

# Application of Lagrangian techniques for calculating the on-axis rotational transform

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The Floquet exponents of periodic field lines are studied through the variations of the magnetic action on the magnetic axis, which is assumed to be elliptical. The near-axis formalism developed by Mercier, Solov'ev and Shafranov is combined with a Lagrangian approach. The on-axis Floquet exponent is shown to coincide with the on-axis rotational transform. A discrete solution suitable for numerical implementation is introduced, which gives the Floquet exponents as solutions to an eigenvalue problem. This discrete formalism expresses the exponents as the eigenvalues of a  $6 \times 6$  matrix.

**Key words:** magnetic action, rotational transform, near-axis expansion, Lagrangian integration, Mercier, Solov'ev-Shafranov, variation

## 1. The need for and the origin of the rotational transform

Magnetostatic equilibria are characterized by the following equations:

$$\nabla p = \mathbf{J} \times \mathbf{B}, \quad \nabla \times \mathbf{B} = \mathbf{J}, \quad \nabla \cdot \mathbf{B} = 0, \quad (1.1a-c)$$

where  $\mathbf{B}$  is the magnetic field,  $\mathbf{J}$  is the current density and  $p$  is the scalar pressure. With the assumption that  $p$  is not constant in a small region and the surface lies in a bounded volume of space, the boundary must be topologically toroidal (Kruskal & Kulsrud 1958). The phase portrait of  $\mathbf{B}$ , where the magnetic field lines are treated like integral curves of a Hamiltonian dynamical system, is characterized by the topology of the level sets of  $p$ . A magnetic field line action can be defined (Cary & Littlejohn 1983), which serves as a starting point for the Lagrangian integration carried out in this paper.

For magnetic confinement of plasmas in toroidal geometries, that the magnetic field lines rotate poloidally (the short way) as they rotate toroidally (the long way) around the torus is essential for cancelling charged particle drifts, which would otherwise lead to loss (Spitzer 1958). The number of poloidal rotations that a field line achieves per toroidal period is called the rotational transform,  $\iota$  (Spitzer 1958).

Kruskal & Kulsrud consider the case where ' $p$  is reasonably smooth and not constant in any region', and this entails that the magnetic field lines lie on nested flux surfaces. In

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the following, we do not place such a constraint on the pressure and we do not require that the magnetic field has nested flux surfaces. The existence of nested magnetic surfaces, their link with continuous symmetries and the preservation of magnetic surfaces under small symmetry-destroying perturbation are widely discussed in Kolmogorov (1954), Arnol'd (1963) and Moser (1962) and also in texts and reviews: Moser (1973), Arnold (1978), Lichtenberg & Lieberman (1992) and Meiss (1992). Herein, we only require that the magnetic field is 'toroidal', which we describe as follows. We consider a magnetic configuration in the domain  $\Omega$ , which is assumed to be a solid torus, with  $\mathbf{B} \cdot \nabla \phi > 0$  everywhere where  $\phi$  is the toroidal angle, and with  $\mathbf{B} \cdot \mathbf{n} = 0$  where  $\mathbf{n}$  is normal to the boundary. For any poloidal section  $\Sigma$  of  $\Omega$ , the Poincaré return map is defined by the intersection of field lines and  $\Sigma$  after one toroidal period. Brouwer's fixed-point theorem (Brouwer 1910) ensures that there will be at least one fixed point of the Poincaré first return map.

Taking such a point as the origin of polar-like coordinates, e.g.  $(r, \theta)$ , we may consider the effect of iterating the mapping on nearby points, where the point  $(r_n, \theta_n)$  gets mapped to  $(r_{n+1}, \theta_{n+1})$ . If, upon iterating the return map, the points rotate around the fixed point, then the fixed point is called elliptic. (A more rigorous definition of elliptic fixed points is given in Meiss (1992).) There may be more than one fixed point, and not all fixed points are *elliptic*. Even though the methods of Lagrangian integration as described below can be applied more generally, for clarity of exposition we restrict our attention to the case of an isolated elliptic fixed point. Stellarators are typically designed to have one easily identifiable elliptic fixed point about which most field lines rotate, and this is called the *magnetic axis*.

This sequence of angular displacements,  $\Delta\theta = \theta_{n+1} - \theta_n$ , enables one to define the rotational transform of a field line:

$$\iota := \lim_{N \uparrow \infty} \frac{1}{2\pi N} \sum_n^N \Delta\theta_n. \quad (1.2)$$

Even though the angle coordinate becomes degenerate at the origin in polar-like coordinates, the common approach is to define the on-axis rotational transform by taking the limit as the starting point gets closer to the axis.

Note that this definition of the rotational transform is purely geometric, in that the rotational transform effectively measures how many times a given magnetic field line links the magnetic axis. A discussion of the rotational transform as an asymptotic linking of neighbouring field lines can be found in Arnold (2014).

Mercier (see Mercier 1964; Helander 2014) expressed the on-axis rotational transform as an integral along the axis:

$$\iota = N + \frac{1}{2\pi} \oint \frac{d\ell}{\cosh \eta} \left( \frac{J_0}{2B_0} + \delta' - \tau \right), \quad (1.3)$$

with  $\ell$  denoting the arc length. The on-axis current density and magnetic field are denoted by  $J_0$  and  $B_0$ , respectively, and the torsion of the axis by  $\tau$ . The eccentricity of flux surfaces is described by  $\eta$  and  $\delta$  is a parameter describing their rotation around the axis, with  $\delta' := d\delta/d\ell$ . Here  $N$  is an integer coming from the phase of the rotation term  $\delta$  (see Pfefferlé *et al.* 2018). Note that Spitzer (1958) had identified independently that a way to generate some rotational transform is to give torsion to an axis. This expression provides invaluable insight, showing how rotational transform can be produced by plasma currents, as is used by tokamaks, or by geometrical shaping, as is used by stellarators.

One may also encounter different definitions of the rotational transform. Assuming that the magnetic configuration possesses magnetic surfaces, ensuring that the toroidal and poloidal magnetic fluxes  $\psi$  and  $\chi$  respectively, can be properly defined, then as shown by Mercier *et al.* (1974),  $\iota$  can also be expressed as the ratio of differential fluxes:

$$\iota = \frac{d\chi}{d\psi}. \quad (1.4)$$

Mercier's expression (1.3) for instance was derived starting from the flux definition (1.4). The fluxes were expanded in a power series of the distance to the axis by constructing custom polar coordinates, so-called Mercier coordinates, which are described in Appendix C.

In this paper, we present a derivation of Mercier's formula using Lagrangian integration. This approach expresses the rotational transform as a Floquet exponent of the field lines. The relationship between the rotational transform and the Floquet exponent was described by Greene (1979), and more recently by Duignan & Meiss (2021). In § 2, the magnetic field-line action is defined. Stationary curves of the action are shown to be field lines, enabling one to identify a magnetic axis. Assumption is made in this paper that the axis is elliptical, but the formalism can be applied to the hyperbolic case. In § 3, Mercier's formula (1.3) is derived from the second variation of the action. The result is obtained through a near-axis expansion of the null eigenspace of the second variation operator. The periodicity enables the use of a Floquet description of the solutions. This same result is derived through the theory of the Hill infinite determinant, analogous to Schrödinger's equation in a periodic potential. In both cases, the rotational transform is shown to be a Floquet exponent of the null eigenspace of the second variation operator. In § 4, a discrete formalism is introduced so that the Floquet exponents can be solved for numerically.

## 2. Description of the magnetic field-line action

Let us start by defining the magnetic field line action, as introduced by Cary & Littlejohn (1983). This is defined as a line integral that, for a given magnetic field, depends only the integration contour. Herein, the integration contours are assumed to be curves that close after one toroidal period. Let us consider a closed, differentiable curve  $\mathcal{C} \subset \mathbb{R}^3$  with total length  $L$ , closing after one toroidal transit. The latter assumption that the curve closes after one toroidal period is a necessary condition for a magnetic axis. Curve  $\mathcal{C}$  is parametrized by the  $C^1$ ,  $L$ -periodic vector-valued function

$$\left. \begin{aligned} \mathbf{x} : [0, L] &\rightarrow \mathbb{R}^3 \\ \ell &\mapsto \mathbf{x}(\ell) \in \mathcal{C}, \end{aligned} \right\} \quad (2.1)$$

$\ell$  being the arc length and we note  $\mathbf{x}' := d\mathbf{x}/d\ell$ . The action is defined as the circulation along  $\mathcal{C}$  of the magnetic vector potential  $\mathbf{A}$ , with  $\mathbf{B} = \nabla \times \mathbf{A}$  in  $\Omega$ :

$$\mathcal{S} := \oint_{\mathcal{C}} d\ell \mathbf{A} \cdot \mathbf{x}'. \quad (2.2)$$

Properties of the magnetic field are accessible through calculus of variations, performed on the action. The variations are performed with respect to changes in the geometry of the

curve. The first variation with respect to a variation  $\delta \mathbf{x}$  is

$$\delta \mathcal{S}[\delta \mathbf{x}] = \oint_C d\ell \mathbf{x}' \times \mathbf{B} \cdot \delta \mathbf{x}, \quad (2.3)$$

which shows that stationary curves are tangential to the magnetic field and hence field lines. We used that  $\delta \mathcal{A}[\delta \mathbf{x}] = \delta \mathbf{x} \cdot \nabla \mathcal{A}$ . From (2.3), a magnetic axis can be identified. We focus in particular on elliptical axes, but one can apply the following formalism to hyperbolic axes as well. The difference would be seen in the flux-surface functions around the axis, and in the nature of the Floquet exponents. Let us denote an elliptical axis by  $\mathcal{C}_a$ . Additional properties of the field appear at higher orders of variations of  $\mathcal{S}$ . For the rotational transform in particular, the second-order variation  $\delta^2 \mathcal{S}$  is of interest.

### 3. Verification of Mercier's formula for the Floquet exponent

#### 3.1. The second variation as an operator

In order to express the on-axis rotational transform, the second-order variation of the action applied to the axis needs to be derived. Assume that an elliptical axis  $\mathcal{C}_a$  has been found as a stationary curve of the action. The formalism is identical for hyperbolic axes, but in this paper, we focus on the elliptic case. The second variation of the action performed from the axis  $\mathcal{C}_a$  is

$$\delta^2 \mathcal{S}[\delta \mathbf{x}] = \oint_{\mathcal{C}_a} d\ell \delta(\mathbf{x}' \times \mathbf{B} \cdot \delta \mathbf{x}) = \oint_{\mathcal{C}_a} d\ell \delta \mathbf{x} \cdot (\delta \mathbf{x}' \times \mathbf{B} + \mathbf{x}' \times \delta \mathbf{B}), \quad (3.1)$$

where  $\mathbf{f}' := d\mathbf{f}/d\ell$  for any  $\mathbf{f}$ . Using  $\delta \mathbf{B} = \delta \mathbf{x} \cdot \nabla \mathbf{B}$  and the Einstein summation convention, we write the second variation as an operator:

$$\delta^2 \mathcal{S} = \oint_{\mathcal{C}_a} d\ell \delta x^i \frac{\delta^2 \mathcal{S}}{\delta x^i \delta x^j} \delta x^j, \quad (3.2)$$

where

$$\frac{\delta^2 \mathcal{S}}{\delta x^i \delta x^j} = \epsilon_{ijk} \mathbf{B}^k \frac{d}{d\ell} + \epsilon_{imk} x'^m \partial_j \mathbf{B}^k, \quad (3.3)$$

and  $i, j, k, m \in \{1, 2, 3\}$ , which in matrix form reads

$$\mathbf{M} := \frac{\delta^2 \mathcal{S}}{\delta \mathbf{x} \delta \mathbf{x}} = -(\mathbf{I} \times \mathbf{B}) \frac{d}{d\ell} + \mathbf{x}' \times (\nabla \mathbf{B})^\top. \quad (3.4)$$

The identity tensor is denoted by  $\mathbf{I}$ . Note that the covariant tensor  $\mathbf{M}$  can be easily symmetrized (see Hudson & Dewar 2009). However, the direction in which the derivative  $d/d\ell$  is taken has to be chosen carefully. We continue with the non-symmetric form (3.4).

