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# A CERTAIN CLASS OF GENERATING FUNCTIONS INVOLVING BILATERAL SERIES

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**Abstract**

The authors derive a general theorem on partly bilateral and partly unilateral generating functions involving multiple series with essentially arbitrary coefficients. By appropriately specialising these coefficients, a number of (known or new) results are shown to follow as applications of the theorem.

## 1. Introduction, definitions and notation

In the usual notation ${}_pF_q$ for a generalised hypergeometric function with $p$ numerator and $q$ denominator parameters, let

$$F_n^m(x) := \frac{1}{m!n!}\,{}_1F_1(-n; m+1; x)$$
$$= \frac{1}{(m+n)!}\,L_n^{(m)}(x), \tag{1.1}$$

where $L_n^{(\alpha)}(x)$ denotes the classical Laguerre polynomials defined by (see, for example, [15, Chapter 5])

$$L_n^{(\alpha)}(x) := \sum_{k=0}^{n} \binom{n+\alpha}{n-k} \frac{(-x)^k}{k!}. \tag{1.2}$$

An interesting (partly bilateral and partly unilateral) generating function for $F_n^m(x)$, due to Exton [2, p. 147, Equation (3)], is recalled here in the following (*modified*)

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form (see [8, 9]):

$$\exp\left(y + z - \frac{xz}{y}\right) = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} F_n^m(x)\, y^m z^n \tag{1.3}$$

or, equivalently,

$$\exp\left(y + z - \frac{xz}{y}\right) = \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} {}_1F_1(-n; m+1; x) \frac{y^m}{m!} \frac{z^n}{n!}, \tag{1.4}$$

where, and in what follows,

$$m^* := \max(0, -m) \quad (m \in \mathbb{Z} := \{0, \pm 1, \pm 2, \dots\}). \tag{1.5}$$

Exton's generating function (1.3) has since been extended by a number of workers including (for example) Pathan and Yasmeen ([8] and [9]), Kamarujjama *et al.* [7], Srivastava *et al.* [14], and Gupta *et al.* [4]. The present sequel to these earlier papers is motivated largely by the aforementioned work of Kamarujjama *et al.* [7] in which the generating function in (1.3) was extended to hold true for the product of *three* Hubbell-Srivastava functions $\omega_N^\nu(x)$ defined by [6, p. 351, Equation (3.1)]

$$\omega_N^\nu(x) := (\nu)_N \sum_{k=0}^{\infty} \frac{\Omega_k\, x^{N-2k}}{(1-\nu-N)_k}, \tag{1.6}$$

where $(\lambda)_\mu := \Gamma(\lambda+\mu)/\Gamma(\lambda)$ denotes the Pochhammer symbol, $\{\Omega_n\}_{n=0}^{\infty}$ is a suitably bounded sequence of complex numbers, and the parameters $\nu$ and $N$ are unrestricted, in general. We aim here at presenting a general theorem on partly bilateral and partly unilateral generating functions involving multiple series with essentially arbitrary coefficients. We also show how a number of (known or new) results can be deduced from the theorem by appropriately specialising these coefficients. For an indication of applications of the various special hypergeometric functions and polynomials in one, two and more variables, which are involved in the results presented in this paper, we refer the interested reader to the works of (for example) Srivastava *et al.* ([12, 13] and [14]) and indeed *also* to many of the references which are already cited in these earlier works.

For convenience, a few conventions and some notation are introduced here:

(1) Boldface letters denote vectors of dimension $r$; for instance, we have

$$\mathbf{m} = (m_1, \dots, m_r), \quad \mathbf{n} = (n_1, \dots, n_r) \quad \text{and} \quad \mathbf{k} = (k_1, \dots, k_r).$$

(2) The symbol $\{\Omega(m, n, k)\}$ denotes a triple sequence and the symbol $\{\Omega(\mathbf{m}, \mathbf{n}, \mathbf{k})\}$ denotes a multiple ($3r$-dimensional) sequence:

$$\{\Omega(m_1, \dots, m_r; n_1, \dots, n_r; k_1, \dots, k_r)\}.$$

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Sufficient conditions to ensure absolute convergence are understood to hold true, but each of these sequences is otherwise arbitrary.

