$, and $\mathcal{A}$ be the set of those $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ such that every periodic reflecting ray on $f(X)$ has zero defect Then $\mathcal{A}$ contains a residual subset of $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$*

Using different methods Lazutkin [5] proved that for generic strictly convex X in $\mathbb{R}^2$ the inequality $d(\gamma) \leq 2$ is fulfilled for every periodic reflecting ray γ on X

THEOREM B *Let X be as in theorem A, and $\mathcal{B}$ be the set of those $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ such that any two different periodic reflecting rays on $f(X)$ have no common reflection point Then $\mathcal{B}$ contains a residual subset of $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$*

Roughly speaking the idea of the proof of theorem A is the following Let γ be a periodic reflecting ray on X with s different reflection points $x_1, x_2, \quad, x_s$ Then γ passes through the points x_i in a certain pattern For fixed s the number of all those patterns is finite, so we can restrict our considerations to periodic reflecting rays γ having s different reflection points (s being fixed) which are visited by γ in a fixed pattern Every such ray is determined by the points x_i, that is by $s(n-1)$ unknowns since $\dim X = n-1$ On the other hand, any two successive segments of γ determine lines which are symmetric with respect to the normal to X at their common end x_i, and this gives $n-1$ conditions for the points $x_1, \quad, x_s$ Therefore, if γ passes only once through each of its reflection points, then the general number of the conditions is $s(n-1)$ In the other case we have more conditions than unknowns Applying transversality type arguments we see that for generic X this system has no solution

The same idea can be exploited to prove theorem B Now for fixed s we consider pairs (γ, δ) of periodic reflecting rays on X having together exactly s different reflection points, and such that any of the rays γ, δ passes in a certain fixed pattern through its reflection points Again the number of the unknowns is $s(n-1)$ and, if γ and δ have at least one common reflection point, then we get more than $s(n-1)$

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conditions which are of the same type as those considered above. This enables us to use the arguments from the proof of theorem A.

Throughout the paper smooth means C^∞, although everything is true replacing C^∞ with C^k for $k = 1, 2, \ldots$. Note also that our results remain true for non-compact X, if we consider only such periodic reflecting rays γ on X that for every reflection point x of γ all segments of γ passing through x make acute angles with one and the same normal vector to X at x.

3. *Preliminaries*

Here we give a construction from [9] which describes analytically the periodic reflecting rays and provides a simple classification of them.

3.1. We use $\langle\ ,\ \rangle$ to denote the standard inner product in $\mathbb{R}^n$, and by $\|\ \|$ the induced norm in $\mathbb{R}^n$. Given a map $g: Y \to Z$ and an integer $s > 0$, the product $g^s: Y^s \to Z^s$ is defined by $g^s(y_1, \ldots, y_s) = (g(y_1), \ldots, g(y_s))$. For every set A we put

$$A^{(s)} = \{(a_1, \ldots, a_s) \in A^s : a_i \neq a_j \text{ whenever } i \neq j\}.$$

3.2. Let $k \geq s \geq 2$ be integers and

$$\omega: \{1, 2, \ldots, k\} \to \{1, 2, \ldots, s\} \tag{3.1}$$

be a map with

$$\omega(i) \neq \omega(i+1) \qquad (i = 1, 2, \ldots, k). \tag{3.2}$$

We set for convenience $\omega(p) = \omega(i)$ for $p = i + mk$, $i = 1, 2, \ldots, k$, m an integer. If we have

$$\{\omega(i), \omega(i+1)\} \neq \{\omega(j), \omega(j+1)\} \tag{3.3}$$

whenever $1 \leq i < j \leq k$, then ω will be called a *non-symmetric map*. If $k = 2m$ and there is $i_0 = 1, 2, \ldots, k$ such that (3.3) holds for $i_0 \leq i < j \leq i_0 + m$, and

$$\omega(i_0 + m + j) = \omega(i_0 + m - j) \qquad (j = 1, 2, \ldots, m-1), \tag{3.4}$$

then the map ω will be called *symmetric*. By *admissible map* we mean a map (3.1) with (3.2) which is either symmetric or non-symmetric.

