Appendix B
Light-front coordinates, rapidity, etc.

The use of light-front variables, rapidity and pseudo-rapidity is very common in treating high-energy scattering, particularly in hadron-hadron and lepton-hadron collisions. The essential features of these collisions that make these variables of utility are the presence of ultra-relativistic particles and a preferred axis.

B.1 Definition

Light-front coordinates are defined by a change of variables from the usual (t, x, y, z) [or (0, 1, 2, 3)] coordinates. Given a vector V^μ, its light-front components are defined by

$$V^+ = \frac{V^0 + V^3}{\sqrt{2}}, \quad V^- = \frac{V^0 - V^3}{\sqrt{2}}, \quad \boldsymbol{V}_\mathrm{T} = (V^1, V^2), \tag{B.1}$$

and I will write the components in the order $V^\mu = (V^+, V^-, \boldsymbol{V}_\mathrm{T})$. Some authors prefer to omit the $1/\sqrt{2}$ factor in (B.1), but among the reasons not to is that the change of variable from ordinary coordinates has unit Jacobian. Thus the element of volume is simply

$$\mathrm{d}^4 k = \mathrm{d}k^+ \, \mathrm{d}k^- \, \mathrm{d}^2\boldsymbol{k}_\mathrm{T} \,. \tag{B.2}$$

What are the motivations for defining such coordinates, which evidently depend on a particular choice of the z axis? One is that these coordinates transform very simply under boosts along the z axis. Another is that when a vector is highly boosted along the z axis, light-front coordinates nicely show what are the large and small components of momentum. Typically one uses light-front coordinates in a situation like high-energy hadron scattering. In that situation, there is a natural choice of an axis, the collision axis, and one frequently needs to transform between different frames related by boosts along the axis. Commonly used frames include the rest frame of one of the incoming particles, the overall center-of-mass frame, and the center-of-mass frame of a partonic subprocess.

It can easily be verified that Lorentz-invariant scalar products have the form

$$\begin{aligned} V \cdot W &= V^+ W^- + V^- W^+ - \boldsymbol{V}_\mathrm{T} \cdot \boldsymbol{W}_\mathrm{T}, \\ V \cdot V &= 2V^+ V^- - V_\mathrm{T}^2. \end{aligned} \tag{B.3}$$

It follows that the metric tensor has as its non-zero components $g_{+-} = g_{-+} = 1$, $g_{ij} = -\delta_{ij}$, where the indices i and j refer to the two transverse coordinates.

It is important to make a distinction between contravariant vectors, whose indices are superscripts, and covariant vectors, whose indices are subscripts. Indices are contracted by the Einstein summation convention only between upper and lower indices, as in $g_{\mu\nu} V^\mu W^\nu$. Contravariant and covariant vectors are transformed into each other by the metric tensor, e.g., $V_\mu = g_{\mu\nu} V^\nu$. It is readily checked that the components of the metric tensor do not change their values when

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both indices are changed from contravariant to covariant, $g^{+-} = g^{-+} = 1$, $g^{ij} = -\delta_{ij}$, but that the mixed tensor g^{μ}_{ν} is just a Kronecker delta.

We will choose to treat ordinary coordinate vectors and momentum vectors as naturally contravariant. Derivatives with respect to these are then naturally covariant:

$$\partial_\mu f(x) \stackrel{\text{def}}{=} \frac{\partial f}{\partial x^\mu}, \tag{B.4}$$

so that a Taylor expansion can be written without any metric tensor: $f(a+x) = f(a) + a^\mu \partial_\mu f(a) + O(a^2)$. Notice also, for example, that $\partial_+ f = \partial f / \partial x^+$. Thus the corresponding contravariant derivative (with upstairs indices) has the slightly counterintuitive of being with respect to the opposite coordinate, $\partial^+ f = \partial f / \partial x^-$, and similarly for $\partial^- f$.

B.2 Boosts

Let us make a boost in the z direction to make a new vector V'^μ. In the ordinary (t, x, y, z) components we have the well-known formulae

$$V'^0 = \frac{V^0 + vV^z}{\sqrt{1-v^2}}, \quad V'^z = \frac{vV^0 + V^z}{\sqrt{1-v^2}}, \quad V'^x = V^x, \quad V'^y = V^y. \tag{B.5}$$

It is easy to derive the following for the light-front components:

$$V'^+ = V^+ e^\psi, \quad V'^- = V^- e^{-\psi}, \quad \boldsymbol{V'}_\mathrm{T} = \boldsymbol{V}_\mathrm{T}, \tag{B.6}$$

where the hyperbolic angle ψ is $\frac{1}{2} \ln \frac{1+v}{1-v}$, so that $v = \tanh \psi$.