We show that the null eigenspace of  $\mathbf{M}$  is of particular interest. Indeed, by periodicity of the field lines considered to compute the action, the eigenspaces of the operator  $\mathbf{M}$  are related to geometric properties of the neighbouring field lines, including the rotational transform. Let  $\mathbf{v}$  be a null eigenfunction of  $\mathbf{M}$  such that

$$\mathbf{M}\mathbf{v} = \mathbf{0}. \quad (3.5)$$

For  $\mathbf{v}$  to be non-trivial, (3.5) rewrites

$$\det(\mathbf{M}) = 0. \quad (3.6)$$

We will demonstrate that the condition (3.5) is satisfied by solutions  $\mathbf{v}$ , such that their Floquet exponent is the on-axis rotational transform, and (3.6) will serve in the discrete

formalism introduced in § 4. To solve (3.5), the near-axis formalism developed by Mercier, Solov'ev and Shafranov (Solov'ev & Shafranov 1970; Mercier *et al.* 1974) is used. Near-axis formalism in the inverse coordinate approach, where the flux surface  $\psi$  is used as a coordinate, has proved to be very successful in understanding quasi-symmetry (see Garren & Boozer 1991; Landreman & Sengupta 2018, 2019; Rodríguez, Sengupta & Bhattacharjee 2023). However, for this work, the near-axis formalism in direct or Mercier–Solov'ev–Shafranov coordinates (Jorge, Sengupta & Landreman 2020b,a; Sengupta *et al.* 2024) is more relevant.

### 3.2. Derivation of the Floquet exponent from the second variation

The operator is expanded in the Solov'ev–Shafranov coordinates, introduced in Appendix C.2. The latter set of coordinates is closely related to the Mercier coordinates, presented in Appendix C.1.

In what follows, it is assumed that the magnetic axis is a closed curve  $\mathcal{C}_a \subset \mathbb{R}^3$  parametrized by the vector-valued function  $\mathbf{r}_0$ , with the arc length  $\ell$  as parameter, and a total length  $L$ . The basis of expansion is taken to be  $\{\mathbf{e}_1, \mathbf{e}_2, \mathbf{e}_3\}$ , whose expressions in terms of  $\mathcal{N}$ ,  $\mathcal{B}$  and  $\mathbf{t}$  are given in Appendix C.2, but the results will always be expressed in terms of the Solov'ev–Shafranov vectors  $\{\mathcal{N}, \mathcal{B}, \mathbf{t}\}$ . Additionally, the following notation is adopted:  $f' = df/d\ell$  for any  $f$ .

#### 3.2.1. Magnetic field expansion near an elliptic magnetic axis

From (3.4), the magnetic field  $\mathbf{B}$  needs to be described in the vicinity of the axis  $\mathbf{r}_0$  for  $\mathbf{M}$  to be expanded in the near-axis formalism. The expansion of  $\mathbf{B}$  in Solov'ev–Shafranov coordinates is carried in the limit  $x, y \ll 1$ , or in other words, is limited to linear terms only. The magnetic field can be written in contravariant form as

$$\sqrt{|g|}\mathbf{B} = \sqrt{|g|}B^1\mathbf{e}_1 + \sqrt{|g|}B^2\mathbf{e}_2 + \sqrt{|g|}B^3\mathbf{e}_3, \quad (3.7)$$

with

$$\sqrt{|g|}B^1 = a_1x + a_2y, \quad \sqrt{|g|}B^2 = b_1x + b_2y, \quad \sqrt{|g|}B^3 = B_0 + c_1x + c_2y, \quad (3.8a-c)$$

where  $a_1, a_2, b_1, b_2, c_1, c_2$  are periodic functions of  $\ell$ ,  $B_0$  the zeroth-order magnitude of  $\mathbf{B}$  and  $\sqrt{|g|} = h$  as defined in Appendix C.2. Therefore, in the limit  $x, y \ll 1$ ,  $h = 1$  and the magnetic field reads

$$\begin{aligned} \mathbf{B} = & (B_0 + c_1x + c_2y)\mathbf{t} + [a_1x + (a_2 + u'B_0)y]\mathcal{N} \\ & + [(b_1 - u'B_0)x + b_2y]\mathcal{B} + \mathcal{O}(x^2 + y^2). \end{aligned} \quad (3.9)$$

A direct calculation shows that

$$\left. \begin{aligned} \nabla \mathbf{B} = & \kappa B_0 \mathbf{t} \mathbf{n} + a_1 \mathcal{N} \mathcal{N} + b_2 \mathcal{B} \mathcal{B} + B'_0 \mathbf{t} \mathbf{t} + (a_2 + u'B_0) \mathcal{B} \mathcal{N} \\ & + (b_1 - u'B_0) \mathcal{N} \mathcal{B} + (c_1 \mathcal{N} + c_2 \mathcal{B}) \mathbf{t}, \\ \mathbf{x}' \times (\nabla \mathbf{B})^\top = & \kappa B_0 \mathbf{b} \mathbf{t} + \mathcal{B} [a_1 \mathcal{N} + (a_2 + u'B_0) \mathcal{B}] - \mathcal{N} [(b_1 - u'B_0) \mathcal{N} + b_2 \mathcal{B}], \end{aligned} \right\} \quad (3.10)$$

which yields, using dyadic algebra

$$\left. \begin{aligned} \nabla \cdot \mathbf{B} = & \mathbf{I} : \nabla \mathbf{B} = B'_0 + a_1 + b_2, \\ \nabla \times \mathbf{B} = & -\mathbf{I} \times \nabla \mathbf{B} = (b_1 - a_2 - 2u'B_0) \mathbf{t} + (c_2 - \kappa B_0 \sin \delta) \mathcal{N} \\ & - (c_1 - \kappa B_0 \cos \delta) \mathcal{B}. \end{aligned} \right\} \quad (3.11)$$

Here  $\kappa$  denotes the local curvature of the axis  $\mathbf{r}_0$ . Additionally, (C10a,b) has been used to rewrite  $\mathbf{b}$  in terms of  $\mathcal{N}$  and  $\mathcal{B}$ . Requiring that  $\nabla \cdot \mathbf{B} = 0$  and  $\nabla \times \mathbf{B} = J_0 \mathbf{t}$ , one finds

$$a_1 + b_2 = -B'_0, \quad \frac{b_1 - a_2}{2B_0} = u' + \frac{J_0}{2B_0}, \quad c_1 = \kappa B_0 \cos \delta, \quad c_2 = \kappa B_0 \sin \delta. \quad (3.12a-d)$$

So far, only two constraints have been derived for the four expansion functions  $a_1, a_2, b_1, b_2$ . Turning to the Mercier representation (D2a,b) and (D7a-c) of  $\mathbf{B}$  brings in additional conditions. Together with the definitions of  $\mathcal{N}$  and  $\mathcal{B}$ ,

$$\begin{aligned} \frac{\mathbf{B}}{B_0} = & \mathbf{t} \left[ 1 + \kappa \sqrt{x^2 + y^2} \cos(u - \delta) \right] \\ & + \mathcal{N} \left[ -\frac{1}{2} \left( \frac{B'_0}{B_0} + \eta' \right) x + \left( \frac{-J_0}{2B_0} + \frac{\frac{J_0}{2B_0} - \tau + \delta'}{\cosh \eta} \sinh \eta \right) y \right] \\ & + \mathcal{B} \left[ \left( \frac{J_0}{2B_0} + \frac{\frac{J_0}{2B_0} - \tau + \delta'}{\cosh \eta} \sinh \eta \right) x - \frac{1}{2} \left( \frac{B'_0}{B_0} - \eta' \right) y \right] \\ & + \mathcal{O}(x^2 + y^2), \end{aligned} \quad (3.13)$$

where  $\delta$ , as explained in Appendix C, describes the rotation of elliptical flux surfaces around the expansion axis  $\mathbf{r}_0$ . As for  $\eta$ , it describes the eccentricity of the flux surfaces around  $\mathbf{r}_0$ . Defining

$$\Omega_0(\ell) = \frac{\frac{J_0}{2B_0} - \tau + \delta'}{\cosh \eta}, \quad (3.14)$$

the comparison between the Mercier representation (3.13) and the Solov'ev–Shafranov representation (3.9) yields for the expansion coefficients

$$\left. \begin{aligned} \frac{a_1}{B_0} &= -\frac{1}{2} \left( \frac{B'_0}{B_0} + \eta' \right), & \frac{a_2}{B_0} &= -\left( u' + \frac{J_0/2}{B_0} \right) + \Omega_0 \sinh \eta, \\ \frac{b_2}{B_0} &= -\frac{1}{2} \left( \frac{B'_0}{B_0} - \eta' \right), & \frac{b_1}{B_0} &= \left( u' + \frac{J_0/2}{B_0} \right) + \Omega_0 \sinh \eta, \end{aligned} \right\} \quad (3.15)$$

which satisfy the relations (3.12a–d). From the definition of  $\Omega_0$ , we can further simplify  $a_2$  and  $b_1$  as

$$\frac{a_2}{B_0} = -\Omega_0 e^{-\eta}, \quad \frac{b_1}{B_0} = \Omega_0 e^{\eta}. \quad (3.16a,b)$$

As a check of correctness for the above expressions, one can show that  $\mathbf{B} \cdot \nabla \psi = 0$ , where

$$\begin{aligned} \sqrt{g} \mathbf{B} \cdot \nabla &= \sqrt{g} B^1 \partial_x + \sqrt{g} B^2 \partial_y + \sqrt{g} B^3 \partial_\ell \\ &= B_0 \partial_\ell + (a_1 x + a_2 y) \partial_x + (b_1 x + b_2 y) \partial_y. \end{aligned} \quad (3.17)$$

The flux surface function  $\psi$  for the elliptic case is given in the Mercier representation by (D10) or equivalently in the Solov'ev–Shafranov representation by

$$\psi(x, y) = B_0 (e^{\eta} x^2 + e^{-\eta} y^2). \quad (3.18)$$

Note that in the case of a hyperbolic axis, the plus sign from the flux function (3.18) would be replaced by a minus sign.

### 3.2.2. The second variation tensor and its null eigenspace

Now that the magnetic field has been properly expressed and expanded in Solov'ev–Shafranov coordinates, the obtained  $\mathbf{B}$  and  $\nabla\mathbf{B}$  can be substituted in (3.4) to expand the second variation tensor. From (3.10), we find that it is given by

$$\left. \begin{aligned} \mathbf{M} &:= \mathbf{M}_1 \frac{d}{d\ell} + \mathbf{M}_2, \\ \mathbf{M}_1 &= (\mathcal{N}\mathcal{B} - \mathcal{B}\mathcal{N})B_0, \\ \mathbf{M}_2 &= \kappa B_0 [\mathcal{B} \cos \delta - \mathcal{N} \sin \delta] \mathbf{t} + \mathcal{B} [a_1 \mathcal{N} + (a_2 + u'B_0)\mathcal{B}] \\ &\quad - \mathcal{N} [(b_1 - u'B_0)\mathcal{N} + b_2 \mathcal{B}]. \end{aligned} \right\} \quad (3.19)$$

Observing the dyadic form of  $\mathbf{M}$ , it can be shown that tangential components  $v^t$  of the eigenvectors  $\mathbf{v} = (v^t, v^{\mathcal{N}}, v^{\mathcal{B}})^T$  will not contribute to  $\mathbf{M}\mathbf{v} = \mathbf{0}$ . Alternatively, the underlying argument is that the arc length parametrization of  $\mathcal{C}$  constrains the admissible variations  $\delta\mathbf{x} = \mathbf{v}$ , removing the dependence in the tangential component. Therefore,  $v^t$  can be absorbed by redefining  $v^{\mathcal{N}}$  and  $v^{\mathcal{B}}$ . Thus, the null eigenvectors are chosen to be of the form  $\mathbf{v} = v^{\mathcal{N}}\mathcal{N} + v^{\mathcal{B}}\mathcal{B}$ . We note that

$$\frac{d}{d\ell}\mathbf{v} = (v^{\mathcal{N}'} + u'v^{\mathcal{B}})\mathcal{N} + (v^{\mathcal{B}'} - u'v^{\mathcal{N}})\mathcal{B} + \kappa [v^{\mathcal{N}} \cos(u - \delta) + v^{\mathcal{B}} \sin(u - \delta)] \mathbf{t}. \quad (3.20)$$

The tangential terms  $\propto \mathbf{t}$  are not relevant since they do not contribute to  $\mathbf{M}\mathbf{v} = \mathbf{0}$ . The equations for  $v^{\mathcal{N}}, v^{\mathcal{B}}$  then read

$$B_0 \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \frac{d}{d\ell} \begin{pmatrix} v^{\mathcal{N}} \\ v^{\mathcal{B}} \end{pmatrix} + \begin{pmatrix} -b_1 & -b_2 \\ a_1 & a_2 \end{pmatrix} \begin{pmatrix} v^{\mathcal{N}} \\ v^{\mathcal{B}} \end{pmatrix} = 0. \quad (3.21)$$

From (3.21), by periodicity of  $a_i/B_0$  and  $b_i/B_0$ ,  $i = 1, 2$ , the system can immediately be rewritten in the form of a periodic system:

$$\frac{d\mathbf{v}}{d\ell} = \mathbf{A}(\ell)\mathbf{v}, \quad \mathbf{v} = \begin{pmatrix} v^{\mathcal{N}} \\ v^{\mathcal{B}} \end{pmatrix}. \quad (3.22a,b)$$

Here,  $\mathbf{A}(\ell)$  is periodic in  $\ell$ . The Floquet theorem, (A6), can be applied to conclude that the solution must be of the form

$$\mathbf{v} = \mathbf{U}(\ell)\mathbf{e}^{\mathbf{C}\ell/L}, \quad (3.23)$$

where  $\mathbf{U}$  is a symplectic periodic matrix (with period  $L$ ) and  $\mathbf{C}$  is a constant Hamiltonian matrix (Duignan & Meiss 2021). The eigenvalues of  $\mathbf{C}$  are the Floquet exponents, which must be purely imaginary near an elliptic axis and real in the hyperbolic case.