(3.3) Let (3.1) be an admissible map. Define

$$I_i = I_i(\omega) = \{j : \text{there is } t = 1, \ldots, k \text{ with } \{i, j\} = \{\omega(t), \omega(t+1)\}\} \tag{3.5}$$

for $i \in \operatorname{Im} \omega$, and

$$U_\omega = \{(y_1, \ldots, y_s) \in (\mathbb{R}^n)^{(s)} : y_i \notin \text{convex hull } \{y_j : j \in I_i\}, i \in \operatorname{Im} \omega\}. \tag{3.6}$$

Clearly, U_ω is an open subset of $(\mathbb{R}^n)^{(s)}$. Define the map $F: (\mathbb{R}^n)^{(s)} \to \mathbb{R}$ by

$$F(y_1, \ldots, y_s) = \sum_{i=1}^{k} \|y_{\omega(i)} - y_{\omega(i+1)}\|. \tag{3.7}$$

Suppose Y is a smooth compact $(n-1)$-dimensional submanifold of $\mathbb{R}^n$. A periodic reflecting ray γ on Y will be said to be of *type* ω if there exist s different points $y_1, \ldots, y_s$ of Y so that

$$y_{\omega(1)}, y_{\omega(2)}, \ldots, y_{\omega(k)}, y_{\omega(k+1)} = y_{\omega(1)}$$

are all successive reflection points of γ. In this case we have $y = (y_1, \ldots, y_s) \in U_\omega$, and the length of γ is equal to $F(y)$ Note also that $I_i(\omega)$ is just the set of those j such that there is a segment of γ joining y_i and y_j

If $Y=f(X)$ for some $f\in C^{\infty}_{\text{emb}}(X,\mathbb{R}^n)$, then there exists an element $x=(x_1, \quad , x_s)$ of $X^{(s)}$ with $f(x_i)=y_i$ $(i=1, \quad , s)$ Clearly $y=f^s(x)\in U_\omega$ and x is a critical point of $F\circ f^s$

3 4 LEMMA ([9]) *For every $y\in U_\omega$ and every $i\in \operatorname{Im}\omega$ there exists $t=1,2, \quad , n$ with* $(\partial F/\partial y_i^{(t)})(y)\neq 0$ $(y_i=(y_i^{(1)}, \quad , y_i^{(n)})\in\mathbb{R}^n)$

(3 5) Fix an admissible map (3 1) and consider the *s-fold bundle of the* 1*-jets* $J^1_s(X,\mathbb{R}^n)$ (cf [1, p 57]) For every $f\in C^\infty(X,\mathbb{R}^n)$ the map $j^1_sf\ X^{(s)}\to J^1_s(X,\mathbb{R}^n)$ is given by $j^1_sf(x_1, \quad , x_s)=(j^1f(x_1), \quad , j^1f(x_s))$, where $j^1f(x)\in J^1(X,\mathbb{R}^n)$ is the 1-jet generated by f and $x\in X$ Denote by V_ω the set of those

$$\tau=(j^1f_1(x_1), \quad , j^1f_s(x_s))\in J^1_s(X,\mathbb{R}^n)$$

such that for every $i\in\operatorname{Im}\omega$, rank $df_i(x_i)=n-1$ and the vector $f_j(x_j)-f_i(x_i)$ is not tangent to $f_i(X)$ at the point $f_i(x_i)$ for all $j\in I_i(\omega)$ Consider the open submanifold M of $J^1_s(X,\mathbb{R}^n)$ given by

$$M=(\alpha^s)^{-1}(X^{(s)})\cap(\beta^s)^{-1}(U_\omega)\cap V_\omega, \tag{3 8}$$

where $\alpha\ J^1(X,\mathbb{R}^n)\to X$, $\beta\ J^1(X,\mathbb{R}^n)\to\mathbb{R}^n$ are defined by

$$\alpha(j^1f(x))=x, \qquad \beta(j^1f(x))=f(x)$$

(3 6) Given $\tau\in M$ there are coordinate neighbourhoods V_i of elements of X such that $V_i\cap V_j=\varnothing$ for $i\neq j$ and τ belongs to

$$\Omega=M\cap\left(\prod_{i=1}^{s} J^1(V_i,\mathbb{R}^n)\right) \tag{3 9}$$

Take arbitrary charts $\theta_i\ V_i\to\mathbb{R}^{n-1}$, and consider the chart

$$\theta\ \Omega\to(\mathbb{R}^{n-1})^{(s)}\times(\mathbb{R}^n)^{(s)}\times\mathbb{R}^{(n-1)ns}, \tag{3 10}$$

defined by

$$\theta(j^1f(x_1), \quad , j^1f_s(x_s))=(u_1, \quad , u_s, v_1, \quad , v_s, (a_{ij}^{(t)}))$$
$$1\le t\le n, 1\le i\le s, 1\le j\le n-1, \tag{3 11}$$

where for all $i=1, \quad , s$, $j=1, \quad , n-1$, $t=1, \quad , n$ we have

$$u_i=\theta_i(x_i), \qquad v_i=f_i(x_i) \tag{3 12}$$

and

$$a_{ij}^{(t)}=\frac{\partial(f_i^{(t)}\circ\theta_i^{-1})}{\partial u_i^{(j)}}(u_i) \tag{3 13}$$

