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DEGREE SPECTRA OF ANALYTIC COMPLETE EQUIVALENCE RELATIONS

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Abstract. We study the bi-embeddability and elementary bi-embeddability relation on graphs under Borel reducibility and investigate the degree spectra realized by these relations. We first give a Borel reduction from embeddability on graphs to elementary embeddability on graphs. As a consequence we obtain that elementary bi-embeddability on graphs is a $\mathbf{\Sigma}^1_1$ complete equivalence relation. We then investigate the algorithmic properties of this reduction. We obtain that elementary bi-embeddability on the class of computable graphs is Σ^1_1 complete with respect to computable reducibility and show that the elementary bi-embeddability and bi-embeddability spectra realized by graphs are related.

§1. Introduction. Equivalence relations on countable structures are among the most heavily studied objects in descriptive set theory and computability theory. In descriptive set theory, starting with the work of Friedman and Stanley [10], the complexity of equivalence relations on spaces of structures under Borel reducibility has seen much interest by experts, culminating in results by Louveau and Rosendal [16], who showed that, among others, the bi-embeddability relation on graphs is $\mathbf{\Sigma}^1_1$-complete. Since then there has been a constant stream of work on the complexity of the bi-embeddability relation, both on other classes of structures, see for instance [4], and refinements of completeness notions, e.g., in [5].

Equivalence relations are also one of the main objects of study in computability theory. Here, the equivalence relations are usually on the set of natural numbers and their complexity is established using computable reducibility. Identifying a computable structure with the index of the algorithm computing it, one can obtain completeness results like the ones in descriptive set theory for equivalence relations on computable structures [6, 9]. One object of study in computable structure theory which also takes non-computable structures into account is degree spectra of structures, introduced by Knight [15]. The degree spectrum of a given structure is the set of sets of natural numbers Turing equivalent to one of its isomorphic copies. They provide a measure of the algorithmic complexity of countable structures.

Recently, researchers initiated the study of degree spectra with respect to other model theoretic equivalence relations such as bi-embeddability [7], elementary bi-embeddability [20], elementary equivalence [1–3], or Σ_n equivalence [8]. One of the

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main goals in this line of research is to distinguish these equivalence relations with respect to the degree spectra they realize. While for elementary equivalence and Σ_n equivalence examples that separate them from each other and from isomorphism and elementary bi-embeddability are known, so far all attempts to separate isomorphism, bi-embeddability, and elementary bi-embeddability have been unsuccessful.

There seem to be various reasons for this. That we can separate elementary equivalence and Σ_n equivalence is the case because they have different levels in the Borel hierarchy while isomorphism and bi-embeddability are not even Borel. On the other hand bi-embeddability preserves very little structural properties and it is thus difficult to construct interesting examples. The aim of this article is to investigate the relationship between the degree spectra realized by the bi-embeddability relation and by the elementary bi-embeddability relation. First, we establish that elementary bi-embeddability on graphs is $\mathbf{\Sigma}^1_1$ complete with respect to Borel reducibility. We then proceed to establish a relationship between the degree spectra realized by the bi-embeddability and elementary bi-embeddability relation on graphs. Our main results are as follows.

THEOREM 1.1. *The elementary embeddability relation on graphs $\preccurlyeq_{\mathfrak{G}}$ is a complete $\mathbf{\Sigma}^1_1$ quasi-order. In particular, the elementary bi-embeddability relation on graphs $\approxeq_{\mathfrak{G}}$ is a complete $\mathbf{\Sigma}^1_1$ equivalence relation.*

As a corollary of Theorem 1.1 we obtain the corresponding result for elementary bi-embeddability on computable structures.

THEOREM 1.2. *The elementary embeddability relation on the class of computable graphs is a Σ^1_1 complete quasi-order with respect to computable reducibility. In particular, the elementary bi-embeddability relation on computable graphs is a Σ^1_1 complete equivalence relation with respect to computable reducibility.*

The following result establishes a relationship between bi-embeddability spectra of graphs and elementary bi-embeddability spectra of graphs.

THEOREM 1.3. *Let $\mathcal{G}$ be an automorphically non-trivial graph, then there is a graph $\hat{\mathcal{G}}$ such that*

$$DgSp_{\approxeq}(\hat{\mathcal{G}}) = \{X : X' \in DgSp_{\approx}(\mathcal{G})\}.$$

Note that in Theorem 1.3 we deal only with automorphically non-trivial graphs. This might seem like a shortcoming; however, automorphically trivial structures are not interesting from a computability theoretic point of view. In particular, every structure bi-embeddable with an automorphically trivial graph is computable and thus both its bi-embeddability spectrum and elementary bi-embeddability spectrum are the set of all computable sets.

The proofs of Theorems 1.1 and 1.2 are the topic of Section 3. In Section 4 we build on these results to prove Theorem 1.3. In Section 2 we give the necessary background and definitions.

