SETS OF CONVERGENCE FOR SERIES DEFINED BY ITERATION[1]

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Let $f(x)$ be a real-valued function defined on an interval $I_a:[0,a]$. For each point x in I_a we form the series $\sum_{n=0}^{\infty} u_n$, where $u_o = x$ and $u_{n+1} = f(u_n)$ for $n \geq 0$. If the series $\sum_{n=0}^{\infty} u_n$ converges, x will be called a point of convergence; if this series diverges, x will be called a point of divergence. In this note several properties of sets of convergence[2] will be obtained. We shall always assume that:

(1) f is continuous on I_a,

(2) $f(0) = 0$, $0 \leq f(x) < x$ for $0 < x \leq a$.

Fort and Schuster [1] showed that if f satisfies (1) and (2) as well as the following additional conditions on an interval I_b:

(3) f is differentiable in I_b,

(4) there exists a positive constant c such that $f'(x) \geq c$ in I_b,

(5) if $0 < x_1 < x_2 < b$, $f(x_1)/x_1 \geq f(x_2)/x_2 > 0$,

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[2] The set of convergence is the set of points of convergence.

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then, for each point $x = u_0$ in I_b, the series $\sum_{n=0}^{\infty} u_n$ converges or diverges according as the integral $\int_0^b \frac{y}{y - f(y)} dy$ converges or diverges.

THEOREM 1. *If the function* f *satisfies conditions (1) and (2) in the interval* I_a *and there exists a number* b, $0 < b \le a$, *such that* f *is nondecreasing in* I_b, *then the set of convergence for* f *is either the entire interval* I_a *or it is a closed set containing* 0 *as an isolated point. Furthermore each closed set in* I_a *which contains* 0 *as an isolated point is the set of convergence for a function satisfying (1) and (2) in* I_a *as well as (3), (4), and (5) in some interval* I_b.

Proof: If b is a point of convergence and $0 \le y \le b$, then, since f is nondecreasing in I_b, $f^{(n)}(y) \le f^{(n)}(b)$ for all n; consequently y is a point of convergence. (We shall use the symbol $f^{(n)}$ to denote the n^{th} iterate of f.) In this case all points of I_b are points of convergence. If b is a point of divergence and $0 \le y \le b$, then, since $\{f^{(n)}(y)\}$ is a null sequence, there exists a number k such that $f^{(k)}(b) \le y$. Again, since f is nondecreasing in I_b, $f^{(n+k)}(b) \le f^{(n)}(y)$ for all n; hence y is a point of divergence. Thus, if b is a point of divergence, all points of the interval $(0, b]$ are points of divergence.

Now let x be an arbitrary point in I_a. The sequence $\{f^{(n)}(x)\}$ is monotone nonincreasing in n, and it tends to zero for each x. By a well-known theorem of Dini, the sequence $\{f^{(n)}(x)\}$ tends uniformly to zero on I_a. Thus, there exists a natural number N, independent of x, such that if $n > N$ then each point $f^{(n)}(x)$ lies in I_b. If b is a point of convergence, then $f^{(N)}(x)$ is a point of convergence for each x in I_a; certainly the point x is likewise a point of convergence. If b is a point of divergence, then the only point of convergence in I_b is 0; the point x is a point of convergence if and only if it lies in one of the sets

$$F_n = \{x : f^{(n)}(x) = 0\} .$$

Since f is continuous, each set F_n is closed. The set of convergence is the union of the sets $F_n (n = 0, 1, \ldots, N)$, and therefore it is closed. Since f does not vanish in $(0, b]$, 0 is an isolated point of convergence.

This concludes the proof of the first part of the theorem. We note that if (3), (4), and (5) hold in I_b, then, if $\int_0^b \frac{y}{y - f(y)} dy$ converges, the set of convergence is I_a, while if this integral diverges, the set of convergence is a closed set containing 0 as an isolated point.