Here $f_i=(f_i^{(1)}, \quad , f_i^{(n)})$ and $u_i=(u_i^{(1)}, \quad , u_i^{(n-1)})\in\mathbb{R}^{n-1}$

We shall write the elements ξ of $\theta(\Omega)$ in the form

$$\xi=(u, v, a), \tag{3 14}$$

where

$$u=(u_1, \quad , u_s)\in(\mathbb{R}^{n-1})^{(s)}, \quad v=(v_1, \quad , v_s)\in(\mathbb{R}^n)^{(s)}, \quad a=(a_{ij}^{(t)})\in\mathbb{R}^{ns(n-1)} \tag{3 15}$$

4 *Proof of theorem A*

We shall consider in detail the case of non-symmetric rays

Suppose $Y=f(X)$, $f\in C^{\infty}_{\mathrm{emb}}(X,\mathbb{R}^n)$, and γ is a non-symmetric periodic reflecting ray on Y with $d(\gamma)>0$ Then (cf 3 3) there exists an admissible surjection ω of type (3 1) and an element $x=(x_1, \quad, x_s)\in X^{(s)}$ such that $y_{\omega(1)}, \quad, y_{\omega(k)}, y_{\omega(k+1)}=y_{\omega(1)}$ are all successive reflection points of γ We may assume

$$\omega(1)=1, \tag{4 1}$$

and

$$r=\operatorname{card}\omega^{-1}(1)>1 \tag{4 2}$$

Let $\imath_1<\imath_2<\quad<\imath_r$ be all the elements of $\omega^{-1}(1)$ It is clear that $f^s(x)\in U_\omega$, and

$$\operatorname{grad}_{x'} F\circ f^s(x)=0 \tag{4 3}$$

for

$$x'=(x_2, \quad, x_s)\in X^{(s-1)} \tag{4 4}$$

Moreover, for every $l=1, \quad, r$, if $\imath=\omega(\imath_l-1)$ and $\jmath=\omega(\imath_l+1)$, then the vectors $w_1=(y_\imath-y_1)/\|y_\imath-y_1\|$ and $w_2=(y_\jmath-y_1)/\|y_\jmath-y_1\|$ lie in a common (two-dimensional) plane with a unit normal vector N_1 to $Y=f(X)$ at the point y_1, and w_1 and w_2 make equal angles with N_1 This is equivalent to

$$\frac{f(x_\imath)-f(x_1)}{\|f(x_\imath)-f(x_1)\|}+\frac{f(x_\jmath)-f(x_1)}{\|f(x_\jmath)-f(x_1)\|}=\left\langle\frac{f(x_\imath)-f(x_1)}{\|f(x_\imath)-f(x_1)\|}+\frac{f(x_\jmath)-f(x_1)}{\|f(x_\jmath)-f(x_1)\|}, N_1\right\rangle N_1 \tag{4 5}$$

Now fix a non-symmetric surjection (3 1) with (4 1) and (4 2), and denote by T_ω the set of all $f\in C^{\infty}_{\mathrm{emb}}(X,\mathbb{R}^n)$ such that for every $x\in X^{(s)}$ with $f^s(x)\in U_\omega$ and (4 3), there exists at least one $l=1,2, \quad, r$ for which (4 5) is not satisfied for $\imath=\omega(\imath_l-1)$, $\jmath=\omega(\imath_l+1)$, N_1 being a unit normal vector to $f(X)$ at $f(x_1)$ We are going to prove T_ω contains a residual subset of $C^{\infty}_{\mathrm{emb}}(X,\mathbb{R}^n)$ To this end we shall use the open submanifold M of $J^1_s(X,\mathbb{R}^n)$, defined by (3 8)

Introduce the set Σ of those $\tau=(\jmath^1 f_1(x_1), \quad, \jmath^1 f_s(x_s))\in M$ so that

$$\operatorname{grad}_{x'} F\circ(f_1\times \quad \times f_s)(x)=0, \tag{4 6}$$

and for any $l=1, \quad, r$ we have

$$\frac{f_\imath(x_\imath)-f_1(x_1)}{\|f_\imath(x_\imath)-f_1(x_1)\|}+\frac{f_\jmath(x_\jmath)-f_1(x_1)}{\|f_\jmath(x_\jmath)-f_1(x_1)\|}=\left\langle\frac{f_\imath(x_\imath)-f_1(x_1)}{\|f_\imath(x_\imath)-f_1(x_1)\|}+\frac{f_\jmath(x_\jmath)-f_1(x_1)}{\|f_\jmath(x_\jmath)-f_1(x_1)\|}, N\right\rangle N, \tag{4 7}$$

for $\imath=\omega(\imath_l-1)$, $\jmath=\omega(\imath_l+1)$, N being a unit normal vector to $f_1(X)$ at $f_1(x_1)$ It follows from the definitions of T_ω and Σ that

$$T_\omega=\{f\in C^{\infty}_{\mathrm{emb}}(X,\mathbb{R}^n) \ \ \jmath^1_s f(X^{(s)})\cap\Sigma=\varnothing\} \tag{4 8}$$