(3) Quite frequently, multiple series are written in simplified notation. Thus, for $p, q \in \mathbb{Z}$,

$$\sum_{\mathbf{m}=p}^{q} \text{ means } \sum_{m_1=p}^{q} \cdots \sum_{m_r=p}^{q}, \qquad \sum_{\mathbf{m},\mathbf{n}=p}^{q} \text{ means } \sum_{m_1=p}^{q} \cdots \sum_{m_r=p}^{q} \cdot \sum_{n_1=p}^{q} \cdots \sum_{n_r=p}^{q}$$

and

$$\sum_{\mathbf{m},\mathbf{n},\mathbf{k}=p}^{q} \text{ means } \sum_{m_1=p}^{q} \cdots \sum_{m_r=p}^{q} \cdot \sum_{n_1=p}^{q} \cdots \sum_{n_r=p}^{q} \cdot \sum_{k_1=p}^{q} \cdots \sum_{k_r=p}^{q},$$

with the usual meaning when $p$ or $q$ (or both $p$ and $q$) are replaced by $r$-dimensional vectors with integer elements, so that (for example) $\sum_{\mathbf{k}=\mathbf{p}}^{\mathbf{q}}$ means $\sum_{k_1=p_1}^{q_1} \cdots \sum_{k_r=p_r}^{q_r}$.

## 2. Generating functions involving bilateral series

Our main results on generating functions involving bilateral series are given by the following theorem.

THEOREM. *Let* $\{\Omega(m, n, k)\}$ *be a suitably bounded triple sequence of complex numbers. Also let* $m^*$ *be defined by* (1.5). *Then*

$$\sum_{m,n,k=0}^{\infty} \Omega(m, n, k) \frac{y^m}{m!} \frac{z^n}{n!} \frac{(-xz/y)^k}{k!}$$
$$= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{y^m}{m!} \frac{z^n}{n!} \sum_{k=0}^{n} \binom{n}{k} \Omega(m+k, n-k, k) \frac{(-x)^k}{(m+1)_k}, \tag{2.1}$$

*provided that each member of* (2.1) *exists.*

*More generally, for a suitably bounded multiple* ($3r$*-dimensional*) *sequence*

$$\{\Omega(\mathbf{m}, \mathbf{n}, \mathbf{k})\}$$

*of complex numbers, if* $\mathbf{m}^* = (m_1^*, \ldots, m_r^*)$ *with* $m_j^* := \max(0, -m_j)$ $(m_j \in \mathbb{Z};$ $j = 1, \ldots, r)$, *then*

$$\sum_{\mathbf{m},\mathbf{n},\mathbf{k}=0}^{\infty} \Omega(\mathbf{m}, \mathbf{n}, \mathbf{k}) \prod_{j=1}^{r} \left\{ \frac{y_j^{m_j}}{m_j!} \frac{z_j^{n_j}}{n_j!} \frac{(-x_j z_j/y_j)^{k_j}}{k_j!} \right\}$$
$$= \sum_{\mathbf{m}=-\infty}^{\infty} \sum_{\mathbf{n}=\mathbf{m}^*}^{\infty} \prod_{j=1}^{r} \left\{ \frac{y_j^{m_j}}{m_j!} \frac{z_j^{n_j}}{n_j!} \right\} \sum_{\mathbf{k}=0}^{\mathbf{n}} \Omega(\mathbf{m}+\mathbf{k}, \mathbf{n}-\mathbf{k}, \mathbf{k}) \prod_{j=1}^{r} \left\{ \binom{n_j}{k_j} \frac{(-x_j)^{k_j}}{(m_j+1)_{k_j}} \right\}, \tag{2.2}$$

*provided that each member of* (2.2) *exists.*

PROOF. Denote, for convenience, the first member of the assertion (2.1) by $\mathcal{S}(x, y, z)$. Then it is easily seen that

$$\mathcal{S}(x, y, z) = \sum_{m,n,k=0}^{\infty} \Omega(m, n, k) \frac{y^{m-k}}{m!} \frac{z^{n+k}}{n!} \frac{(-x)^k}{k!}. \tag{2.3}$$

Upon replacing the summation indices $m$ and $n$ in (2.3) by $m + k$ and $n - k$, respectively, if we rearrange the resulting triple series (which can be justified by absolute convergence of the series involved), we are led finally to the generating function (2.1)

The derivation of the (*multidimensional*) assertion (2.2) runs parallel to that of (2.1) and we skip the details.