Although the system is already in a form that allows one to solve for the exponents  $\nu$  by means of (3.23), identifying the matrices  $\mathbf{C}$  and  $\mathbf{U}$  can be somewhat troublesome. For that reason, (3.21) can be alternatively rewritten in the following ordinary differential equation



(ODE) form:

$$\frac{dv^{\mathcal{N}}}{a_1 v^{\mathcal{N}} + a_2 v^{\mathcal{B}}} = \frac{dv^{\mathcal{B}}}{b_1 v^{\mathcal{N}} + b_2 v^{\mathcal{B}}} = \frac{d\ell}{B_0}, \quad (3.24)$$

which shows that the eigenvector  $\mathbf{v}$  satisfies the general characteristic equation along the magnetic field:

$$\frac{d\mathcal{X}}{\sqrt{g}B^1(\mathcal{X}, \mathcal{Y}, \ell)} = \frac{d\mathcal{Y}}{\sqrt{g}B^2(\mathcal{X}, \mathcal{Y}, \ell)} = \frac{d\ell}{\sqrt{g}B^3(\mathcal{X}, \mathcal{Y}, \ell)}. \quad (3.25)$$

The eigenvector  $\mathbf{v}$  components  $v^{\mathcal{N}}, v^{\mathcal{B}}$  can be identified with  $\mathcal{X}, \mathcal{Y}$ , the displacements of the magnetic field line from the closed field line  $\mathbf{r}_0$  along the rotated normal and binormal directions. The expansion coordinates  $(x, y)$  ought not to be confused with  $(\mathcal{X}, \mathcal{Y})$ , which are the solutions of the characteristic ODEs (3.25). The equations for  $(\mathcal{X}, \mathcal{Y})$  are

$$\left. \begin{aligned} \mathcal{X}' + \frac{1}{2} \left( \frac{B'_0}{B_0} + \eta' \right) \mathcal{X} + \Omega_0 e^{-\eta} \mathcal{Y} &= 0, \\ \mathcal{Y}' + \frac{1}{2} \left( \frac{B'_0}{B_0} - \eta' \right) \mathcal{Y} - \Omega_0 e^{+\eta} \mathcal{X} &= 0. \end{aligned} \right\} \quad (3.26)$$

Introducing the variables  $X, Y$  defined through

$$\mathcal{X} = \frac{1}{\sqrt{B_0}} e^{-\eta/2} X(\ell), \quad \mathcal{Y} = \frac{1}{\sqrt{B_0}} e^{+\eta/2} Y(\ell), \quad (3.27a,b)$$

which is possible since the periodicity is conserved, the system reduces to the simple harmonic oscillator:

$$X' + \Omega_0 Y = 0, \quad Y' - \Omega_0 X = 0, \quad \Omega_0 = \frac{\frac{J_0/2}{B_0} - \tau + \delta'}{\cosh \eta}, \quad (3.28a-c)$$

with ‘time-dependent’ frequency  $\Omega_0(\ell)$ ,  $\ell$  being the time-like parameter. Using the complex variable  $Z = X + iY$ , the system reshapes as a single complex ODE:

$$Z' - i\Omega_0 Z = 0 \quad \Rightarrow \quad Z(\ell) = Z_0 \exp \int_0^\ell i\Omega_0(s) ds. \quad (3.29)$$

Separating the periodic and non-periodic parts of the exponential, we obtain

$$\left. \begin{aligned} Z(\ell) &= Z_p(\ell) e^{i\nu\ell/L}, \quad Z_p(\ell) = Z_0 \exp \int_0^\ell i\tilde{\Omega}_0(s) ds, \\ \tilde{\Omega}_0 &:= \frac{1}{L} \int_0^L \Omega_0(s) ds, \quad \tilde{\Omega}_0 = \Omega_0 - \tilde{\Omega}_0. \end{aligned} \right\} \quad (3.30)$$

Comparing (3.30) with the Floquet form (3.23), we find that  $\nu$  is given by

$$\nu = \int_0^L \Omega_0(s) ds \pmod{2\pi} = \oint \frac{\frac{J_0(s)/2}{B_0(s)} - \tau(s) + \delta'(s)}{\cosh \eta(s)} ds \pmod{2\pi} = 2\pi\iota. \quad (3.31)$$

This matches the expression for the rotational transform (1.3) up to a factor  $2\pi$ , as  $\nu$  represents an angle and  $\iota$  the number of turns that this angle constitutes. It follows



that solving for the null eigenspace of the second variation tensor expanded in the Solov'ev–Shafranov near-axis formalism yields the correct on-axis rotational transform. The exact same approach can be followed in the hyperbolic case.

### 3.2.3. Derivation of the Floquet exponent from the Hill's determinant equation

Here we pursue an alternative approach to obtain the Floquet exponent, using the theory of the Hill's infinite determinant. This way of solving for the Floquet exponents of a periodic system has been known for a long time (Magnus 1953); however, to the authors' knowledge, it is the first time that such an approach is used from the magnetic action. We start with the system (3.26) written as

$$\mathcal{X}' = \frac{1}{B_0}(a_1\mathcal{X} + a_2\mathcal{Y}), \quad \mathcal{Y}' = \frac{1}{B_0}(b_1\mathcal{X} + b_2\mathcal{Y}). \quad (3.32a,b)$$

Eliminating  $\mathcal{Y}$  from (3.32a,b), we obtain the following second-order ODE for  $\mathcal{X}$ :

$$\mathcal{X}'' + 2C_1\mathcal{X}' + C_2\mathcal{X} = 0, \quad (3.33)$$

where

$$2C_1 = -\frac{a_2'}{a_2} + 2\frac{B_0'}{B_0}, \quad C_2 = \mathcal{D} - \frac{a_2}{B_0}\left(\frac{a_1}{a_2}\right)', \quad \mathcal{D} = \frac{1}{B_0^2}(a_1b_2 - a_2b_1). \quad (3.34a-c)$$

We can eliminate the first derivative term from (3.33) by the change of variables

$$\mathcal{X} = \exp\left(-\int C_1 d\ell\right)\Psi = \frac{\sqrt{a_2}}{B_0}X, \quad (3.35)$$

which leads to

$$\Psi'' + \omega^2\Psi = 0, \quad \omega^2 \equiv C_2 - C_1' - C_1^2. \quad (3.36a,b)$$

Equation (3.36a,b) is in the form of Hill's equation or a Schrödinger equation with a periodic potential  $\omega^2$ . We note that the linear transformation (3.35) implies that both  $\mathcal{X}$  and  $\Psi$  have the same Floquet exponent. This is because the multiplication factor  $\sqrt{a_2}/B_0$  is periodic in nature and therefore does not change the Floquet exponent.

Leveraging the periodicity of  $\omega^2$ ,  $\omega^2(\ell + L) = \omega^2(\ell)$ , we can Fourier expand

$$\omega^2(\ell) = \sum_{k \in \mathbb{Z}} \Omega_k e^{i\ell(2\pi/L)k}, \quad (3.37)$$

where  $\{\Omega_k\}_{k \in \mathbb{Z}}$  denote the Fourier coefficients of  $\omega^2$ . Let the fundamental solutions of (3.36a,b) be given by  $\Psi_{\pm}(\ell)$  such that

$$\Psi_+(0) = 1, \quad \Psi_-(0) = 0, \quad \Psi'_+(0) = 0, \quad \Psi'_-(0) = 1, \quad (3.38a-d)$$

where these conditions have been chosen so that the basis functions  $\Psi_{\pm}$  are linearly independent and with unit Wronskian. The Floquet solutions are given by

$$\left. \begin{aligned} \Psi_+(\ell) &= e^{+i\nu\ell}\sigma_+(\ell), & \sigma_+(\ell + L) &= \sigma_+(\ell), & \sigma_+(0) &= 1, \\ \Psi_-(\ell) &= e^{-i\nu\ell}\sigma_-(\ell), & \sigma_-(\ell + L) &= \sigma_-(\ell), & \sigma_-(0) &= 0, \end{aligned} \right\} \quad (3.39)$$

where  $\sigma_+$  and  $\sigma_-$  satisfy the periodicity condition and the linear independence, and  $\nu$  denotes the Floquet exponents for which we seek. Nothing more needs to be known about

the functions  $\sigma_{\pm}$  since their coefficients do not appear in the final expression for the Floquet exponents. Let us now Fourier expand the Floquet solution as

$$\Psi_+ = e^{i\nu\ell} \sum_{n \in \mathbb{Z}} b_n e^{i(2\pi n/L)\ell} = \sum_{n \in \mathbb{Z}} b_n \exp\left(i\ell \left(\frac{2\pi n}{L} + \nu\right)\right), \quad (3.40)$$

where  $\{b_n\}_{n \in \mathbb{Z}}$  is the set of Fourier coefficients of  $\sigma_+$ . The Fourier expansion of (3.36a,b) reads

$$\begin{aligned} \Psi_+''(\ell) + \omega^2(\ell)\Psi_+(\ell) &= - \sum_{n \in \mathbb{Z}} b_n \left(\nu + \frac{2\pi n}{L}\right)^2 \exp\left(i\ell \left(\nu + \frac{2\pi n}{L}\right)\right) \\ &\quad + \left(\sum_{k \in \mathbb{Z}} \Omega_k e^{i\ell(2\pi/L)k}\right) \left(\sum_{n \in \mathbb{Z}} b_n \exp\left(i\ell \left(\frac{2\pi n}{L} + \nu\right)\right)\right) \\ &= \sum_{n \in \mathbb{Z}} \left(\sum_k \Omega_k b_{n-k} - \left(\frac{2\pi n}{L} + \nu\right)^2 b_n\right) \exp\left(i\ell \left(\frac{2\pi n}{L} + \nu\right)\right) \\ &= 0. \end{aligned} \quad (3.41)$$

In order for (3.41) to be verified, the following condition has to hold:

$$\sum_{k \in \mathbb{Z}} \Omega_k b_{n-k} - \left(\frac{2\pi n}{L} + \nu\right)^2 b_n = 0, \quad n \in \mathbb{Z}, \quad (3.42)$$

which can be rewritten in the following matrix form, after dividing (3.42) by  $\Omega_0 - (2\pi n/L + \nu)^2$ :

$$\sum_{m \in \mathbb{Z}} B_{nm} b_m = 0, \quad n \in \mathbb{Z}, \quad (3.43)$$

where the matrix  $B$  is defined as

$$B_{nn} = 1, \quad B_{nm} = \frac{\Omega_{n-m}}{\Omega_0 - \left(\frac{2\pi n}{L} + \nu\right)^2}. \quad (3.44)$$

This yields the following determinant equation:

$$\det B_{nm} = 0. \quad (3.45)$$

The above determinant can be considered a function of the Floquet exponents  $\nu$ :

$$\det B_{nm}(\nu) := \Delta(\nu) \equiv \det \left( \delta_{nm} + \frac{\Omega_{n-m}}{\Omega_0 - \left(\frac{2\pi n}{L} + \nu\right)^2} \right), \quad n, m \in \mathbb{Z}. \quad (3.46)$$