The most important result of this section is the following

LEMMA 4 1 *There exist smooth submanifolds W_m $(m=1,2, \quad)$ of M with*

$$\operatorname{codim} W_m=s(n-1)+(r-1)(n-1), \qquad (m=1,2, \quad) \tag{4 9}$$

and

$$\Sigma\subset\bigcup_{m=1}^{\infty} W_m \tag{4 10}$$

Proof Consider a coordinate neighbourhood Ω of the type (3 9) of an element of Σ and a chart (3 10) on Ω defined by (3 11), (3 12) and (3 13) It is sufficient to show that $\theta(\Omega\cap\Sigma)$ is contained in a smooth submanifold of $\theta(\Omega)$ with codimension $s(n-1)+(r-1)(n-1)$

Take $l=1, \quad, r$ and set $i=\omega(i_l-1), j=\omega(i_l+1)$ For $m\le n$ define $d_l^{(m)} \ \theta(\Omega)\to\mathbb{R}$ by

$$d_l^{(m)}(\xi)=\frac{v_i^{(m)}-v_1^{(m)}}{\|v_i-v_1\|}+\frac{v_j^{(m)}-v_1^{(m)}}{\|v_j-v_1\|}-\left\langle\frac{v_i-v_1}{\|v_i-v_1\|}+\frac{v_j-v_1}{\|v_j-v_1\|}, N(\xi)\right\rangle \ N^{(m)}(\xi), \tag{4 11}$$

where $\xi\in\theta(\Omega)$ is given by (3 14) and (3 15), and $N(\xi)=N'/\|N'\|$,

$$N'(\xi)=\det\begin{vmatrix} e_1 & e_2 & & e_n \\ a_{11}^{(1)} & a_{11}^{(2)} & & a_{11}^{(n)} \\ a_{12}^{(1)} & a_{12}^{(2)} & & a_{12}^{(n)} \\ & & & \\ a_{1n-1}^{(1)} & a_{1n-1}^{(2)} & & a_{1n-1}^{(n)} \end{vmatrix},$$

$e_p=(0, \quad, 0, 1, 0, \quad, 0)$ being the pth standard basis vector in $\mathbb{R}^n$ Consider also the maps $b_{pq} \ \theta(\Omega)\to\mathbb{R}$ ($p=2, \quad, s$, $q=1, \quad, n-1$), defined by

$$b_{pq}(\xi)=\sum_{t=1}^{n}\frac{\partial F}{\partial y_p^{(t)}}(v)a_{pq}^{(t)} \tag{4 12}$$

Finally, for every $m=1, \quad, n$ set

$$O_m=\{\xi\in\theta(\Omega) \ N^{(m)}(\xi)\neq 0\} \tag{4 13}$$

Clearly, O_m are open subsets of $\theta(\Omega)$ and $\bigcup_{m=1}^n O_m=\theta(\Omega)$ So it is sufficient to prove that for every m, $\theta(\Omega\cap\Sigma)\cap O_m$ is contained in a smooth submanifold of O_m with codimension $s(n-1)+(r-1)(n-1)$

Fix $m_0=1, \quad, n$ and consider the map $K \ O_{m_0}\to(\mathbb{R}^{n-1})^{s-1+r}$, given by

$$K(\xi)=((b_{pq}(\xi))_{p=2, \ ,s}^{q=1, \ ,n-1}, (d_l^{(m)}(\xi))_{l=1, \ r}^{m=1 \ ,n,m\neq m_0})$$