§2. Background.

Our definitions follow for the most part [12, 19]. We assume that all structures have universe ω and are relational. Let $\mathcal{L}$ be a relational language $(R_i)_{i\in\omega}$ where without loss of generality R_i has arity i. Then each element $\mathcal{A}$ of

$Mod(\mathcal{L})$ can be viewed as an element of the product space

$$X_{\mathcal{L}} = \prod_{i\in\omega} 2^{\omega^i},$$

and thus $Mod(\mathcal{L})$ becomes a compact Polish space on which we can define the Borel and projective hierarchy in the usual way.

Let $\mathcal{A}$ be an $\mathcal{L}$-structure and $(\varphi_i^{at})_{i\in\omega}$ be a computable enumeration of the atomic $\mathcal{L}$-sentences with variables in $\{x_1, x_2, \dots\}$. The *atomic diagram* $D(\mathcal{A})$ of $\mathcal{A}$ is the element of Cantor space defined by

$$D(\mathcal{A})(i) = \begin{cases} 1 & \text{if } \mathcal{A} \models \varphi_i^{at}[x_j \to j : j \in \omega], \\ 0 & \text{otherwise.} \end{cases}$$

The Turing degree of a structure $\mathcal{A}$ is the degree of $D(\mathcal{A})$. We will in general not distinguish between a structure as an element of $Mod(\mathcal{L})$ and its atomic diagram and assume that what is meant is clear from the context.

Variations of the following definition were independently suggested in [8, 17, 22].

DEFINITION 2.1. Let E be an equivalence relation on $Mod(\mathcal{L})$ and $\mathcal{A} \in Mod(\mathcal{L})$. Then the *degree spectrum of* $\mathcal{A}$ *with respect to* E, or, short E-spectrum of $\mathcal{A}$, is the set

$$DgSp_E(\mathcal{A}) = \{X : \exists \mathcal{B}\ E\ \mathcal{A}\ D(\mathcal{B}) \equiv_T X\}.$$

We write $\mathcal{A} \hookrightarrow \mathcal{B}$ to say that $\mathcal{A}$ is *embeddable* in $\mathcal{B}$, and $\mathcal{A} \approx \mathcal{B}$ to say that $\mathcal{A}$ is *bi-embeddable* with $\mathcal{B}$, i.e., $\mathcal{A} \hookrightarrow \mathcal{B}$ and $\mathcal{B} \hookrightarrow \mathcal{A}$. Further, we write $\mathcal{A} \preccurlyeq \mathcal{B}$ to say that $\mathcal{A}$ is *elementary embeddable* in $\mathcal{B}$ and $\mathcal{A} \approxeq \mathcal{B}$ to say that $\mathcal{A}$ is *elementary bi-embeddable* with $\mathcal{B}$, i.e., $\mathcal{A} \preccurlyeq \mathcal{B}$ and $\mathcal{B} \preccurlyeq \mathcal{A}$.

DEFINITION 2.2. Let R, S be binary relations on a set X. The relation R is *reducible* to S if there is a function $f : X \to X$ such that for all $x, y \in X$

$$xRy \Leftrightarrow f(x)Sf(y).$$

Assume $X = Mod(\mathcal{L})$. Then

(1) R is *Borel reducible* to S if f is Borel on $Mod(\mathcal{L}) \times Mod(\mathcal{L})$,
(2) R is *computably reducible* to S if there is a computable operator Φ such that for all $\mathcal{A} \in Mod(\mathcal{L})$, $\Phi^{D(\mathcal{A})} = D(f(\mathcal{A}))$.

Assume X is ω and that $(\mathcal{A}_i)_{i\in\omega}$ is a computable enumeration of all partial computable $\mathcal{L}$ structures. Then R is *computably reducible* to S if f is a computable function.

We say that an equivalence relation (quasi-order) $R \in \Gamma$ is a Γ complete equivalence relation (quasi-order) for a complexity class Γ with respect to x-reducibility if all equivalence relations (quasi-orders) in Γ are x-reducible to R.

A standard reference on Borel reducibility is [12]. Computable reducibility on the natural numbers can be seen as a natural effectivization of Borel reducibility where one only considers computable structures. Fokina and Friedman [6] showed that bi-embeddability on trees and thus also graphs is Σ_1^1 complete with respect to computable reducibility, and in [9] it is shown that isomorphism on graphs

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is Σ^1_1 complete with respect to computable reducibility. This contrasts with Borel reducibility; it is well known that isomorphism on graphs is not $\boldsymbol{\Sigma}^1_1$ complete.