## 3. Applications of the theorem

First of all, in its special case when

$$\Omega(m, n, k) \equiv 1, \tag{3.1}$$

the assertion (2.1) would obviously correspond to the generating functions (1.3) and (1.4). Secondly, upon setting

$$\Omega(m, n, k) = (\lambda)_L(\mu)_M(\nu)_N \frac{m!\Omega'_m}{(1-\lambda-L)_m} \frac{n!\Omega''_n}{(1-\mu-M)_n} \frac{k!\,\Omega'''_k}{(1-\nu-N)_k} \tag{3.2}$$

in terms of the sequences $\{\Omega_n^{(j)}\}$ $(j = 1, 2, 3)$ and the (essentially unrestricted) parameters $\lambda$, $\mu$, $\nu$ and $L$, $M$, $N$, if we make the following variable changes: $x \mapsto -x^{-2}$, $y \mapsto y^{-2}$ and $z \mapsto z^{-2}$, and apply the definition (1.6), we shall obtain a partly bilateral and partly unilateral generating function for the product of three Hubbell-Srivastava functions in the following form:

$$\begin{aligned} &\omega_L^{\lambda}(y)\omega_M^{\mu}(z)\omega_N^{\nu}\left(-\frac{xz}{y}\right) \\ &\quad = (\lambda)_L\,(\mu)_M\,(\nu)_N \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{y^{L-N-2m} z^{M+N-2n}}{(1-\lambda-L)_m(1-\mu-M)_n} \\ &\qquad \times \sum_{k=0}^{n} \frac{(-1)^k(\mu+M-n)_k}{(1-\lambda-L+m)_k(1-\nu-N)_k} \Omega'_{m+k}\Omega''_{n-k}\Omega'''_k(-x)^{N-2k}, \end{aligned} \tag{3.3}$$

provided that each member of (3.3) exists.

The generating function (3.3) corresponds to the *main* result of Kamarujjama *et al.* [7, p. 361, Equation (1.8)] in which the $k$-summand should be *corrected* to read $n$ in place of $\eta$. Two *further* special cases of the generating function (3.3), associated with the products of three generalised hypergeometric polynomials, also appear erroneously in the work of Kamarujjama *et al.* [7, p. 362, Equations (3.1) and (3.2)].

Yet another special case of the assertion (2.1) would occur when we set

$$\Omega(m,n,k) = \Omega'_m \Omega''_n \Omega'''_k, \tag{3.4}$$

so that the left-hand side of (2.1) becomes a product of three series with essentially arbitrary coefficients. If, in this special case, we further let

$$\Omega'_m = \frac{(\alpha_1)_m \cdots (\alpha_p)_m}{(\beta_1)_m \cdots (\beta_q)_m}, \quad \Omega''_n = \frac{(\lambda_1)_n \cdots (\lambda_r)_n}{(\mu_1)_n \cdots (\mu_s)_n} \quad \text{and} \quad \Omega'''_k = \frac{(\rho_1)_k \cdots (\rho_u)_k}{(\sigma_1)_k \cdots (\sigma_v)_k},$$

we shall obtain the following hypergeometric generating function:

$$\begin{aligned} &{}_pF_q\left[\begin{matrix}\alpha_1,\ldots,\alpha_p; \\ \beta_1,\ldots,\beta_q;\end{matrix}\; y\right] {}_rF_s\left[\begin{matrix}\lambda_1,\ldots,\lambda_r; \\ \mu_1,\ldots,\mu_s;\end{matrix}\; z\right] {}_uF_v\left[\begin{matrix}\rho_1,\ldots,\rho_u; \\ \sigma_1,\ldots,\sigma_v;\end{matrix}\; -\frac{xz}{y}\right] \\ &= \sum_{m=-\infty}^{\infty} \sum_{n=m^*}^{\infty} \frac{\prod_{j=1}^{p}(\alpha_j)_m \prod_{j=1}^{r}(\lambda_j)_n}{\prod_{j=1}^{q}(\beta_j)_m \prod_{j=1}^{s}(\mu_j)_n} \frac{y^m}{m!} \frac{z^n}{n!} \\ &\times {}_{p+s+u+1}F_{q+r+v+1}\left[\begin{matrix}-n, & (\alpha_p)+m, 1-(\mu_s)-n, (\rho_u); \\ m+1, & (\beta_q)+m, 1-(\lambda_r)-n, (\sigma_v);\end{matrix}\; (-1)^{r-s}x\right], \end{aligned} \tag{3.5}$$

where, for convenience, $(\alpha_p)+m$ abbreviates the array of $p$ parameters: $\alpha_1+m,\ldots,$ $\alpha_p+m$, with similar interpretations for $(\beta_q)+m$, *et cetera.*

The generating function (3.5) can also be deduced by appropriately specialising (3.3); indeed, except for some *obvious* notational variations, it is the *main* result of Pathan and Yasmeen [9, p. 3, Equation (1.5)].