It follows from (4 6), (4 7), (4 11), (4 12) and the definition of K that $O_{m_0}\cap\theta(\Omega\cap\Sigma)\subset K^{-1}(0)$ Therefore the proof will be complete if we show that K is a submersion at any point of O_{m_0} Take $\xi\in O_{m_0}$, ξ being given by (3 14) and (3 15), and suppose

$$\sum_{p=2}^{s}\sum_{q=1}^{n-1}B_{pq}\operatorname{grad}b_{pq}(\xi)+\sum_{l=1}^{r}\sum_{\substack{m=1\\ m\neq m_0}}^{n}D_l^{(m)}\operatorname{grad}d_l^{(m)}(\xi)=0 \tag{4 14}$$

for some real numbers B_{pq} and $D_l^{(m)}$ For fixed p and q by 3 3 there is $t=1, \quad, n$ with $(\partial F/\partial y_p^{(t)})(v)\neq 0$ By (4 14), taking into account the derivatives with respect to $a_{pq}^{(t)}$, we get $B_{pq}(\partial F/\partial y_p^{(t)})(v)=0$, and therefore $B_{pq}=0$ Thus the first double sum in (4 14) is trivial Now fix $l=1, \quad, r$ and $m=1, \quad, n$, $m\neq m_0$, and set $i=\omega(i_l-1)$, $j=\omega(i_l+1)$, $N=N(\xi)$ For convenience put

$$w=(v_i-v_1)/\|v_i-v_1\|, \tag{4 15}$$

Then by (4 14) for $t = 1, \quad , n$, $t \neq m$, we have

$$\frac{\partial d_l^{(m)}}{\partial v_i^{(t)}}(\xi) = \frac{1}{\|v_i - v_1\|}[N^{(m)}w^{(t)}\langle w, N\rangle - N^{(m)}N^{(t)} - w^{(m)}w^{(t)}] \qquad (t \neq m) \tag{4 16}$$

Similarly, we get

$$\frac{\partial d_l^{(t)}}{\partial v_i^{(t)}}(\xi) = \frac{1}{\|v_i - v_1\|}[1 + N^{(t)}w^{(t)}\langle w, N\rangle - (N^{(t)})^2 - (w^{(t)})^2] \tag{4 17}$$

Note that $(\partial d_l^{(m')}/\partial v_i^{(m)})(\xi) = 0$ if $l' \neq l$, so considering in (4 14) the derivatives with respect to $v_i^{(t)}$ we obtain

$$\sum_{\substack{m=1\\ m \neq m_0}}^{n} D_l^{(m)} \frac{\partial d_l^{(m)}}{\partial v_i^{(t)}}(\xi) = 0 \qquad (t = 1, \quad , n)$$

For convenience set

$$D_l^{(m_0)} = 0, \qquad D_l = (D_l^{(1)}, \quad , D_l^{(n)}) \in \mathbb{R}^n \tag{4 18}$$

Then we have

$$\sum_{m=1}^{n} D_l^{(m)} \frac{\partial d_l^{(m)}}{\partial v_i^{(t)}}(\xi) = 0 \qquad (t = 1, \quad , n),$$

and (4 16) and (4 17) imply

$$D_l^{(t)} + \sum_{m=1}^{n} D_l^{(m)}[N^{(m)}w^{(t)}\langle w, N\rangle - N^{(m)}N^{(t)} - w^{(m)}w^{(t)}] = 0,$$

that is

$$D_l^{(t)} + \langle D_l, N\rangle\langle w, N\rangle w^{(t)} - \langle D_l, N\rangle N^{(t)} - \langle D_l, w\rangle w^{(t)} = 0$$

$(t = 1, \quad , n)$, and equivalently

$$D_l + \langle D_l, N\rangle\langle w, N\rangle w - \langle D_l, N\rangle N - \langle D_l, w\rangle w = 0 \tag{4 19}$$

Taking the inner product of the left hand side of (4 19) with N we find

$$\langle D_l, N\rangle\langle w, N\rangle^2 - \langle D_l, w\rangle\langle w, N\rangle = 0$$

It is not difficult to see that $\langle w, N\rangle \neq 0$ Indeed, since $\xi \in \theta(\Omega)$, $\Omega \subset M$, we have $\xi = \theta(\tau)$ for some element τ of M, $\tau = (j^1 f_1(x_1), \quad , j^1 f_s(x_s))$ Then u, v and a have the form (3 15) with (3 12) and (3 13) By (3 8), $M \subset V_\omega$, and the definition of V_ω implies $N'(\xi) \neq 0$ and $N(\xi) = N'(\xi)/\|N'(\xi)\|$ is a unit normal vector to $f_1(X)$ at $v_1 = f_1(x_1)$ On the other hand, by $i \in I_1(\omega)$ and the definition of V_ω we have $\langle v_i - v_1, N(\xi)\rangle \neq 0$ Thus, by (4 15) we get $\langle w, N\rangle \neq 0$ Now (4 20) implies $\langle D_l, N\rangle \langle w, N\rangle = \langle D_l, w\rangle$, and combining this with (4 19) we find $D_l = \langle D_l, N\rangle N$ On the other hand, by (4 18)