For (3.45) to be verified, we seek for the Floquet exponents  $\nu$  such that  $\Delta(\nu) = 0$ . Following Wang, Guo & Xia (1989), we now show that the exponents can be deduced from the very simple expression (3.50). One notes that  $\Delta(\nu)$  is a  $2\pi$ -periodic function, with

poles in  $\nu = \pm\sqrt{\Omega_0} - 2\pi n/L$ ,  $n \in \mathbb{Z}$ . It can be shown that the determinant is absolutely convergent in the whole  $\nu$ -plane, except in these poles. Therefore,  $\Delta$  is meromorphic. Moreover, as the imaginary part  $\text{Im}(\nu) \rightarrow \pm\infty$ ,  $\Delta(\nu) \rightarrow 1$ . Let us now define the complex function  $f$  as

$$f(\nu) := \cot \frac{L}{2}(\nu - \sqrt{\Omega_0}) - \cot \frac{L}{2}(\nu + \sqrt{\Omega_0}). \quad (3.47)$$

It is useful to introduce  $f$  since it has the same poles and periodicity as  $\Delta$ . Moreover, it is bounded as  $\text{Im}(\nu) \rightarrow \pm\infty$ . This way, there has to exist a constant  $K \in \mathbb{C}$  such that

$$D(\nu) \equiv \Delta(\nu) + Kf(\nu) \quad (3.48)$$

has no singularity in the whole  $\nu$  plane. Together with the fact that it is bounded as  $|\nu| \rightarrow \infty$ , according to Liouville's theorem,  $D$  is a constant function. In the limit  $|\nu| \rightarrow \infty$ , we see that  $D = 1$ . To determine  $K$ , take  $\nu = 0$ :

$$\nu = 0 \implies K = \frac{1 - \Delta(0)}{f(0)} = \frac{1 - \Delta(0)}{2 \cot \frac{\sqrt{\Omega_0}L}{2}}. \quad (3.49)$$

Using the value of  $K$  from (3.49), the Floquet exponent equation reduces to

$$\sin^2 \nu \frac{L}{2} = \Delta(0) \sin^2 \frac{\sqrt{\Omega_0}L}{2}. \quad (3.50)$$

Although (3.50) is very simple, one obstacle remains to compute the exponents: one has to evaluate the infinite determinant,  $\Delta(0)$ . We refer to Wang *et al.* (1989) for some approximations of  $\Delta(0)$ . For instance, when  $\Omega_n$  are sufficiently small,  $\Delta(0)$  can be approximated by the order-3 determinant with  $B_{00}$  as central element (we recall that the determinant involves summation over all  $\mathbb{Z}$ ), providing

$$\Delta(0) \simeq 1 + \frac{2\Omega_1^2}{\Omega_0(4 - \Omega_0)^2} + \frac{2\Omega_1^2\Omega_2}{\Omega_0(4 - \Omega_0)^2} - \frac{\Omega_2^2}{(4 - \Omega_0)^2}. \quad (3.51)$$

However, this approximation breaks down when the coefficients become too large as  $|n|$  increases. As of the exponents computed, they might not all be suitable for  $\iota$ . One has to discard the results that are not relevant. Moreover, we emphasize that the exponents may be shifted by  $2k\pi/L$ , with  $k$  an integer, without changing the mathematics of the system, by periodicity. However, one has to carefully choose the adapted value for  $\iota$ , by setting the appropriate phase shift – see (1.3).

In a nutshell, the second variation of the magnetic action, at an elliptical axis, has been expanded in the Solov'ev–Shafranov near-axis formalism. It has been shown that the null eigenspace of this operator yields the correct on-axis rotational transform, upon applying Floquet theory to solve for the latter. On the other hand, solving for the null eigenspace has led to a system that could be rewritten in the form of a Hill equation. The Floquet exponents appear as solutions of an infinite determinant equation, that can only be approximated analytically. The derivation of the on-axis rotational transform as a Floquet exponent of the null eigenvectors of  $\delta^2\mathcal{S}$  has also been carried on in Mercier coordinates, to support the previous conclusions – see Appendix D.

Magnetic confinement device design needs fast and accurate computation of  $\iota$ . Currently, the most widely used method to compute  $\iota$  is field-line tracing (Todoroki 2003).

Field-line tracing methods compute  $\iota$  as an infinite time limit of following a field line. In practice, it can be done by following a field line sufficiently long to achieve convergence. As an alternative, one can also compute Greene's residue of the axis, which is related to  $\iota$  (see Greene 1979; Hanson & Cary 1984; Hudson 2004). Even though it still involves ODE integration, we can limit ourselves to just one circuit around the torus.

In the following section, we introduce a new method to compute  $\iota$  as a solution to a discrete problem involving the magnetic action. The action is discretized in a way similar to that in Mackay & Meiss (1983) and Hudson & Suzuki (2014).

#### 4. Discrete formalism: piecewise action

We have seen that the problem described in the previous section, involving the second-order variation of the magnetic action to determine the on-axis Floquet exponent, leads to the operator equation (3.5). In another approach, the magnetic axis can be discretized, and the rotational transform can be determined from the multipliers of the latter curve. The multipliers have been shown to be linked to the residue of the curve (see Greene 1979; Mackay & Meiss 1983). This discrete approach is explored in what follows. We consider

$$\begin{aligned} \mathcal{S} &= \sum_{i=1}^{n-1} \int_{C_i} \mathbf{A} \cdot d\mathbf{l} \\ &= \sum_{i=1}^{n-1} S(\mathbf{x}_i, \mathbf{x}_{i+1}), \end{aligned} \quad (4.1)$$

meaning that the curve has been discretized with  $n \in \mathbb{N}$  points. So far, the way that  $C_i$  are defined is not important. Only the endpoints matter. The particular case where  $C_i$  are segments is given in Appendix E. The following notation is adopted for the derivatives:

$$\left. \begin{aligned} S_1^{[i,i+1]} &:= \nabla_{\mathbf{x}_i} S^{[i,i+1]} = \nabla_{\mathbf{x}_i} S(\mathbf{x}_i, \mathbf{x}_{i+1}), \\ S_2^{[i,i+1]} &:= \nabla_{\mathbf{x}_{i+1}} S^{[i,i+1]} = \nabla_{\mathbf{x}_{i+1}} S(\mathbf{x}_i, \mathbf{x}_{i+1}), \end{aligned} \right\} \quad (4.2)$$

as well as for the second-order derivatives:

$$\mathbf{S}_{21}^{[i,i+1]} := \nabla_{\mathbf{x}_{i+1}} \mathbf{S}_1^{[i,i+1]}, \quad (4.3)$$

and similarly for  $\mathbf{S}_{12}$ ,  $\mathbf{S}_{11}$  and  $\mathbf{S}_{22}$ . Let us also define generalized periodic orbits of type  $(q)$ , as orbits with

$$\mathbf{x}_{i+q} = \mathbf{x}_i, \quad (4.4)$$

for some  $q \in \mathbb{N}$ . Therefore, the magnetic axis as discretized above is a general periodic orbit of type  $(n)$ . The terminology 'orbit' for field lines is justified by the Hamiltonian behaviour of the magnetic field. Moreover, the discretized action (4.1) satisfies the periodicity condition

$$\sum_{i=1}^{n-1} S(\mathbf{x}_{i+n}, \mathbf{x}_{i+n+1}) = \sum_{i=1}^{n-1} S(\mathbf{x}_i, \mathbf{x}_{i+1}) + C, \quad (4.5)$$

where the constant  $C = 0$ . In the one-dimensional case, as dealt with in Mackay & Meiss (1983), the existence of extrema of the action is ensured by an additional convexity

condition on the Lagrangian,  $-L_{12} > 0$ . In our three-dimensional case, this can be generalized in that the second-order derivative tensors  $\mathbf{S}_{12}^{[i,i+1]}$  ought to be negative definite for all  $i$ :

$$\mathbf{x}^\top \mathbf{S}_{12}^{[i,i+1]} \mathbf{x} < 0, \quad \mathbf{x} \in \mathbb{R}^{*3}, \quad 1 \leq i \leq n. \quad (4.6)$$

The fact that  $C = 0$  together with the condition that  $\mathbf{S}_{12}^{[i,i+1]}$  guarantees that the action of periodic orbits of type  $(n)$  is bounded below and the Poincaré–Birkhoff theorem (Birkhoff 1913) ensures the existence of at least two stationary trajectories among the space of all generalized periodic paths of type  $(n)$ , one that minimizes the action and is elliptic, and one that is a saddle called *minimax* and is usually hyperbolic but can be alternating hyperbolic, where hyperbolic and elliptic describe the behaviour of nearby trajectories.

For an  $(n)$ -periodic curve that extremizes the action, the latter has to be stationary with respect to an arbitrary variation in its geometry  $\delta \mathbf{x}_i$ , and the stationarity condition can be expressed in terms of the previously defined derivatives:

$$\left. \begin{aligned} \delta S[\delta \mathbf{x}_i] &= [\nabla_{\mathbf{x}_i} S^{[i-1,i]} + \nabla_{\mathbf{x}_i} S^{[i,i+1]}] \cdot \delta \mathbf{x}_i = 0, \\ \Leftrightarrow \quad \mathbf{S}_2^{[i-1,i]} + \mathbf{S}_1^{[i,i+1]} &= \mathbf{0}. \end{aligned} \right\} \quad (4.7)$$

Note that (4.7) holds for any  $1 \leq i \leq n$ , and the stationarity condition is expressed for each point in terms of the two nearest neighbours. This way, a magnetic axis can be identified. Similarly to the continuous case dealt with in §3, the neighbouring field lines have to satisfy

$$\nabla_{\mathbf{x}_{i-1}} \mathbf{S}_2^{[i-1,i]} \delta \mathbf{x}_{i-1} + \nabla_{\mathbf{x}_{i+1}} \mathbf{S}_1^{[i,i+1]} \delta \mathbf{x}_{i+1} + \nabla_{\mathbf{x}_i} (\mathbf{S}_2^{[i-1,i]} + \mathbf{S}_1^{[i,i+1]}) \delta \mathbf{x}_i = \mathbf{0}, \quad (4.8)$$

which we rewrite in terms of the second-order derivatives of the action as

$$\mathbf{S}_{12}^{[i-1,i]} \delta \mathbf{x}_{i-1} + \mathbf{S}_{21}^{[i,i+1]} \delta \mathbf{x}_{i+1} + (\mathbf{S}_{22}^{[i-1,i]} + \mathbf{S}_{11}^{[i,i+1]}) \delta \mathbf{x}_i = \mathbf{0}, \quad 1 \leq i \leq n. \quad (4.9)$$

From Mackay & Meiss (1983), we know that the multipliers  $\lambda$  of an  $(n)$ -periodic orbit are defined by the existence of a tangent orbit satisfying

$$\delta \mathbf{x}_{i+n} = \lambda \delta \mathbf{x}_i, \quad (4.10)$$

so the following holds:

$$\delta \mathbf{x}_{n+1} = \lambda \delta \mathbf{x}_1, \quad \delta \mathbf{x}_0 = \lambda^{-1} \delta \mathbf{x}_n. \quad (4.11a,b)$$

They can be written in their Floquet form (Greene 1979):

$$\lambda = e^{i\nu}, \quad (4.12)$$

where the Floquet exponent  $\nu$  describes the average rotation angle per period of the  $(n)$ -orbit, therefore, the rotational transform  $\iota$ . Since (4.9) is valid for any  $1 \leq i \leq n$ ,

together with (4.11a,b), it can be rewritten in tensor form:

$$\begin{pmatrix} (\mathbf{s}_{22}^{[01]} + \mathbf{s}_{11}^{[12]}) & \mathbf{s}_{12}^{[12]} & & \lambda^{-1} \mathbf{s}_{21}^{[01]} \\ & \mathbf{s}_{21}^{[12]} & \mathbf{s}_{12}^{[23]} & \ddots \\ & & \ddots & \ddots \\ \lambda \mathbf{s}_{12}^{[n,n+1]} & & \mathbf{s}_{21}^{[n-1,n]} & (\mathbf{s}_{22}^{[n-1,n]} + \mathbf{s}_{11}^{[n,n+1]}) \end{pmatrix} \begin{pmatrix} \delta \mathbf{x}_1 \\ \vdots \\ \vdots \\ \delta \mathbf{x}_n \end{pmatrix} = \mathbf{0}, \quad (4.13)$$

where the block-tridiagonal form with corners arises naturally. Denoting by  $\mathbf{M}$  the matrix of second derivatives, we get an equation for the multipliers:

$$\mathbf{M}(\lambda) \delta \mathbf{x} = \mathbf{0}. \quad (4.14)$$

The blank spaces in  $\mathbf{M}$  are blocks of 0. Note that each block in  $\mathbf{M}$  is of size  $3 \times 3$  since  $\mathbf{s}_{lm}$  contain the second derivatives of a line integral embedded in three-dimensional space. Thus, for an  $(n)$ -periodic curve,  $\mathbf{M} \in \mathcal{M}_{3n \times 3n}$ , and  $\delta \mathbf{x} \in \mathbb{R}^{3n}$ . For (4.14) to hold, the determinant of  $\mathbf{M}$  ought to be zero to avoid the trivial solution  $\delta \mathbf{x} = \mathbf{0}$ .