Next we consider some multivariable applications of the assertion (2.2). First of all, by setting $\Omega(\mathbf{m},\mathbf{n},\mathbf{k}) \equiv 1$, (2.2) immediately yields the following simple consequence of Exton's generating function (1.4):

$$\begin{aligned} &\exp\left(\sum_{j=1}^{r}(y_j + z_j - x_j z_j/y_j)\right) \\ &= \sum_{\mathbf{m}=-\infty}^{\infty} \sum_{\mathbf{n}=\mathbf{m}^*}^{\infty} \prod_{j=1}^{r} \left\{ {}_1F_1(-n_j; m_j+1; x_j) \frac{y_j^{m_j}}{m_j!} \frac{z_j^{n_j}}{n_j!} \right\}. \end{aligned} \tag{3.6}$$

If, in the assertion (2.2), we set

$$\Omega(\mathbf{m},\mathbf{n},\mathbf{k}) = \Omega_1(\mathbf{m})\,\Omega_2(\mathbf{n})\,\Omega_3(\mathbf{k}), \tag{3.7}$$

the left-hand side of (2.2) would reduce at once to a product of three multiple series with essentially arbitrary coefficients. Thus, by assigning suitable special values to

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the coefficients $\Omega_1(\mathbf{m})$, $\Omega_2(\mathbf{n})$ and $\Omega_3(\mathbf{k})$, we can derive a number of generating functions involving the products of such multivariable hypergeometric functions as the familiar Lauricella functions $F_A^{(r)}$, $F_B^{(r)}$, $F_C^{(r)}$ and $F_D^{(r)}$ of $r$ variables [12, p. 33] and their generalisation introduced and studied by Srivastava and Daoust ([10, 11]; see also [5]). The details involved in these derivations are fairly straightforward and are being left as an exercise for the interested reader.

In view of the multinomial expansion (see, for example, [12, Equation 9.4(220)]):

$$(1-z_1-\cdots-z_r)^{-\lambda}=\sum_{\mathbf{m}=0}^{\infty}(\lambda)_{m_1+\cdots+m_r}\frac{z_1^{m_1}}{m_1!}\cdots\frac{z_r^{m_r}}{m_r!} \tag{3.8}$$
$$(\lambda\in\mathbb{C};\ |z_1+\cdots+z_r|<1),$$

in (2.2) and (3.7) we set

$$\Omega_1(\mathbf{m})=\begin{cases}1, & (\mathbf{m}=0)\\ 0, & (\mathbf{m}\neq 0),\end{cases}\qquad \Omega_2(\mathbf{n})=(\lambda_1)_{n_1}\cdots(\lambda_r)_{n_r}\quad\text{and}\quad \Omega_3(\mathbf{k})=(\mu)_{k_1+\cdots+k_r},$$

and replace $x_j$ by $x_j y_j$ $(j=1,\dots,r)$. We thus find for the Lauricella first function $F_A^{(r)}$ that (see [1, p. 25, Equation (3.1)]; see also [13, p. 494, Problem 8 (i)])

$$\begin{aligned}&(1-z_1)^{-\lambda_1}\cdots(1-z_r)^{-\lambda_r}(1+x_1z_1+\cdots+x_rz_r)^{-\mu}\\ &=\sum_{\mathbf{n}=0}^{\infty}(\lambda_1)_{n_1}\cdots(\lambda_r)_{n_r}\frac{z_1^{n_1}}{n_1!}\cdots\frac{z_r^{n_r}}{n_r!}\\ &\quad\times F_A^{(r)}[\mu,-n_1,\dots,-n_r;1-\lambda_1-n_1,\dots,1-\lambda_r-n_r;-x_1,\dots,-x_r].\end{aligned} \tag{3.9}$$