$$0 = D_l^{(m_0)} = \langle D_l, N\rangle N^{(m_0)} \tag{4 20}$$

then $\langle D_l, N\rangle = 0$, because $N^{(m_0)} \neq 0$ by (4 13) and $\xi \in O_{m_0}$ Hence $D_l = 0$ which means by (4 18), $D_l^{(m)} = 0$ for every $m = 1, \quad , n$ Thus we have proved the second double sum in (4 14) is trivial which shows K is a submersion in O_{m_0} Therefore $K^{-1}(0)$ is a smooth submanifold of O_{m_0} with codimension $(n-1)(s-1) + r(n-1) = s(n-1) + (r-1)(n-1)$ □

COROLLARY 4 2 *T_ω contains a residual subset of $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$*

Proof Since M is open in $J^1_s(X, \mathbb{R}^n)$, any W_m is a smooth submanifold of $J^1_s(X, \mathbb{R}^n)$ with the same codimension By the multijet transversality theorem ([1, ch II]) for every m the set

$$S_m = \{f \in C^\infty(X, \mathbb{R}^n)\ j^1_s f \pitchfork W_m\} \cap C^\infty_{\text{emb}}(X, \mathbb{R}^n)$$

is residual in $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ On the other hand, (4 2) and (4 9) imply

$$\operatorname{codim} W_m \geq s(n-1) + (n-1) > s(n-1) = \dim X^{(s)}$$

Since $j^1_s f\ X^{(s)} \to J^1_s(X, \mathbb{R}^n)$, we obtain

$$S_m = \{f \in C^\infty_{\text{emb}}(X, \mathbb{R}^n)\ j^1_s f(X^{(s)}) \cap W_m = \varnothing\}$$

Combining this with (4 8) and (4 10) we get $\bigcap_{m=1}^\infty S_m \subset T_\omega$ which proves the corollary ☐

COROLLARY 4 3 *The set T' of those $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ such that every non-symmetric periodic reflecting ray on $f(X)$ has zero defect, contains a residual subset of $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$*

Proof The assertion follows from $T' \supset \bigcap_\omega T_\omega$, where ω runs over the non-symmetric surjections (3 1) with (4 1) and (4 2) ☐

In a similar way one can deal with the symmetric rays We shall point out only a few differences Suppose γ is a symmetric periodic reflecting ray on $Y = f(X)$ Then we can find a symmetric surjection (3 1) such that γ is of type ω By the definition of symmetric map we have $k = 2m$, moreover we can choose ω in such a way that (3 3) is fulfilled for $i_0 = 1$ Suppose $d(\gamma) > 0$ Then for some $j_0 = 1, \ldots, s$ we have

$$r = \operatorname{card}(\omega^{-1}(j_0) \cap \{1, 2, \ldots, m+1\}) > 1 \tag{4 21}$$

Let $i_1 < i_2 < \cdots < i_r$ be the elements of $\omega^{-1}(j_0) \cap \{1, \ldots, m+1\}$ Given $x = (x_1, \ldots, x_s) \in X^{(s)}$, define x' by $x' = (x_1, \ldots, x_{j_0-1}, x_{j_0+1}, \ldots, x_s)$ Further, we go on as in the non-symmetric case, proving that for every symmetric surjection ω with the properties listed above (including (4 21) for some $j_0 = 1, \ldots, s$) the set T_ω contains a residual subset of $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ Thus we establish the set T'' of those $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ so that every symmetric periodic reflecting ray on $f(X)$ has zero defect, contains a residual subset of $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$

To conclude the proof of theorem A, we mention that $T' \cap T'' \subset \mathscr{A}$ Therefore $\mathscr{A}$ also contains a residual subset of $C^\infty_{\text{emb}}(X, \mathbb{R}^n)$

5 *Proof of theorem B*

Suppose γ_1 and γ_2 are two different non-symmetric periodic reflecting rays on $Y = f(X)$, $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ Let $y_1, \ldots, y_s$ be all different reflection points of γ_1 and γ_2 taken together, $y_i = f(x_i)$, $x_i \in X$, $i = 1, \ldots, s$ Then there exist admissible non-symmetric maps

$$\omega\ \{1, \ldots, k\} \to \{1, \ldots, s\}, \qquad \delta\ \{1, \ldots, l\} \to \{1, \ldots, s\}$$

with

$$\operatorname{Im} \omega \cup \operatorname{Im} \delta = \{1, \ldots, s\} \tag{5 1}$$

and

$$\{\omega(i), \omega(i+1)\} \neq \{\delta(j), \delta(j+1)\} \qquad (1 \le i \le k, 1 \le j \le l) \tag{5.2}$$

such that γ_1 is of type ω and γ_2 is of type δ (cf. 3.3). Note especially that (5.2) expresses the fact that γ_1 and γ_2, being different, have no common segments.