For a block-tridiagonal matrix defined as in (4.13) with  $\lambda \in \mathbb{C}$ , the determinant can be expressed analytically (Molinari 2008):

$$\left. \begin{aligned} \det \mathbf{M}(\lambda) &= \frac{(-1)^{3n}}{(-\lambda)^3} \det(\mathbf{T}_S - \lambda \mathbf{I}_6) \det \left( \prod_{i=1}^n \mathbf{s}_{12}[i, i+1] \right), \\ \mathbf{T}_S &= \prod_{i=1}^n \begin{pmatrix} -\mathbf{s}_{12}^{-1[i,i+1]} (\mathbf{s}_{22}^{[i-1,i]} + \mathbf{s}_{11}^{[i,i+1]}) & -\mathbf{s}_{12}^{-1[i,i+1]} \mathbf{s}_{12}^{[i-1,i]} \\ \mathbf{I}_3 & \mathbf{0} \end{pmatrix}, \end{aligned} \right\} \quad (4.15)$$

which requires one to define the so-called transfer matrix  $\mathbf{T}_S$ . Our generalized convexity condition (4.6) that ensures the existence of minimizing orbits guarantees that the determinant of the transfer matrix can be computed as  $\mathbf{s}_{12}^{[i,i+1]}$  is invertible  $\forall i$ . The multipliers  $\lambda$  are then given by the solutions of

$$\lambda^{-3} \det(\mathbf{T}_S - \lambda \mathbf{I}) = 0, \quad (4.16)$$

so they are the non-zero eigenvalues of  $\mathbf{T}_S$ . Once the multipliers have been computed, the Floquet exponent can be easily determined from (3.31).

Alternatively, note that (4.13) gives a recursive relation for  $\delta \mathbf{x}_i$ :

$$\left. \begin{aligned} (\mathbf{s}_{22}^{[01]} + \mathbf{s}_{11}^{[12]}) \delta \mathbf{x}_1 + \mathbf{s}_{12}^{[12]} \delta \mathbf{x}_2 + \lambda^{-1} \mathbf{s}_{21}^{[01]} \delta \mathbf{x}_n &= 0, \\ \lambda \mathbf{s}_{12}^{[n,n+1]} \delta \mathbf{x}_1 + \mathbf{s}_{21}^{[n-1,n]} \delta \mathbf{x}_{n-1} + (\mathbf{s}_{22}^{[n-1,n]} + \mathbf{s}_{11}^{[n,n+1]}) \delta \mathbf{x}_n &= 0, \\ \mathbf{s}_{21}^{[k-1,k]} \delta \mathbf{x}_{k-1} + (\mathbf{s}_{22}^{[k-1,k]} + \mathbf{s}_{11}^{[k,k+1]}) \delta \mathbf{x}_k + \mathbf{s}_{12}^{[k,k+1]} \delta \mathbf{x}_{k+1} &= 0; \quad 2 \leq k \leq n-1. \end{aligned} \right\} \quad (4.17)$$

Equation (4.17) can be rewritten as

$$\begin{pmatrix} \delta \mathbf{x}_{n+1} \\ \delta \mathbf{x}_n \end{pmatrix} = \mathbf{J} \begin{pmatrix} \delta \mathbf{x}_1 \\ \delta \mathbf{x}_0 \end{pmatrix}, \quad (4.18)$$

with

$$\mathbf{J} := \prod_{k=1}^n \begin{pmatrix} -\mathbf{s}_{12}^{-1[k,k+1]} (\mathbf{s}_{22}^{[k-1,k]} + \mathbf{s}_{11}^{[k,k+1]}) & -\mathbf{s}_{12}^{-1[k,k+1]} \mathbf{s}_{12}^{[k-1,k]} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}. \quad (4.19)$$

The multipliers of a periodic orbit are the eigenvalues of the derivatives of the return map around the orbit,  $\mathbf{J}$  (Mackay & Meiss 1983), so upon comparison with (4.15),  $\mathbf{J} = \mathbf{T}_S$  confirming the result from (4.16).

The strength of this method resides in the fact that the problem is reduced to finding the eigenvalues of a  $6 \times 6$  matrix. In fact, the size of the operator matrix  $\mathbf{M}$  increases linearly with the number of discretization points, but of interest are solely the eigenvalues of the matrix  $\mathbf{T}_S$ , whose size is  $6 \times 6$ , no matter how many discretization points have been used. Although the number of matrices in the product  $\mathbf{J}$  also increases linearly with the number of discretization points, the most commonly used algorithms to determine the eigenvalues of a matrix have a cost that is generally cubic (Francis 1961, 1962; Kublanovskaya 1962). Since an  $m \times m$  matrix has at most  $m$  distinct eigenvalues, reducing its size is crucial for computational efficiency. The derivation of the aforementioned results in the case where the curve of interest is discretized by piecewise linears is given in Appendix E.

## 5. Conclusion

In this paper, after having introduced the rotational transform, the Hamiltonian behaviour of toroidal magnetic fields was used to motivate the definition of a magnetic action. The latter action served as starting point to express the on-axis rotational transform from a novel approach. The focus has been on elliptical magnetic axes, but this method applies to hyperbolic axes as well, and more generally to any periodic field line.

The action and resulting properties were studied through the lens of the calculus of variations, where variations of the curves' geometry were performed. The first variation led to the result that extremizing curves are magnetic field lines, enabling one to identify a magnetic axis. The second variation sheds light on the geometry of the neighbouring field lines. Studying the null eigenspace of the second variation enabled the derivation of the on-axis rotational transform, with a focus on elliptical axes. The second variation, seen as an operator, was expanded in the near-axis formalism developed by Mercier, Solov'ev and Shafranov, to yield a system of periodic differential equations. The periodicity of the system allowed the use of Floquet theory to solve for the eigenspace. The key result is that the Floquet exponents of the axis were shown to match Mercier's expression (1.3) of the rotational transform. Additionally, solving for the null eigenspace was shown to lead to a Hill equation, from which the Floquet exponents were expressed as solutions of an infinite determinant equation.

Following this continuous derivation of the on-axis rotational transform through the Floquet exponents of the null eigenspace of the second variation, a discrete approach was introduced. It consists of discretizing the field line of interest and by linearity, to define the action as a sum of piecewise actions. This approach was described by Mackay & Meiss (1983) for one-dimensional Lagrangian systems. We provide a generalization of this method as our action is based on field lines which are in essence three-dimensional. Solving for the Floquet exponents was shown to be closely related to solving for the multipliers of the curve, described by Mackay & Meiss (1983) and Greene (1979), and the parallel between the two approaches was made as a consistency check. The on-axis Floquet multiplier is computed by finding the eigenvalues of a  $6 \times 6$  matrix, made of second derivatives of the action. Its efficiency in comparison with field-line tracing will be studied in future work.

Finally, this paper applies the near-axis formalism to recovering the rotational transform from a Lagrangian approach. Although the method has been introduced for elliptical axes, also called *O*-points, it can be applied to hyperbolic axes, or *X*-points, with the only



difference being that the Floquet exponents are real, not purely imaginary; and, more generally, to all periodic orbits. The discrete approach stands as an alternative to compute the rotational transform instead of following field lines. We aim at using this method in stellarator optimization.

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### Declaration of interests

The authors report no conflict of interest.

### Data availability

The data that support the findings of this study are available from the corresponding author upon reasonable request.

### Appendix A. Floquet theory

Floquet theory was formulated by Gaston Floquet towards the end of the 19th century, in his attempt to solve linear differential equations with periodic coefficients (Floquet 1883). Suppose one needs to solve the linear system

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}, \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad (\text{A1a},b)$$

where  $\mathbf{A} \in \mathcal{M}_{n \times n}$  is periodic in  $t$  with period  $T$ . Considering  $n$  linearly independent solutions  $\{\mathbf{x}_1, \dots, \mathbf{x}_n\}$  of (A1a,b), it is useful to introduce the so-called fundamental matrix  $\mathbf{X}$  by grouping  $\mathbf{x}_i$  together:

$$\mathbf{X}(t, t_0) := (\mathbf{x}_1; \mathbf{x}_2; \dots; \mathbf{x}_n), \quad (\text{A2})$$

where  $\mathbf{x}_i$  are column vectors, and the second argument has been added to specify that the initial condition occurs at time  $t_0$ . Thus, (A1a,b) can be rewritten as

$$\frac{d}{dt}\mathbf{X}(t, t_0) = \mathbf{A}(t)\mathbf{X}(t, t_0). \quad (\text{A3})$$

If  $\mathbf{X}(t_0, t_0) = \mathbf{I}$ ,  $\mathbf{X}$  is called the *principal* fundamental matrix. We now state some well-known theorems of Floquet theory, the proofs of which can be found in Meiss (2007).

THEOREM A.1 (Abel). *The determinant of the fundamental matrix  $\mathbf{X}$  is*

$$\det \mathbf{X}(t, t_0) = \exp \int_{t_0}^t \text{tr} \mathbf{A}(s) \, ds. \quad (\text{A4})$$

Moreover, when  $(t_0, t) = (0, T)$ , it can be rewritten as the product of the so-called Floquet multipliers:

$$\det \mathbf{X}(T, 0) = \prod_{i=1}^n e^{v_i T}, \quad (\text{A5})$$

where  $v_i$  are the Floquet exponents.

This second property simply states that the Floquet multipliers are the eigenvalues of the monodromy matrix  $\mathbf{m} := \mathbf{X}(T, 0)$  of the linear system (A3).

THEOREM A.2 (Floquet–Lyapunov). *The fundamental matrix  $\mathbf{X}$  solution of the system (A3) is of the form*

$$\mathbf{X}(t, t_0) = \mathbf{P}(t) e^{(t-t_0)\mathbf{B}}, \quad (\text{A6})$$

where the matrix  $\mathbf{P}$  is symplectic and  $T$ -periodic, with  $\mathbf{P}(t_0) = \mathbf{I}$ , and  $\mathbf{B}$  is a constant Hamiltonian matrix, that is,

$$\mathbf{J}\mathbf{B} = \mathbf{B}^\top \mathbf{J}^\top, \quad \mathbf{J} = \begin{pmatrix} \mathbf{0} & -\mathbf{I} \\ \mathbf{I} & \mathbf{0} \end{pmatrix}. \quad (\text{A7a,b})$$

## Appendix B. Frenet–Serret frame

The Frenet–Serret frame is a local basis that spans the three-dimensional space  $\mathbb{R}^3$ . The terminology ‘local’ arises from the fact that this frame is defined locally along a curve  $\mathcal{C} \subset \mathbb{R}^3$ . Let us define the basis vectors and some essential properties of the frame. We refer to Duignan & Meiss (2021) as they provide the reader with a brief yet efficient description of this frame. The original mathematical developments can be found in Frenet (1852).

The frame is composed of the three well-known *tangent*, *normal* and *binormal* vectors, defined and related to each other as follows. Provided that the curve  $\mathcal{C}$  is described by the vector-valued function  $\mathbf{r}_0 \in \mathbb{R}^3$  and parametrized by the arc length  $\ell$ , the tangent vector  $\mathbf{t}$  is defined as

$$\mathbf{t} := \frac{d\mathbf{r}_0}{d\ell} = \mathbf{r}'_0. \quad (\text{B1})$$

The normal vector accounts for the normalized rate of change of the tangent along the curve:

$$\mathbf{n} := \mathbf{t}' |\mathbf{t}'|^{-1}. \quad (\text{B2})$$

Finally, the binormal vector is the cross product of the tangent and the normal vector:

$$\mathbf{b} := \mathbf{t} \times \mathbf{n}. \quad (\text{B3})$$

The curvature  $\kappa$  and the torsion  $\tau$  of the curve  $\mathcal{C}$  can be expressed from the derivatives of  $\mathbf{r}_0$ :

$$\kappa := |\mathbf{r}''_0| = |\mathbf{t}'|, \quad \tau := \frac{(\mathbf{r}'_0 \times \mathbf{r}''_0) \cdot \mathbf{r}'''_0}{\kappa^2}, \quad (\text{B4a,b})$$

provided that the curvature is non-vanishing. The rate of change of the Frenet–Serret frame along the curve  $\mathcal{C}$  is written in terms of the curvature and the torsion. The resulting

expressions are the so-called Frenet–Serret formulae:

$$\frac{d}{d\ell} \begin{pmatrix} \mathbf{n} \\ \mathbf{b} \\ \mathbf{t} \end{pmatrix} = \begin{pmatrix} 0 & \tau(\ell) & -\kappa(\ell) \\ -\tau(\ell) & 0 & 0 \\ \kappa(\ell) & 0 & 0 \end{pmatrix} \begin{pmatrix} \mathbf{n} \\ \mathbf{b} \\ \mathbf{t} \end{pmatrix}. \quad (\text{B5})$$

With the previous relations, the reader is equipped with what is necessary to delve into near-axis expansion coordinates, based on the Frenet–Serret frame.