§3. Elementary bi-embeddability is analytic complete. In this section we prove Theorems 1.1 and 1.2 and some lemmas needed for Theorem 1.3. The section is structured as follows. In Section 3.1 we give a reduction from embeddability on the class of graphs $\mathfrak{G}$ to elementary embeddability on a Borel class $\mathfrak{C}$ of structures in an infinite relational language. In Section 3.2 we show that graphs are complete for elementary embeddability. That is, for every Borel class, elementary embeddability on this class can be reduced to elementary embeddability on graphs. Theorem 1.1 then follows by composing the reductions given in Sections 3.1 and 3.2. For Theorem 1.2 we need a few more observations made at the end of this section.

3.1. The reduction from $\hookrightarrow_{\mathfrak{G}}$ to $\preccurlyeq_{\mathfrak{C}}$. The main idea of the construction is that for any given graph $\mathcal{G}$ we replace the edge relation with structures having the property that they are minimal under elementary embeddability.

Definition 3.1. A structure $\mathcal{A}$ is *minimal* if it does not have proper elementary substructures.

Minimal structures were investigated by Fuhrken [11] who showed that there is a theory with $2^{\aleph_0}$ minimal models, and Shelah [21] who showed that for every $n \leq \aleph_0$, there is a theory with n minimal models. Later, Ikeda [14] investigated minimal models of minimal theories. Notice that a prime model is not necessarily minimal, as it might contain elementary substructures isomorphic to itself.

Given a graph $\mathcal{G}$, if $x, y \in G$ and xEy, then we associate a copy of a structure $\mathcal{A}$ with the pair (x, y) and otherwise we associate a copy $\mathcal{B}$ with (x, y). The structures $\mathcal{A}$ and $\mathcal{B}$ will be elementary equivalent and minimal.

Before we formally state the reduction let us describe $\mathcal{A}$ and $\mathcal{B}$. They will be models of the theory of the following structure studied by Shelah [21]. The language of the theory contains countably many unary functions F_ν and unary relation symbols R_ν, one for each $\nu \in 2^{<\omega}$. Consider the structure

$$\mathcal{S} = (2^\omega, \langle F_\nu\rangle_{\nu\in 2^{<\omega}}, \langle R_\nu\rangle_{\nu\in 2^{<\omega}}),$$

where F_ν is defined by $F_\nu(\sigma)(x) = \sigma(x) + \nu(x) \bmod 2$ where we assume that $\nu(x) = 0$ for $x \geq |\nu|$ and $R_\nu(\sigma)$ if and only if $\nu \prec \sigma$. Shelah showed that the theory of $\mathcal{S}$ has quantifier elimination and that each element of $\mathcal{S}$ generates an elementary substructure that is minimal.

Let $\hat{\mathcal{S}}_0$ be the substructure of $\mathcal{S}$ generated by $\bar{0}$, the constant string of 0's, and $\hat{\mathcal{S}}_1$ be the substructure generated by $\bar{1}$, the constant string of 1's. These structures are countable and by Shelah's argument, $\hat{\mathcal{S}}_0 \equiv \hat{\mathcal{S}}_1 \equiv \mathcal{S}$. Furthermore, $\hat{\mathcal{S}}_0$ and $\hat{\mathcal{S}}_1$ are minimal models of $Th(\mathcal{S})$. To see this, let $x \in \hat{S}_0$, then $x = F_\nu(\bar{0})$ and in particular, $\bar{0} = F_\nu(x)$ for some $\nu \in 2^{<\omega}$. So, the substructure of $\hat{\mathcal{S}}_0$ generated by x is already $\hat{\mathcal{S}}_0$.

As we require our structures in $\mathfrak{C}$ to be of relational syntax we will let $\mathcal{S}_0$ and $\mathcal{S}_1$ be the structures corresponding to $\hat{\mathcal{S}}_0$, respectively $\hat{\mathcal{S}}_1$, after we replace each $F_\nu^{\mathcal{S}_i}$ by its graph $graph^{\mathcal{S}_i}_{F_\nu} = \{(\sigma, F_\nu^{\mathcal{S}_i}(\sigma)) : \sigma \in S_i\}$. We may assume without loss of generality that the universes of $\mathcal{S}_0$ and $\mathcal{S}_1$ are ω and let $\mathcal{A} = \mathcal{S}_0$ and $\mathcal{B} = \mathcal{S}_1$.

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Let us describe the structures in the class $\mathfrak{C}$ more formally. The class of structures $\mathfrak{C}$ consists of all countable structures with universe ω in the language consisting of a unary relation W, binary relations R_v, and $graph_{F_v}$ for all $v \in 2^{<\omega}$, and a ternary relation O. We are now ready to give the function $f : \mathfrak{G} \to \mathfrak{C}$ witnessing the reduction.