As in [9], the symbol $\Gamma = (k, l, s, \omega, \delta)$ with the above properties will be called a configuration, and the pair of rays (γ_1, γ_2) will be said to be of type Γ. In this case we have $y = (y_1, \ldots, y_s) \in U_\omega \cap U_\delta$ and the periods (lengths) of γ_1 and γ_2 coincide with $F(y)$ and $G(y)$, respectively, where $F, G: (\mathbb{R}^n)^{(s)} \to \mathbb{R}$ are given by

$$F(z_1, \ldots, z_s) = \sum_{i=1}^{k} \|z_{\omega(i)} - z_{\omega(i+1)}\|, \tag{5.3}$$

$$G(z_1, \ldots, z_s) = \sum_{i=1}^{l} \|z_{\delta(i)} - z_{\delta(i+1)}\|. \tag{5.4}$$

As for ω, set $\delta(p) = \delta(i)$ if $p = i + ml$, $1 \le i \le l$, m an integer.

Suppose γ_1 and γ_2 have at least one common reflection point. Without loss of generality we may assume y_1 is a reflection point for both γ_1 and γ_2, that is $1 \in \operatorname{Im} \omega \cap \operatorname{Im} \delta$. Moreover, we may assume

$$\omega(1) = \delta(1) = 1. \tag{5.5}$$

Set

$$i' = \omega(k), \quad j' = \omega(2), \quad i'' = \delta(l), \quad j'' = \delta(2). \tag{5.6}$$

Then (4.5) is satisfied for $i = i', j = j'$ and for $i = i'', j = j''$, where $x = (x_1, \ldots, x_s) \in X^{(s)}$ and N_1 is a unit normal vector to $Y = f(X)$ at y_1. Introduce the function

$$H: (\mathbb{R}^n)^{(s)} \to \mathbb{R}$$

defined by $H(z) = F(z) + G(z)$. Since x is a critical point for both $F \circ f^s$ and $G \circ f^s$, it is a critical point for $H \circ f^s$ too. In particular,

$$\operatorname{grad}_{x'} H \circ f^s(x) = 0, \tag{5.7}$$

where x' is defined by (4.4).

We are going to use the argument from the proof of theorem A, replacing F by H. To do this we need a property of H similar to 3.4. In fact, what we need is equivalent to

$$\sum_{j \in I_i(\omega)} \frac{y_i - y_j}{\|y_i - y_j\|} + \sum_{j \in I_i(\delta)} \frac{y_i - y_j}{\|y_i - y_j\|} \neq 0$$

for $i = 2, \ldots, s$, $i \in \operatorname{Im} \omega \cap \operatorname{Im} \delta$ and $y = (y_1, \ldots, y_s) \in U_\omega \cap U_\delta$. However, this is not true in general (see for example figure 2). That is why we have to make a little modification of the argument from § 4.

Denote by $\mathcal{B}_\Gamma$ the set of those $f \in C^\infty_{\text{emb}}(X, \mathbb{R}^n)$ such that if $x \in X^{(s)}$ is a critical point of $F \circ f^s$ with $f^s(x) \in U_\omega \cap U_\delta$ and (5.7), then (4.5) is not satisfied either for $i = i'$ and $j = j'$ or for $i = i''$ and $j = j''$. It follows from above that

$$\bigcap_\Gamma \mathcal{B}_\Gamma \subset \mathcal{B},$$

where $\mathcal{B}$ is the set defined in theorem B, and $\Gamma = (k, l, s, \omega, \delta)$ runs over those configurations with (5.5).

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Fix a configuration $\Gamma = (k, l, s, \omega, \delta)$ with (5 5) and non-symmetric ω and δ, and define $F, G, H, i', j', i'', j''$ as above We have to show $\mathcal{B}_\Gamma$ contains a residual subset of $C^\infty_{\mathrm{emb}}(X, \mathbb{R}^n)$ For $i = 1, \quad , s$ define

$$\mathcal{U}_i = \{y \in U_\omega \cap U_\delta \ \ \mathrm{grad}_{y_i} H(y) \neq 0\}$$

and set $\tilde{\mathcal{U}} = \bigcap_{i=1}^s \mathcal{U}_i$ All $\mathcal{U}_i$ are open subsets of $U_\omega \cap U_\delta$ (therefore $\tilde{\mathcal{U}}$ is also open), and $\mathcal{U}_i = U_\omega \cap U_\delta$ if $i \notin \mathrm{Im}\, \omega \cap \mathrm{Im}\, \delta$ (cf 3 4)