### Appendix C. Near-axis expansion coordinates

This appendix is dedicated to the description of two sets of coordinates that have been proven to be powerful in the expansion of operators or physical quantities in the vicinity of field lines. The terminology *near-axis* expansion is used here, as in this paper expansions are carried out around magnetic axes, but those coordinates remain suitable for any near-field-line expansion. [Appendix C.1](#) is dedicated to the description of Mercier coordinates, introduced by Mercier (1964), and [Appendix C.2](#) deals with the Solov’ev–Shafranov coordinates, as described in Solov’ev & Shafranov (1970). They are both closely related, and a correspondence can easily be established between the two. In addition, they both are based on the Frenet–Serret frame, as described in [Appendix B](#). In what follows, the coordinate systems are introduced in the context of magnetic fields, and each curve that is dealt with is assumed to be a field line. In addition, we emphasize that the following two subsections do not describe new material, introduced in the above references. The reader might want to refer to Jorge, Sengupta & Landreman (2020c) and Sengupta *et al.* (2024) for more insight on how near-axis expansion has been used in the context of plasma physics.

#### C.1. Mercier coordinates

Mercier’s frame  $\{\boldsymbol{\rho}, \boldsymbol{\omega}, \mathbf{t}\}$  is based on the Frenet–Serret frame  $\{\mathbf{n}, \mathbf{b}, \mathbf{t}\}$ , and related to the latter by a rotation of the normal and binormal vectors by a polar angle  $\theta$ , which is a purely geometric quantity, as shown in [figure 1](#). The field line (that can be considered to be a magnetic axis here)  $\mathcal{C} \subset \mathbb{R}^3$  and described by  $\mathbf{r}_0$  is assumed closed and parametrized by the arc length  $\ell$  with a total length  $L$ . Following Mercier *et al.* (1974), the near-axis expansion is based on the construction of a tube of radius  $\rho$  around the axis, such that any neighbouring point can be described by  $\rho(\ell)\boldsymbol{\rho}(\ell)$ :

$$\mathbf{r}(\ell) = \mathbf{r}_0(\ell) + \rho(\ell)\boldsymbol{\rho}(\ell), \quad \mathbf{r}'_0(\ell) = \mathbf{t}(\ell), \quad (\text{C1a,b})$$

where the dependence on  $\ell$  has been made explicit, but is omitted in what follows for the sake of readability. However, it is made clear whenever the dependence on quantities is not obvious. Therefore  $\{\boldsymbol{\rho}, \boldsymbol{\omega}, \mathbf{t}\}$  is a right-handed triad related to  $\{\mathbf{n}, \mathbf{b}, \mathbf{t}\}$  by

$$\boldsymbol{\rho} = \mathbf{n} \cos \theta + \mathbf{b} \sin \theta, \quad \boldsymbol{\omega} = \mathbf{b} \cos \theta - \mathbf{n} \sin \theta. \quad (\text{C2a,b})$$

It can be shown by direct computation that  $(\rho, \omega = \theta + \int \tau d\ell', \ell)$ , with  $\tau$  denoting the torsion of the axis  $\mathbf{r}_0$  (see [Appendix B](#)), forms an orthogonal coordinate system with metric

$$ds^2 = d\rho^2 + \rho^2 d\omega^2 + h^2 d\ell^2, \quad h = 1 - \kappa\rho \cos \theta, \quad (\text{C3a,b})$$

where  $\kappa$  denotes the curvature of  $\mathcal{C}$ . The product  $\kappa\rho$  can be seen as a measure of the ratio between the radius of curvature of  $\mathbf{r}_0$  and the radius of expansion around the latter. The

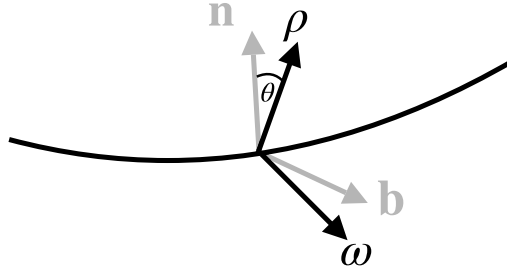


FIGURE 1. Mercier's triad  $\{\rho, \omega, t\}$  related to the usual Frenet–Serret  $\{n, b, t\}$ . The solid black line is a field line.

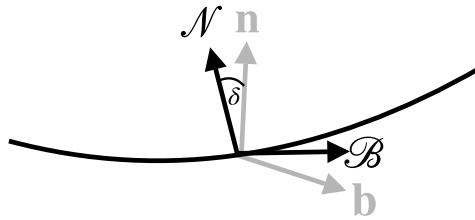


FIGURE 2. Solov'ev–Shafranov triad  $\{\mathcal{N}, \mathcal{B}, t\}$  related to the usual Frenet–Serret  $\{n, b, t\}$ . The solid black line is a field line.

metric tensor reads

$$g_{ij} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \rho^2 & 0 \\ 0 & 0 & h^2 \end{pmatrix}. \quad (\text{C4})$$

Therefore, from the previous characterization of the Mercier coordinates, a few differential identities can be derived. Since the basis considered for Mercier coordinates is the triad  $\{\rho, \omega, t\}$ , the identity tensor is trivially written as

$$I = \rho\rho + \omega\omega + tt. \quad (\text{C5})$$

By means of the metric expression (C1a,b), the gradient can be derived:

$$\nabla = \rho\partial_\rho + \frac{\omega}{\rho}\partial_\omega + \frac{t}{h}\partial_\ell, \quad (\text{C6})$$

where the subscripts denote with respect to which coordinate the partial derivative is taken. Moreover, some pseudo-Poisson formulae can be derived for the derivatives of  $\rho$  and  $\omega$ , to read

$$\rho_{,\omega} = \omega, \quad \rho_{,\ell} = -t\kappa \cos \theta, \quad \omega_{,\omega} = -\rho, \quad \omega_{,\ell} = t\kappa \sin \theta, \quad (\text{C7a–d})$$

where the subscript comma represents partial differentiation. Finally, using (C6), the gradient of each basis vector can be easily expressed:

$$\nabla t = \frac{\kappa}{h}tn, \quad \nabla \rho = \frac{1}{\rho}\omega\omega - \frac{\kappa \cos \theta}{h}tt, \quad \nabla \omega = \frac{\kappa \sin \theta}{h}tt - \frac{1}{\rho}\omega\rho. \quad (\text{C8a–c})$$

In the formalism developed by Mercier, when expanding quantities in powers of  $\rho$ , it can be useful to have an additional ‘phase’ term  $\delta$  that may simplify expressions. The latter

phase is a function of the arc length  $\ell$  and enables one to define the so-called Mercier angle  $u$ :

$$u := \theta + \delta = \omega - \int \tau \, d\ell' + \delta. \quad (\text{C9})$$

In fact, the  $\delta$  term describes the rotation of trajectories around the axis  $\mathbf{r}_0$ . One notes that, by periodicity,  $\delta$  has to satisfy  $\delta(\ell + L) = \delta(\ell) + 2\pi n$ , with  $n \in \mathbb{Z}$ . The rotation function  $\delta$  is important for the Solov'ev–Shafranov near-axis expansion as the latter is based on the construction of ellipses around  $\mathbf{r}_0$ , whose rotation is naturally important.

### C.2. Solov'ev–Shafranov coordinates

The Solov'ev–Shafranov coordinate system is closely related to the Mercier triad, but differs in that the expansion is not carried out by constructing a tube of radius  $\rho$  around the axis, but an ellipse of semi-axes varying along  $\mathbf{r}_0$  as depicted in figure 2. Let  $\{\mathcal{N}, \mathcal{B}, \mathbf{t}\}$  be the orthogonal triad related to the Frenet–Serret frame through a rotation by function  $\delta$  introduced in the Mercier formalism such that

$$\mathcal{N} = \mathbf{n} \cos \delta - \mathbf{b} \sin \delta = \boldsymbol{\rho} \cos u - \boldsymbol{\omega} \sin u, \quad \mathcal{B} = \mathbf{n} \sin \delta + \mathbf{b} \cos \delta = \boldsymbol{\rho} \sin u + \boldsymbol{\omega} \cos u, \quad (\text{C10a,b})$$

where  $\{\boldsymbol{\rho}, \boldsymbol{\omega}, \mathbf{t}\}$  form the Mercier basis, with Mercier angle  $u$  as defined in Appendix C.1. Equation (C10a,b) shows how intrinsically related are the two frames. Again, the field line considered,  $\mathbf{r}_0$ , forms a closed curve  $\mathcal{C}$  parametrized by the arc length  $\ell$  and a total length  $L$ . It is obvious that  $\mathcal{N}$  and  $\mathcal{B}$  depend on  $\ell$ , enabling one to write

$$\mathcal{N}' = -\kappa \cos \delta \mathbf{t} - u' \mathcal{B}, \quad \mathcal{B}' = -\kappa \sin \delta \mathbf{t} + u' \mathcal{N}, \quad (\text{C11a,b})$$

where  $\kappa$  denotes the curvature of  $\mathbf{r}_0$ , which is a local property. Following Solov'ev & Shafranov (1970), the position vector  $\mathbf{r}$  of any point in the vicinity of the axis can be expressed as

$$\mathbf{r} = \mathbf{r}_0 + x\mathcal{N} + y\mathcal{B}, \quad (\text{C12})$$

where  $x$  and  $y$  are expansion parameters depending on the position of the frame  $\{\mathcal{N}, \mathcal{B}, \mathbf{t}\}$  through  $\ell$ . In Mercier coordinates,  $x = \rho \cos u$  and  $y = \rho \sin u$ . Differentiating  $\mathbf{r}$  with respect to the length parameter  $\ell$ :

$$\frac{d\mathbf{r}}{d\ell} = (x' + u'y)\mathcal{N} + (y' - u'x)\mathcal{B} + h\mathbf{t}, \quad (\text{C13})$$

where the prime denotes total differentiation with respect to  $\ell$ , and  $\mathbf{r}'_0 = \mathbf{t}$ . Additionally,  $h = 1 - \kappa(x \cos \delta - y \sin \delta) = 1 - \kappa \rho \cos \theta$  and  $u' = \delta' - \tau$ . It induces the following metric:

$$ds^2 = dx^2 + dy^2 + 2u'(y \, dx - x \, dy) \, d\ell + \left[ h^2 + u'^2(x^2 + y^2) \right] d\ell^2. \quad (\text{C14})$$

The covariant and contravariant forms  $g_{ij}$  and  $g^{ij}$  of the metric tensor for  $ds^2$  read

$$g_{ij} = \begin{pmatrix} 1 & 0 & u'y \\ 0 & 1 & -u'x \\ u'y & -u'x & h^2 + u'^2(x^2 + y^2) \end{pmatrix} \quad (\text{C15})$$

and

$$g^{ij} = \frac{1}{h^2} \begin{pmatrix} h^2 + u'^2 y^2 & -u'^2 xy & -u'y \\ -u'^2 xy & 1 & u'x \\ -u'y & u'x & 1 \end{pmatrix} \quad (C16)$$

with  $\sqrt{|g|} = h$ . A direct computation shows that the following basis vectors lead to the same metric tensors:

$$\left. \begin{aligned} e^1 &= \nabla x = \mathcal{N} - \frac{u'y}{h} \mathbf{t}, & e^2 &= \nabla y = \mathcal{B} + \frac{u'x}{h} \mathbf{t}, & e^3 &= \nabla \ell = \frac{1}{h} \mathbf{t}, \\ e_1 &= h(\nabla y \times \nabla \ell) = \mathcal{N}, & e_2 &= h(\nabla \ell \times \nabla x) = \mathcal{B}, & e_3 &= h(\nabla x \times \nabla y) \\ & & & & &= h\mathbf{t} + u'(-x\mathcal{B} + y\mathcal{N}), \end{aligned} \right\} \quad (C17)$$

which will be useful and, in fact, more convenient in the near-axis expansion of the quantities of interest in this paper. The gradient in the basis  $\{e_j\}_j$  is expressed as

$$\nabla \equiv e_j g^{ij} \frac{\partial}{\partial \alpha^i} \quad (C18)$$

with  $\alpha^1 = x$ ,  $\alpha^2 = y$ ,  $\alpha^3 = \ell$  and the Einstein summation convention.