For the other three Lauricella functions, the assertion (2.2) [in conjunction with (3.7)] similarly yields the following generating functions:

$$\begin{aligned}&(1-z_1-\cdots-z_r)^{-\lambda}(1+x_1z_1)^{-\mu_1}\cdots(1+x_rz_r)^{-\mu_r}\\ &=\sum_{\mathbf{n}=0}^{\infty}(\lambda)_{n_1+\cdots+n_r}\frac{z_1^{n_1}}{n_1!}\cdots\frac{z_r^{n_r}}{n_r!}\\ &\quad\times F_B^{(r)}[\mu_1,\dots,\mu_r,-n_1,\dots,-n_r;1-n_1-\cdots-n_r-\lambda;-x_1,\dots,-x_r];\end{aligned} \tag{3.10}$$

$$\begin{aligned}&(1-y_1-\cdots-y_r)^{-\lambda}\left(1+\frac{x_1}{y_1}+\cdots+\frac{x_r}{y_r}\right)^{-\mu}\\ &=\sum_{\mathbf{m}=-\infty}^{\infty}(\lambda)_{m_1+\cdots+m_r}\frac{y_1^{m_1}}{m_1!}\cdots\frac{y_r^{m_r}}{m_r!}\\ &\quad\times F_C^{(r)}[\mu,\lambda+m_1+\cdots+m_r;m_1+1,\dots,m_r+1;-x_1,\dots,-x_r],\end{aligned} \tag{3.11}$$

which provides a multivariable generalisation of a known result [13, p. 325, Equation 6.5(9)];

$$
\begin{aligned}
&(1-z_1-\cdots-z_r)^{-\lambda}(1+x_1z_1+\cdots+x_rz_r)^{-\mu} \\
&\quad =\sum_{\mathbf{n}=0}^{\infty}(\lambda)_{n_1+\cdots+n_r}\frac{z_1^{n_1}}{n_1!}\cdots\frac{z_r^{n_r}}{n_r!} \\
&\qquad \times F_D^{(r)}[\mu,-n_1,\ldots,-n_r;1-n_1-\cdots-n_r-\lambda;-x_1,\ldots,-x_r]. \quad (3.12)
\end{aligned}
$$

Finally, in (2.2) and (3.7) we set

$$
\Omega_1(\mathbf{m})=(\lambda)_{m_1+\cdots+m_r},\qquad \Omega_2(\mathbf{n})=(\mu)_{n_1+\cdots+n_r}\quad\text{and}\qquad \Omega_3(\mathbf{k})=(\nu)_{k_1+\cdots+k_r}, \quad (3.13)
$$

and then apply the multinomial expansion (3.8). If we make use of the familiar notation for multivariable hypergeometric functions (see [12, p. 38, Equation 1.4(24)]), we thus obtain the following multivariable generalisation of another known result given recently by Pathan and Yasmeen [9, p. 7, Equation (3.5)]:

$$
\begin{aligned}
&(1-y_1-\cdots-y_r)^{-\lambda}(1-z_1-\cdots-z_r)^{-\mu}\left(1+\frac{x_1z_1}{y_1}+\cdots+\frac{x_rz_r}{y_r}\right)^{-\nu} \\
&\quad =\sum_{\mathbf{m}=-\infty}^{\infty}\sum_{\mathbf{n}=\mathbf{m}^*}^{\infty}(\lambda)_{m_1+\cdots+m_r}(\mu)_{n_1+\cdots+n_r}\prod_{j=1}^{r}\left\{\frac{y_j^{m_j}}{m_j!}\frac{z_j^{n_j}}{n_j!}\right\} \\
&\qquad \times F^{2:1;\ldots;1}_{1:1;\ldots;1}\left[\begin{matrix}\nu,\lambda+m_1+\cdots+m_r: & -n_1;\ldots; & -n_r;\\ 1-n_1-\cdots-n_r-\mu: & m_1+1;\ldots; & m_r+1;\end{matrix}\ -x_1,\ldots,-x_r\right]. \quad (3.14)
\end{aligned}
$$

A *confluent* case of this last multivariable generating function is worthy of note. Indeed, upon replacing $x_j$ on both sides of (3.14) by $x_j/\nu$ ($j=1,\ldots,r$), if we let $\nu\to\infty$, (3.14) readily yields