Denote by T_Γ the set of those $f \in C^\infty_{\mathrm{emb}}(X, \mathbb{R}^n)y$ such that for every $x \in X^{(s)}$ with $f^s(x) \in \tilde{\mathcal{U}}$ and (5 7), (4 5) is not fulfilled either for $i = i', j = j'$ or for $i = i'', j = j''$

Lemma 5 1 *T_Γ contains a residual subset of $C^\infty_{\mathrm{emb}}(X, \mathbb{R}^n)$*

The proof uses the same argument as those used in § 4 to show T_ω contains a residual subset of $C^\infty_{\mathrm{emb}}(X, \mathbb{R}^n)$

Further, for $i \in \mathrm{Im}\, \omega \cap \mathrm{Im}\, \delta$ let T_i be the set of those f in $C^\infty_{\mathrm{emb}}(X, \mathbb{R}^n)$ such that if $x \in X^{(s)}$ is a critical point of $F \circ f^s$ and $f^s(x) \in U_\omega \cap U_\delta$, then $f^s(x) \in \mathcal{U}_i$ We mention that

$$\bigcap_{i \in \mathrm{Im}\, \omega \cap \mathrm{Im}\, \delta} T_i \cap T_\Gamma \subset \mathcal{B}_\Gamma \tag{5 8}$$

Indeed, suppose f belongs to the left hand side of (5 8) and $x \in X^{(s)}$ is a critical point of $F \circ f^s$ with $f^s(x) \in U_\omega \cap U_\delta$ and (5 7) Then for every $i \in \mathrm{Im}\, \omega \cap \mathrm{Im}\, \delta, f \in T_i$ implies $f^s(x) \in \mathcal{U}$ Thus $f^s(x) \in \tilde{\mathcal{U}}$, and now by $f \in T_\Gamma$ we see that (4 5) is not satisfied either for $i = i', j = j'$ or for $i = i'', j = j''$ Therefore $f \in \mathcal{B}_\Gamma$

Fix $i \in \mathrm{Im}\, \omega \cap \mathrm{Im}\, \delta$ To prove that T_i contains a residual subset of $C^\infty_{\mathrm{emb}}(X, \mathbb{R}^n)$ we shall use the following result which is a part of theorem 3 1 in [**9**]

Theorem 5 2 ([**9**]) *Let $n \geq 2$, $s \geq 2$, $q \geq 1$ be integers, U be an open subset of $(\mathbb{R}^n)^{(s)}$, $H \ \ U \to \mathbb{R}$ and $L \ \ U \to \mathbb{R}^q$ be smooth maps Suppose L has no critical points in U and $\mathrm{grad}_{y_i} H(y) \neq 0$ for all $i = 1, \quad , s$ and $y \in U$ Let X be a smooth $(n-1)$-dimensional submanifold of $\mathbb{R}^n$ and T be the set of those $f \in C^\infty_{\mathrm{emb}}(X, \mathbb{R}^n)$ such that for every $x \in X^{(s)}$ with $f^s(x) \in U$ which is a critical point of $H \circ f^s$, we have $L(f^s(x)) \neq 0$ Then T contains a residual subset of $C^\infty_{\mathrm{emb}}(X, \mathbb{R}^n)$*

To apply the theorem define $L \ \ (\mathbb{R}^n)^{(s)} \to \mathbb{R}^n$ by

$$L^{(t)}(y) = \frac{\partial F}{\partial y_i^{(t)}}(y) + \frac{\partial G}{\partial y_i^{(t)}}(y) \qquad (t = 1, \quad , n),$$

and $L = (L^{(1)}, \quad , L^{(n)})$ The following property of L (established in [**9**]) can easily be proved by direct computations

Lemma 5 3 ([**9**]) *$DL(y) \neq 0$ for every $y \in (\mathbb{R}^n)^{(s)}$*

Now, applying theorem 5 2 for H, $U = U_\omega \cap U_\delta$ and L, we deduce that T_i contains a residual subset of $C^\infty_{\mathrm{emb}}(X, \mathbb{R}^n)$ Hence $\mathcal{B}_\Gamma$ also contains a residual subset of $C^\infty_{\mathrm{emb}}(X, \mathbb{R}^n)$

In case both ω and δ are symmetric or one of them is symmetric and the other is non-symmetric, we use the same arguments with minor changes to see that $\mathcal{B}_\Gamma$ contains a residual subset of $C^\infty_{\mathrm{emb}}(X, \mathbb{R}^n)$ We omit the details

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