#### Appendix D. Derivation of the rotational transform from the action in Mercier coordinates

##### D.1. Near-axis expansion of the second variation tensor

As we did in the Solov'ev–Shafranov geometry in the body of this paper, in order to expand the operator

$$\mathbf{M} := \frac{\delta^2 \mathcal{S}}{\delta \mathbf{x} \delta \mathbf{x}} = -(\mathbf{I} \times \mathbf{B}) \frac{d}{d\ell} + \mathbf{x}' \times (\nabla \mathbf{B})^\top \quad (D1)$$

in Mercier coordinates,  $\mathbf{B}$  is expanded in the parameter  $\rho$  such that  $\kappa\rho \ll 1$ . To evaluate  $\mathbf{M}$  to lowest order in  $\rho$ , the magnetic field needs to be expanded up to first order. Therefore, we set

$$\mathbf{B} = B_0(\ell)\mathbf{t} + \rho\mathbf{B}_1, \quad \mathbf{B}_1 = (B_1^\rho \boldsymbol{\rho} + B_1^\omega \boldsymbol{\omega} + B_1^t \mathbf{t}). \quad (D2a,b)$$

The next step in the expansion of  $\mathbf{M}$  is to compute  $\nabla \mathbf{B}$ . It is straightforward from the differential identity (C6):

$$\nabla \mathbf{B} = \frac{1}{h} (\mathbf{t} B_0'(\ell) + \mathbf{t} n \kappa B_0) + \rho \mathbf{B}_1 + \rho \nabla \mathbf{B}, \quad (D3)$$

such that, to leading order

$$\left. \begin{aligned} \mathbf{x}' \times \nabla \mathbf{B}^\top &= \kappa B_0 \mathbf{b} \mathbf{t} + (B_1^\rho \boldsymbol{\omega} - B_1^\omega \boldsymbol{\rho}) \boldsymbol{\rho} + \boldsymbol{\omega} \boldsymbol{\omega} (\partial_\omega B_1^\rho - B_1^\omega) - \boldsymbol{\omega} \boldsymbol{\rho} (\partial_\omega B_1^\omega + B_1^\rho), \\ -(\mathbf{I} \times \mathbf{B}) &= (\boldsymbol{\rho} \boldsymbol{\omega} - \boldsymbol{\omega} \boldsymbol{\rho}) B_0. \end{aligned} \right\} \quad (D4)$$

Finally,  $\mathbf{M}$  as given by (3.4) simplifies to

$$\left. \begin{aligned} \mathbf{M} &:= \mathbf{M}_1 \frac{d}{d\ell} + \mathbf{M}_2, \\ \mathbf{M}_1 &= (\boldsymbol{\rho} \boldsymbol{\omega} - \boldsymbol{\omega} \boldsymbol{\rho}) B_0, \\ \mathbf{M}_2 &= \kappa B_0 \mathbf{b} \mathbf{t} + (B_1^\rho \boldsymbol{\omega} - B_1^\omega \boldsymbol{\rho}) \boldsymbol{\rho} + \boldsymbol{\omega} \boldsymbol{\omega} (\partial_\omega B_1^\rho - B_1^\omega) - \boldsymbol{\omega} \boldsymbol{\rho} (\partial_\omega B_1^\omega + B_1^\rho). \end{aligned} \right\} \quad (D5)$$

So far, the power expansion of the gradient of the magnetic field relied on a generic expression of  $\mathbf{B}$  in terms of  $B_1^\rho$ ,  $B_1^\omega$  and  $B_1^t$ . The latter functions can be determined enforcing

additional constraints such as  $\nabla \cdot \mathbf{B} = 0$  and  $\nabla \times \mathbf{B} = \mathbf{J}_0 = J_0(\ell)\mathbf{t}$ . The divergence and curl of  $\mathbf{B}$  can be evaluated directly from the  $\nabla\mathbf{B}$  tensor, using dyadic algebra. From

$$\left. \begin{aligned} \nabla \cdot \mathbf{B} &= \mathbf{I} : \nabla\mathbf{B} = B'_0 + 2B_1^\rho + \partial_\omega B_1^\omega, \\ \nabla \times \mathbf{B} &= -\mathbf{I}^\times \nabla\mathbf{B} = \mathbf{t} (2B_1^\omega - \partial_\omega B_1^\rho) + (\omega - \rho \partial_\omega) (B_0 \kappa \cos \theta - B_1^t), \end{aligned} \right\} \quad (\text{D6})$$

where  $\partial_\omega := \partial/\partial\omega$ , it can be easily shown that

$$B_1^t = \kappa B_0 \cos \theta = B_{1t}, \quad B_1^\rho = -\frac{1}{2}(B'_0 + \partial_\omega b_1) = B_{1\rho}, \quad B_1^\omega = \frac{1}{2}J_0 + b_1 = B_{1\omega}, \quad (\text{D7a-c})$$

where  $b_1$  satisfies the harmonic equation

$$(\partial_\omega^2 + 4)b_1 = 0. \quad (\text{D8})$$

Following Mercier *et al.* (1974), the solution of (D8) can be expressed as

$$\frac{b_1}{B_0} = b_{c2} \cos(2u) + b_{s2} \sin(2u) = \tanh \eta(\ell) \left( \delta' - \tau + \frac{J_0}{2B_0} \right) \cos(2u) + \frac{\eta'}{2} \sin(2u), \quad (\text{D9})$$

where  $u$  is the Mercier angle as introduced in Appendix C.1. The functions  $\eta(\ell)$  and  $\delta(\ell)$  represent, respectively, the eccentricity and the rotation of the elliptic flux surfaces winding around the axis, given by the flux function

$$\psi = \rho^2 B_0 [\cosh(\eta) + \sinh(\eta) \sin(2u)]. \quad (\text{D10})$$

## D.2. Floquet exponents and rotational transform

Now that the magnetohydrodynamic constraints have been enforced, let us obtain the null eigenvector  $\mathbf{v} = v^t \mathbf{t} + v^\rho \boldsymbol{\rho} + v^\omega \boldsymbol{\omega}$  of  $\mathbf{M}$  such that  $\mathbf{M}\mathbf{v} = \mathbf{0}$ , in order to determine the Floquet exponents. We take advantage of the orthogonality of Mercier coordinates to work with the covariant representation of  $\mathbf{v}$ , so the notation is less heavy. The metric tensor for this set of coordinates is given by (C4). We state that the components  $v_t, v_\rho, v_\omega$  are functions of  $\omega$  and  $\ell$ . The  $\mathbf{M}_1$  terms can be shown to be

$$\mathbf{M}_1 \frac{d}{d\ell} \mathbf{v} = B_0 [\boldsymbol{\rho} (v'_\omega - \kappa \sin \theta v_t) - \boldsymbol{\omega} (v'_\rho + \kappa \cos \theta v_t)]. \quad (\text{D11})$$

Similarly,

$$\begin{aligned} \mathbf{M}_2 \mathbf{v} &= \boldsymbol{\rho} [B_0 v'_\omega + v_\omega (B'_0 + B_{1\rho}) - B_{1\omega} v_\rho] \\ &\quad + \boldsymbol{\omega} [-B_0 v'_\rho + v_\omega (B_{1\omega} - J_0) + B_{1\rho} v_\rho]. \end{aligned} \quad (\text{D12})$$

Thus, we observe that  $\mathbf{M}\mathbf{v} = \mathbf{0}$  only has  $\boldsymbol{\rho}, \boldsymbol{\omega}$  components with  $\ell$  derivatives of  $v_\rho, v_\omega$ . The tangential component  $v_t$  is thus an arbitrary constant, which can be absorbed by redefining



$v_\rho, v_\omega$ . We therefore set  $v_t = 0$ . The condition  $\mathbf{M}\mathbf{v} = \mathbf{0}$  leads to the system

$$B_0 \begin{pmatrix} 0 & +1 \\ -1 & 0 \end{pmatrix} \frac{d}{d\ell} \begin{pmatrix} v_\rho \\ v_\omega \end{pmatrix} + \begin{pmatrix} B_{1\omega} & [B'_0 + B_{1\rho}] \\ B_{1\rho} & [B_{1\omega} - J_0] \end{pmatrix} \begin{pmatrix} v_\rho \\ v_\omega \end{pmatrix} = 0. \quad (\text{D13})$$

Substituting the constraints (D7a–c) into (D13), we obtain the equivalent system of coupled linear partial differential equations:

$$\left. \begin{aligned} v'_\omega + \left( \frac{B'_0}{2B_0} - \frac{\partial_\omega b_1}{2B_0} \right) v_\omega - \left( \frac{J_0/2}{B_0} + \frac{b_1}{B_0} \right) v_\rho &= 0, \\ v'_\rho + \left( \frac{B'_0}{2B_0} + \frac{\partial_\omega b_1}{2B_0} \right) v_\omega + \left( \frac{J_0/2}{B_0} - \frac{b_1}{B_0} \right) v_\omega &= 0. \end{aligned} \right\} \quad (\text{D14})$$

Therefore, the the system (D14) can be represented in the form

$$\frac{d\tilde{\mathbf{v}}}{d\ell} = \mathbf{A}(\ell)\tilde{\mathbf{v}}, \quad \tilde{\mathbf{v}} = (v_\rho, v_\omega)^\top. \quad (\text{D15})$$

Here,  $\mathbf{A}$  is a matrix with components periodic in  $\ell$  with period  $L$ . Note that this representation is possible since all the quantities between brackets are depending on  $\ell$  along the field line from which the expansion is carried out,  $\omega$  included (Solov'ev & Shafranov 1970; Duignan & Meiss 2021). We can directly use the Floquet theorem on (D15) to conclude that the solution must be of the form

$$\tilde{\mathbf{v}} = \mathbf{U}(\ell)\mathbf{e}^{\mathbf{C}\ell/L}, \quad (\text{D16})$$

where  $\mathbf{U}$  is a symplectic periodic matrix and  $\mathbf{C}$  is a constant Hamiltonian matrix. The eigenvalues of  $\mathbf{C}$  are the Floquet exponents, which must be of the form  $\pm i\nu$  ( $\nu \in \mathbb{R}$ ) near an elliptic axis.

An equivalent approach to solve the system from (D14) and hence identify the exponents  $\nu$  is to use the fact that  $b_1$  only has second harmonics in  $u$ . This allows us to seek a solution where  $\tilde{\mathbf{v}}$  only has first harmonics in  $u$ . From now on, we follow this approach. We show that there exist solutions to  $\mathbf{M}\mathbf{v} = \mathbf{0}$  with

$$\begin{pmatrix} v_\rho(\ell) \\ v_\omega(\ell) \end{pmatrix} = \begin{pmatrix} v_{\rho c}(\ell) \\ v_{\omega c}(\ell) \end{pmatrix} \cos u + \begin{pmatrix} v_{\rho s}(\ell) \\ v_{\omega s}(\ell) \end{pmatrix} \sin u. \quad (\text{D17})$$

Substituting the above into (D14), together with the expressions for  $B_{1\rho}, B_{1\omega}$  from (D7a–c), yields first- and third-order harmonics in  $u$ . Equating the terms with first harmonics in  $u$ , the  $v'_\omega$  equation yields

$$\left. \begin{aligned} v'_{\omega c} - (\tau - \delta')v_{\omega s} - \left( \frac{J_0/2}{B_0} \right) v_{\rho c} + \frac{B'_0}{2B_0} v_{\omega c} + \frac{1}{2}(\mathbf{a}_{c2c} + \mathbf{a}_{s2s}) &= 0, \\ v'_{\omega s} + (\tau - \delta')v_{\omega c} - \left( \frac{J_0/2}{B_0} \right) v_{\rho s} + \frac{B'_0}{2B_0} v_{\omega s} + \frac{1}{2}(\mathbf{a}_{s2c} - \mathbf{a}_{c2s}) &= 0. \end{aligned} \right\} \quad (\text{D18})$$

Equating the third harmonic terms leads to the following constraints:

$$\mathbf{a}_{c2c} = \mathbf{a}_{s2s}, \quad \mathbf{a}_{s2c} = -\mathbf{a}_{c2s}, \quad (\text{D19a,b})$$

where the  $\alpha$  terms have the following definitions:

$$\left. \begin{aligned} \alpha_{c2c} &= -\frac{b_{c2}}{B_0} v_{\rho c} - \frac{b_{s2}}{B_0} v_{\omega c}, & \alpha_{s2s} &= -\frac{b_{s2}}{B_0} v_{\rho s} + \frac{b_{c2}}{B_0} v_{\omega s}, \\ \alpha_{s2c} &= -\frac{b_{s2}}{B_0} v_{\rho c} + \frac{b_{c2}}{B_0} v_{\omega c}, & \alpha_{c2s} &= -\frac{b_{c2}}{B_0} v_{\rho s} - \frac{b_{s2}}{B_0} v_{\omega s}. \end{aligned} \right\} \quad (\text{D20})$$