Notice that if we apply two boosts of parameters ψ_1 and ψ_2 the result is a boost $\psi_1 + \psi_2$. This is clearly simpler than the corresponding result expressed in terms of velocities.

B.3 Rapidity

B.3.1 Boost of particle momentum

Consider a particle of mass m that is obtained by a boost ψ in the z direction from the particle's rest frame. Its momentum is

$$p^\mu = \left(p^+, \frac{m^2}{2p^+}, \mathbf{0}_\mathrm{T} \right) = \left(\frac{m}{\sqrt{2}} e^\psi, \frac{m}{\sqrt{2}} e^{-\psi}, \mathbf{0}_\mathrm{T} \right). \tag{B.7}$$

Notice that if the boost is very large (positive or negative), only one of the two non-zero light-front components of p^μ is large; the other component becomes small. With the usual coordinates, two of the components, p^0 and p^z, become large.

Suppose next that we have two such particles, p_1 and p_2, with the boost for particle 1 being much larger than that for particle 2. Then in the scalar product of the two momenta only one component of each momentum dominates the result, so that for example $(p_1 + p_2)^2 \simeq 2p_1^+ p_2^-$. This implies that, when analyzing the sizes of scalar products of highly boosted particles, it is simpler to use light-front components than to use conventional components.

B.3.2 Definition of rapidity

Since the ratio p^+/p^- gives a measure $e^{2\psi}$ of the boost from the rest frame, we are led to the following definition of a quantity called "rapidity":

$$y = \frac{1}{2} \ln \frac{p^+}{p^-} = \frac{1}{2} \ln \frac{E + p^z}{E - p^z}, \tag{B.8}$$

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which can be applied to a particle of non-zero transverse momentum. The 4-momentum of a particle of rapidity y and transverse momentum $\boldsymbol{p}_{\mathrm{T}}$ is

$$p^{\mu}=\left(e^{y}\sqrt{\frac{m^2+p_{\mathrm{T}}^2}{2}},\ e^{-y}\sqrt{\frac{m^2+p_{\mathrm{T}}^2}{2}},\ \boldsymbol{p}_{\mathrm{T}}\right), \tag{B.9}$$

with $\sqrt{m^2+p_{\mathrm{T}}^2}$ being called the transverse mass m_{T} of the particle. It can be checked that the scalar product of two momenta is

$$p_1\cdot p_2=m_{1\mathrm{T}}m_{2\mathrm{T}}\cosh(y_1-y_2)-\boldsymbol{p}_{\mathrm{T}1}\cdot\boldsymbol{p}_{\mathrm{T}2}. \tag{B.10}$$

In the case where the transverse momenta are negligible, this reduces to $m^2\cosh(y_1-y_2)$, which is like the formula for the product of two Euclidean vectors $\boldsymbol{p}_{\mathrm{T}1}\cdot\boldsymbol{p}_{\mathrm{T}2}=p_1p_2\cos\theta$, with the trigonometric cosine being replaced by the hyperbolic cosine.

B.3.3 Transformation under boosts

Under a boost in the z direction, rapidity transforms additively:

$$y\mapsto y'=y+\psi. \tag{B.11}$$

This implies that in situations where we have a frequent need to work with boosts along the z axis it is economical to label the momentum of a particle by its rapidity and transverse momentum, rather than to use 3-momentum.

B.3.4 Phase-space integration

The standard Lorentz-invariant phase-space integration measure for an on-shell particle of mass m is readily converted to light-front coordinates, or to rapidity and transverse momentum:

$$\frac{\mathrm{d}^3\mathbf{p}}{2E_p(2\pi)^3}=\frac{\mathrm{d}p^+\,\mathrm{d}^2\boldsymbol{p}_{\mathrm{T}}}{2p^+(2\pi)^3}=\frac{\mathrm{d}y\,\mathrm{d}^2\boldsymbol{p}_{\mathrm{T}}}{2(2\pi)^3}, \tag{B.12}$$

where $E_{\boldsymbol{p}}=\sqrt{\boldsymbol{p}^2+m^2}$. Observe that the light-front version does not depend explicitly on the mass, and that there is a restriction to only positive p^+.