$$
\begin{aligned}
&(1-y_1-\cdots-y_r)^{-\lambda}(1-z_1-\cdots-z_r)^{-\mu}\exp\left(-\frac{x_1z_1}{y_1}-\cdots-\frac{x_rz_r}{y_r}\right) \\
&\quad =\sum_{\mathbf{m}=-\infty}^{\infty}\sum_{\mathbf{n}=\mathbf{m}^*}^{\infty}(\lambda)_{m_1+\cdots+m_r}(\mu)_{n_1+\cdots+n_r}\prod_{j=1}^{r}\left\{\frac{y_j^{m_j}}{m_j!}\frac{z_j^{n_j}}{n_j!}\right\} \\
&\qquad \times F^{1:1;\ldots;1}_{1:1;\ldots;1}\left[\begin{matrix}\lambda+m_1+\cdots+m_r: & -n_1;\ldots; & -n_r;\\ 1-n_1-\cdots-n_r-\mu: & m_1+1;\ldots; & m_r+1;\end{matrix}\ -x_1,\ldots,-x_r\right], \quad (3.15)
\end{aligned}
$$

which can also be deduced *directly* from (2.2) and (3.7) by replacing the choice (3.13) by $\Omega_3(\mathbf{k})\equiv 1$.

Formula (3.15) can be shown to reduce to the exponential generating function (3.6) if we first make the following variable changes: $y_j \mapsto \lambda^{-1}y_j$ and $z_j \mapsto \mu^{-1}z_j$ $(j = 1, \ldots, r)$, and then let $\min(\lambda, \mu) \to \infty$.

Finally, we recall here a *further* generalisation of the Hurwitz-Lerch Zeta function $\Phi(z, s, a)$ in the form (see [3, p. 100, Equation (1.5)]):

$$\Phi^*_\mu(z, s, a) := \sum_{n=0}^{\infty} \frac{(\mu)_n}{(n+a)^s} \frac{z^n}{n!} \tag{3.16}$$

$$(a \neq 0, -1, -2, \ldots;\ \mu \in \mathbb{C};\ s \in \mathbb{C} \text{ when } |z| < 1;\ \Re(s) > 1 \text{ when } |z| = 1),$$

so that, obviously,

$$\Phi^*_1(z, s, a) = \Phi(z, s, a) := \sum_{n=0}^{\infty} \frac{z^n}{(n+a)^s} \tag{3.17}$$

$$(a \neq 0, -1, -2, \ldots;\ s \in \mathbb{C} \text{ when } |z| < 1;\ \Re(s) > 1 \text{ when } |z| = 1).$$

In fact, it readily follows from the definitions (3.16) and (3.17) that

$$\Phi^*_\mu(z, s, a) = \frac{1}{\Gamma(\mu)} \int_0^\infty t^{\mu-1} e^{-t} \Phi(zt, s, a)\, dt \quad (\Re(\mu) > 0), \tag{3.18}$$

provided that each member of (3.18) exists.

Equation (3.18) exhibits the fact that $\Phi^*_\mu(z, s, a)$ is essentially an Eulerian integral of the familiar function $\Phi(z, s, a)$. More interestingly, the *main* generating function for $\Phi^*_\mu(z, s, a)$, proven recently by Goyal and Laddha [3, p. 101, Equation (2.4)], is a very *specialised* case of our result (2.2) when $x_j = y_j - t_j = 0$ $(j = 1, \ldots, r)$, $z_1 = z$, $z_j = 0$ $(j = 2, \ldots, r)$ and

$$\Omega(\mathbf{m}, \mathbf{n}, \mathbf{p}) = \Lambda(m_1, \ldots, m_r) \frac{(\mu)_n}{(a+n)^{\nu+m_1+\cdots+m_r}},$$

where the multiple sequence $\{\Lambda(m_1, \ldots, m_r)\}$ is a suitably chosen quotient of Gamma functions [3, p. 102, Equation (2.5)] and, for convenience, $n_1 = n$.

The aforementioned generating function of Goyal and Laddha [3, p. 101, Equation (2.4)] is merely a rewriting of an $(r+1)$-dimensional series as a single sum of the $r$-dimensional series involved. Moreover, in view of the elementary identity [13, p. 52, Equation 1.6(3)]:

$$\sum_{\mathbf{m}=0}^{\infty} f(m_1 + \cdots + m_r) \frac{z_1^{m_1}}{m_1!} \cdots \frac{z_r^{m_r}}{m_r!} = \sum_{m=0}^{\infty} f(m) \frac{(z_1 + \cdots + z_r)^m}{m!},$$

many of the *multiple-series* results, given by Goyal and Laddha [3], are no more general than the corresponding results involving a *single* series. Much more general known families of multiple-series generating functions can be found reproduced (with proper credits) in the work of Srivastava and Manocha [13].

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