The constraint (D19a,b) and the definitions (D20) imply, together with  $b_{s2}^2 + b_{c2}^2 \neq 0$ , that

$$v_{\rho s} = v_{\omega c}, \quad v_{\rho c} = -v_{\omega s}, \quad (\text{D21a,b})$$

which allows us to rewrite (D18) solely in terms of  $v_{\omega c}$ ,  $v_{\omega s}$ . Simplification leads to

$$\left. \begin{aligned} v'_{\omega c} + \left( \frac{B'_0}{2B_0} - \frac{b_{s2}}{B_0} \right) v_{\omega c} + \left( \frac{J_0/2}{B_0} - \tau + \delta' + \frac{b_{c2}}{B_0} \right) v_{\omega s} &= 0, \\ v'_{\omega s} + \left( \frac{B'_0}{2B_0} + \frac{b_{s2}}{B_0} \right) v_{\omega s} - \left( \frac{J_0/2}{B_0} - \tau + \delta' - \frac{b_{c2}}{B_0} \right) v_{\omega c} &= 0. \end{aligned} \right\} \quad (\text{D22})$$

Finally, substituting (D9), we get

$$\left. \begin{aligned} v'_{\omega c} + \left( \frac{B'_0}{2B_0} - \frac{\eta'}{2} \right) v_{\omega c} + \frac{e^{+\eta}}{\cosh(\eta)} \left( \frac{J_0/2}{B_0} - \tau + \delta' \right) v_{\omega s} &= 0, \\ v'_{\omega s} + \left( \frac{B'_0}{2B_0} + \frac{\eta'}{2} \right) v_{\omega s} - \frac{e^{-\eta}}{\cosh(\eta)} \left( \frac{J_0/2}{B_0} - \tau + \delta' \right) v_{\omega c} &= 0. \end{aligned} \right\} \quad (\text{D23})$$

It is possible to further simplify the system by introducing new variables  $X$ ,  $Y$ :

$$v_{\omega c} = \frac{1}{\sqrt{B'_0}} e^{+\eta/2} X(\ell), \quad v_{\omega s} = \frac{1}{\sqrt{B'_0}} e^{-\eta/2} Y(\ell) \quad (\text{D24a,b})$$

such that (D23) reduces to

$$X' + \Omega_0(\ell)Y = 0, \quad Y' - \Omega_0(\ell)X = 0, \quad \Omega_0(\ell) = \frac{\frac{J_0/2}{B_0} - \tau + \delta'}{\cosh \eta}. \quad (\text{D25a-c})$$

One notes that (D24a,b) has the exact same form as (3.27a,b), which describe a harmonic oscillator system with a ‘time’-dependent frequency  $\Omega_0$ . Once again, using the complex variable  $Z = X + iY$ , the system can be rewritten as a single complex ODE:

$$Z' - i\Omega_0 Z = 0 \quad \Rightarrow \quad Z(\ell) = Z_0 \exp \int_0^\ell i\Omega_0(s) ds. \quad (\text{D26})$$

Separating the periodic and non-periodic parts of the exponential we get

$$\left. \begin{aligned} Z(\ell) &= Z_p(\ell) e^{i v \ell / L}, \quad Z_p(\ell) = Z_0 \exp \int_0^\ell i \tilde{\Omega}_0(s) ds, \\ \bar{\Omega} &\equiv \frac{1}{L} \int_0^L \Omega_0(s) ds, \quad \tilde{\Omega}_0 = \Omega_0(\ell) - \bar{\Omega}. \end{aligned} \right\} \quad (\text{D27})$$

Comparing (D27) with (D16), we find that  $\nu$ , given by

$$\nu = \int_0^L \Omega_0(s) ds \pmod{2\pi} = \oint \frac{\frac{J_0(s)/2}{B_0(s)} - \tau(s) + \delta'(s)}{\cosh \eta(s)} ds \pmod{2\pi}, \quad (\text{D28})$$

is the Floquet exponent for the system. It confirms the result obtained in the Solov'ev–Shafranov coordinates, and it is a second confirmation that the second variation of the magnetic field action, when expanded by means of a near-axis formalism and combined with Floquet theory, yields the correct on-axis rotational transform. This was expected since the result is independent of coordinates.

### Appendix E. Discrete formalism: the piecewise linear discretization

A discrete method to compute the on-axis Floquet exponent has been introduced in § 4. Here, we give the computations in the particular case of the curve being broken down to a concatenation of segments. Recall that the discrete action was introduced as

$$\mathcal{S} = \sum_{i=1}^{n-1} \int_{\mathcal{C}_i} \mathbf{A} \cdot d\boldsymbol{\ell}. \quad (\text{E1})$$

Thus, taking  $\mathcal{C}_i$  to be segments, the action sums up to a sum of integrals along piecewise linears  $\mathcal{C}_i := \{\mathbb{R}^3 \ni \mathbf{x} = \zeta(\mathbf{x}_{i+1} - \mathbf{x}_i) + \mathbf{x}_i \mid \zeta \in [0, 1]\}$ :

$$\left. \begin{aligned} \mathcal{S} &= \sum_{i=1}^{n-1} S(\mathbf{x}_i, \mathbf{x}_{i+1}), \\ S(\mathbf{x}_i, \mathbf{x}_{i+1}) &= \int_0^1 d\zeta \mathbf{A}(\zeta(\mathbf{x}_{i+1} - \mathbf{x}_i) + \mathbf{x}_i) \cdot (\mathbf{x}_{i+1} - \mathbf{x}_i) \\ &= \int_0^1 d\zeta \mathbf{A}(\mathbf{v}(\zeta)) \cdot \mathbf{u}, \end{aligned} \right\} \quad (\text{E2})$$

with  $\mathbf{x}(0) = \mathbf{x}_i$  and  $\mathbf{x}(1) = \mathbf{x}_{i+1}$ , and the vector functions  $\mathbf{u}$  and  $\mathbf{v}$  defined as follows:

$$\left. \begin{aligned} \mathbf{u}(\mathbf{x}_i, \mathbf{x}_{i+1}) &= \mathbf{x}_{i+1} - \mathbf{x}_i, \\ \mathbf{v}(\zeta, \mathbf{x}_i, \mathbf{x}_{i+1}) &= \zeta(\mathbf{x}_{i+1} - \mathbf{x}_i) + \mathbf{x}_i, \end{aligned} \right\} \quad (\text{E3})$$

such that  $\nabla_{\mathbf{x}_i} \mathbf{u}(\mathbf{x}_i, \mathbf{x}_{i+1}) = -\mathbf{I}$ ,  $\nabla_{\mathbf{x}_{i+1}} \mathbf{u}(\mathbf{x}_i, \mathbf{x}_{i+1}) = \mathbf{I}$ ,  $\nabla_{\zeta} \mathbf{v}(\zeta, \mathbf{x}_i, \mathbf{x}_{i+1}) = (1 - \zeta)\mathbf{I}$ ,  $\nabla_{\mathbf{x}_{i+1}} \mathbf{v}(\zeta, \mathbf{x}_i, \mathbf{x}_{i+1}) = \zeta\mathbf{I}$ . Since

$$\begin{aligned} \nabla_{\mathbf{x}_i} [\mathbf{A}(\mathbf{v}(\mathbf{x}_i)) \cdot \mathbf{u}(\mathbf{x}_i)] &= \mathbf{J}_{\mathbf{A}\circ\mathbf{v}}^T(\mathbf{x}_i) \mathbf{u}(\mathbf{x}_i) + \mathbf{J}_{\mathbf{u}}^T(\mathbf{x}_i) \mathbf{A}(\mathbf{v}(\mathbf{x}_i)) \\ &= [\mathbf{J}_{\mathbf{v}}^T(\mathbf{x}_i) \mathbf{J}_{\mathbf{A}}^T(\mathbf{v}(\mathbf{x}_i))] \mathbf{u}(\mathbf{x}_i) + \mathbf{J}_{\mathbf{u}}^T(\mathbf{x}_i) \mathbf{A}(\mathbf{v}(\mathbf{x}_i)), \end{aligned} \quad (\text{E4})$$

where  $\mathbf{J}_{\mathbf{a}}^T$  stands for the transpose of the Jacobian matrix of the vector field  $\mathbf{a}$ , one gets the following first-order derivatives of the action:

$$\left. \begin{aligned} \mathcal{S}_1^{[i, i+1]} &= \int_0^1 (1 - \zeta) \mathbf{J}_{\mathbf{A}}^T(\zeta(\mathbf{x}_{i+1} - \mathbf{x}_i) + \mathbf{x}_i) (\mathbf{x}_{i+1} - \mathbf{x}_i) - \mathbf{A}(\zeta(\mathbf{x}_{i+1} - \mathbf{x}_i) + \mathbf{x}_i) d\zeta, \\ \mathcal{S}_2^{[i-1, i]} &= \int_0^1 \zeta \mathbf{J}_{\mathbf{A}}^T(\zeta(\mathbf{x}_i - \mathbf{x}_{i-1}) + \mathbf{x}_{i-1}) (\mathbf{x}_i - \mathbf{x}_{i-1}) + \mathbf{A}(\zeta(\mathbf{x}_i - \mathbf{x}_{i-1}) + \mathbf{x}_{i-1}) d\zeta. \end{aligned} \right\} \quad (\text{E5})$$

A direct computation of the second derivatives reads

$$\left. \begin{aligned} \mathbf{s}_{12}^{[i-1,i]} &= \int_0^1 d\zeta \zeta \left[ (1-\zeta) \nabla_v \mathbf{J}_A^T(\mathbf{v})(\mathbf{x}_i - \mathbf{x}_{i-1}) - \mathbf{J}_A^T(\mathbf{v}) \right] + (1-\zeta) \mathbf{J}_A(\mathbf{v}), \\ \mathbf{s}_{21}^{[i,i+1]} &= \int_0^1 d\zeta (1-\zeta) \left[ \zeta \nabla_v \mathbf{J}_A^T(\mathbf{v})(\mathbf{x}_{i+1} - \mathbf{x}_i) + \mathbf{J}_A^T(\mathbf{v}) \right] + \zeta \mathbf{J}_A(\mathbf{v}), \\ \mathbf{s}_{22}^{[i-1,i]} &= \int_0^1 d\zeta \zeta \left[ \zeta \nabla_v \mathbf{J}_A^T(\mathbf{v})(\mathbf{x}_i - \mathbf{x}_{i-1}) + (\mathbf{J}_A(\mathbf{v}) + \mathbf{J}_A^T(\mathbf{v})) \right], \\ \mathbf{s}_{11}^{[i,i+1]} &= \int_0^1 d\zeta (1-\zeta) \left[ (1-\zeta) \nabla_v \mathbf{J}_A^T(\mathbf{v})(\mathbf{x}_{i+1} - \mathbf{x}_i) - (\mathbf{J}_A(\mathbf{v}) + \mathbf{J}_A^T(\mathbf{v})) \right]. \end{aligned} \right\} \quad (\text{E6})$$

The second-order derivatives above can be easily computed numerically. We recall that the multipliers  $\lambda$  of the curve are given by solutions of

$$\left. \begin{aligned} \det \mathbf{M}(\lambda) &= (-1)^n \lambda^{-3} \det(\mathbf{T}_S - \lambda \mathbf{I}_6) \det \left( \prod_{i=1}^n \mathbf{s}_{12}[i, i+1] \right), \\ \mathbf{T}_S &= \prod_{i=1}^n \begin{pmatrix} -\mathbf{s}_{12}^{-1[i,i+1]} (\mathbf{s}_{22}^{[i-1,i]} + \mathbf{s}_{11}^{[i,i+1]}) & -\mathbf{s}_{12}^{-1[i,i+1]} \mathbf{s}_{12}^{[i-1,i]} \\ \mathbf{I}_3 & 0 \end{pmatrix}, \end{aligned} \right\} \quad (\text{E7})$$

and that they are linked to the Floquet exponent by  $\lambda = e^{i\nu}$ . From the set of expressions (E6), the determinant of  $\mathbf{M}$  can be computed, leading to the multipliers. The above can be implemented numerically as a novel tool to compute the on-axis Floquet exponent.

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