B.3.5 Non-relativistic limit

For a *non*-relativistic particle, rapidity is the same as velocity along the z axis, for then

$$y=\frac{1}{2}\ln\frac{E+p^z}{E-p^z}=\frac{1}{2}\ln\frac{1+v^z}{1-v^z}\simeq v^z. \qquad (v_z\ll 1) \tag{B.13}$$

Non-relativistic velocities transform additively under boosts, and the non-linear change of variable from velocity to rapidity allows this additive rule (B.11) to apply to relativistic particles (but only in one direction of boost).

One way of seeing this is as follows. The relativistic law for addition of velocities in one dimension is

$$\beta_{13}=\frac{\beta_{12}+\beta_{23}}{1+\beta_{12}\beta_{23}}, \tag{B.14}$$

where β_{12} is the velocity of some object 1 measured in the rest frame of object 2, etc. This formula is reminiscent of the following property of hyperbolic tangents:

$$\tanh(A + B) = \frac{\tanh A + \tanh B}{1 + \tanh A \tanh B}. \tag{B.15}$$

So to obtain a linear addition law, we should write $\beta_{12} = \tanh A_{12}$, etc. Then the rule (B.14) for the addition of velocities becomes simply $A_{13} = A_{12} + A_{23}$. The A variables are exactly relative rapidities, since

$$v^z = \frac{p^z}{E} = \frac{p^+ - p^-}{p^+ + p^-} = \tanh y. \tag{B.16}$$

B.3.6 Relative velocity

Rapidity is the natural relativistic velocity variable. Suppose we have a proton and a pion with the same rapidity at $p_{\mathrm{T}} = 0$. Then they have no relative velocity; to see this, one just boosts to the rest frame of one of the particles. But these same particles have very different energies: $E_p = \frac{m_p}{m_\pi} E_\pi$.

B.4 Pseudo-rapidity

As I will now explain, the rapidity of a particle can easily be measured in a situation where its mass is negligible, for then it is simply related to the polar angle of the particle.

First let us define the pseudo-rapidity of a particle by

$$\eta = -\ln \tan \frac{\theta}{2}, \tag{B.17}$$

where θ is the angle of the 3-momentum of the particle relative to the $+z$ axis. It is easy to derive an expression for rapidity in terms of pseudo-rapidity and transverse momentum:

$$y = \ln \frac{\sqrt{m^2 + p_{\mathrm{T}}^2 \cosh^2 \eta} + p_{\mathrm{T}} \sinh \eta}{\sqrt{m^2 + p_{\mathrm{T}}^2}}. \tag{B.18}$$

In the limit that $m \ll p_{\mathrm{T}}$, $y \to \eta$. This accounts both for the name "pseudo-rapidity" and for the ubiquitous use of pseudo-rapidity in high-transverse-momentum physics. Angles, and hence pseudo-rapidity, are easy to measure. But it is really the rapidity that is of physical significance: for example, the distribution of particles in a minimum bias event is approximately uniform in rapidity over the kinematic range available.

The distinction between rapidity and pseudo-rapidity is very clear when one examines the kinematic limits on the two variables. In a collision of a given energy, there is a limit to the energy of the particles that can be produced. This can easily be translated to limits on the rapidities of the produced particles of a given mass. But there is no limit on the pseudo-rapidity, since a particle can be physically produced at zero angle (or at 180°), where its pseudo-rapidity is infinite. The particles for which the distinction is very significant are those for which the transverse momentum is substantially less than the mass. Note: (B.18) implies that $|y| < |\eta|$ always.

B.5 Rapidity distributions in high-energy collisions

In the most common events in high-energy hadronic collisions (the so-called "minimum bias events"), the distribution of final-state hadrons is approximately uniform in rapidity (Alner *et al.*,

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1986; Abe *et al.*, 1990; ATLAS Collaboration, 2010; Khachatryan *et al.*, 2010). That is, the distribution of final-state hadrons is approximately invariant under boosts in the z direction.

In contrast, the distribution in angle $\mathrm{d}N / \mathrm{d}\Omega$ is strongly peaked at forward and backward angles. This follows from the Jacobian between $\cos\theta$ and pseudo-rapidity:

$$\frac{\mathrm{d}\eta}{\mathrm{d}\cos\theta} = \cosh^2\eta = \frac{1}{\sin^2\theta}. \tag{B.19}$$

It follows that rapidity and transverse momentum are appropriate variables for analyzing data, and that detector elements should be approximately uniformly spaced in rapidity. (What is physically possible is to make a detector uniform in the pseudo-rapidity discussed in Sec. B.4.) This is in contrast to the situation for e^+e^- collisions where most of the interest is in events generated via annihilation into an electro-weak boson. Such events are much closer to uniform in solid angle than uniform in rapidity.

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