Now let F denote a closed set which contains 0 as an isolated point. We construct a function f which satisfies (1) and (2) on an interval I_a, as well as (3), (4), and (5) on an interval I_b with $0 < b \leq a$. We take $b \leq 1/2$ and such that the interval I_b contains no point of F except 0. We define f as follows:

$$f(x) = \frac{(x - x^2) d(b, F)}{d(b, F) + 1} \qquad 0 \leq x \leq b ,$$

$$f(x) = \frac{(b - b^2) d(x, F)}{d(x, F) + 1} \qquad b \leq x \leq a ,$$

where $d(x, F)$ denotes the distance of the point x from the set F. We note that the integral $\int_0^b \frac{y}{y - f(y)} dy$ diverges. The set F is the set of convergence for f.

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THEOREM 2. *The set of convergence for a function* f *is a set of type* F_σ.

Proof. The point x is a point of divergence if and only if, for each positive integer j there exists a number n such that

$$\sum_{i=0}^{n} f^{(i)}(x) > j . \tag{6}$$

Let $G_{j,n}$ denote the set of points which satisfy (6). Each set $G_{j,n}$ is open. The set of divergence is the set $\bigcap_j \bigcup_n G_{j,n}$, and this set is of type G_δ. Therefore, the set of convergence is of type F_σ.

THEOREM 3. *If* f *satisfies (1) and (2), and its set of divergence is nonempty, then for each* $x \neq 0$, *the interval* $(f(x), x)$ *contains points of divergence. If, for each positive* δ, *the interval* $(0, \delta)$ *contains points of convergence, then the interval* $(f(x), x)$ *contains points of convergence.*

Proof. We prove only the second part of the theorem; the first part is proved similarly. Again let $u_0 = x$ and $u_{n+1} = f(u_n)$ for $n \geq 0$. It follows from (1) and (2) that $\{u_n\}$ is a null sequence. There is a point of convergence z arbitrarily close to 0. For some positive integer r, z must lie in the interval (u_{r+1}, u_r). Since $f^{(r)}(u_0) = u_r$ and $f^{(r)}(u_1) = u_{r+1}$, there is a point w in the interval (u_1, u_0) such that $f^{(r)}(w) = z$; w is a point of convergence.

THEOREM 4. *Suppose that* f *satisfies (1) and (2) and that the interval* $(0, \delta)$ *contains points of convergence for each positive* δ. *If* x *is a point of divergence and* y *is a point of convergence, then the interval between* x *and* y *contains both points of convergence and points of divergence.*

Proof. Without loss in generality we may take $y < x$.

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Again we let $u_o = x$, $u_{k+1} = f(u_k)$ for $k \geq 0$, $v_o = y$, $v_{k+1} = f(v_k)$ for $k \geq 0$. Since y is a point of convergence it is impossible that $v_k \geq u_{k+1}$ for all values of k, for then we would have $\sum_{k=0}^{\infty} v_k \geq \sum_{k=1}^{\infty} u_k = \infty$, and y would be a point of divergence. Hence there exists a positive integer k such that $v_k < u_{k+1} < u_k$. By the intermediate value theorem there is a point w, $v_o < w < u_o$, such that $f^{(k)}(w) = u_{k+1}$; clearly w is a point of divergence. By Theorem 3 the interval (u_{k+1}, u_k) contains a point of convergence z'; since $v_k < u_{k+1}$, $v_k < z' < u_k$. Again by the intermediate value theorem there is a point z such that $v_o < z < u_o$ and $f^{(k)}(z) = z'$; z is a point of convergence.

We conclude with the following problem.

If f satisfies the conditions (1) and (2), and m_δ denotes the Lebesgue measure of the intersection of the set of divergence with the interval $[0, \delta]$, is it true that $\lim_{\delta \to 0} m_\delta / \delta = 0$ if and only if $\int_0^a \frac{x}{x - f(x)} dx < \infty$?

REFERENCE

1. M. K. Fort, Jr., and Seymour Schuster, Convergence of series whose terms are defined recursively, Amer. Math. Monthly, 71 (1964), 994-998.

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