We formally describe how to obtain a structure in $\mathfrak{C}$ given a graph. Let $\mathcal{G}$ be a graph and partition ω into countably many infinite, coinfinite subsets $(A_i)_{i\in\omega}$. Then

- for every $a_i \in A_0$, $W^{f(\mathcal{G})}(a_i)$ (we will call elements of A_0 the vertices of $f(\mathcal{G})$),
- for every $m, n \in \omega$, if mEn, then for all $v \in 2^{<\omega}$ define $R_v^{f(\mathcal{G})}$ and $graph_{F_v}{}^{f(\mathcal{G})}$ on $A_{\langle m,n\rangle+1}$ such that $(A_{\langle m,n\rangle+1}, \langle graph_{F_v}{}^{f(\mathcal{G})}\rangle_{v\in 2^{<\omega}}, R_v^{f(\mathcal{G})}) \cong \mathcal{S}_0$,
- for every $m, n \in \omega$, if $\neg mEn$, then for all $v \in 2^{<\omega}$ define $R_v^{f(\mathcal{G})}$ and $graph_{F_v}{}^{f(\mathcal{G})}$ on $A_{\langle m,n\rangle+1}$ such that $(A_{\langle m,n\rangle+1}, \langle graph_{F_v}{}^{f(\mathcal{G})}\rangle_{v\in 2^{<\omega}}, R_v^{f(\mathcal{G})}) \cong \mathcal{S}_1$, and
- for every $m, n \in \omega$, let $O^{f(\mathcal{G})}(a_m, a_n, j)$ for all $j \in A_{\langle m,n\rangle+1}$.

This finishes the construction of $f(\mathcal{G})$. We will refer to the substructure on the elements in $A_{\langle m,n\rangle+1}$ as the substructure associated with the pair (a_m, a_n) and with $(A_{\langle m,n\rangle+1}, \langle graph_{F_v}{}^{f(\mathcal{G})}\rangle_{v\in 2^{<\omega}}, R_v^{f(\mathcal{G})})$ as $\mathcal{S}_{(a_m,a_n)}$.

It is easy to see that the function f so defined is Borel; indeed, it is even computable. To see that f is a reduction from $\hookrightarrow_{\mathfrak{G}}$ to $\preccurlyeq_{\mathfrak{C}}$ it remains to prove the following.

Lemma 3.2. *For $\mathcal{G}, \mathcal{H} \in \mathfrak{G}$, $\mathcal{G} \hookrightarrow \mathcal{H}$ if and only if $f(\mathcal{G}) \preccurlyeq f(\mathcal{H})$.*

Proof. That $\mathcal{G} \hookrightarrow \mathcal{H}$ if $f(\mathcal{G}) \preccurlyeq f(\mathcal{H})$ follows trivially from the construction. To show the converse we will use the following model theoretic fact: For two $\mathcal{L}$-structures $\mathcal{A}$ and $\mathcal{B}$, $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$ if and only if for every finite $\mathcal{R} \subseteq \mathcal{L}$, the $\mathcal{R}$ reduct of $\mathcal{A}$ is an elementary substructure of the $\mathcal{R}$ reduct of $\mathcal{B}$. Necessity follows trivially from the fact that $\mathcal{R} \subseteq \mathcal{L}$ and sufficiency is easily seen by noticing that every first-order formula φ is in a finite $\mathcal{R}_\varphi \subseteq \mathcal{L}$.

So, say $\mathcal{G} \hookrightarrow \mathcal{H}$ by h. We get an induced embedding $\hat{h}$ defined such that for all $i, i' \in G$, if $h(i) = j$, then $\hat{h}(a_i) = a_j$ and $\hat{h}$ is the canonic isomorphism between the substructure associated with $(a_i, a_{i'})$ and the one associated with $(\hat{h}(a_i), \hat{h}(a_{i'}))$. Without loss of generality we may assume that $f(\mathcal{G})$ is a substructure of $f(\mathcal{H})$, i.e., that $\hat{h}$ is the identity. We use Ehrenfeucht–Fraïssé games to verify that in every finite $\mathcal{R} \subseteq \mathcal{L}$, $f(\mathcal{G})$ is an elementary substructure of $f(\mathcal{H})$. We assume without loss of generality that $\mathcal{R} = \{O, W, R_{v_0}, \dots, R_{v_k}, Graph_{F_{v_0}}, \dots, Graph_{F_{v_k}}\}$ where v_i is the i^{th} string in the lexicographical ordering of $2^{<\omega}$ and $k \in \omega$. Let us show that Player II has a winning strategy in $G_m((f(\mathcal{G}), g_1, \dots, g_n), (f(\mathcal{H}), g_1, \dots, g_n))$ for arbitrary $m \in \omega$ played in $\mathcal{R}$. First, notice that since $\mathcal{S}_0 \equiv \mathcal{S}_1$, II has a winning strategy for $G_m(\mathcal{S}_0, \mathcal{S}_1)$ in the reduct $\{R_{v_0}, \dots, R_{v_k}, Graph_{F_{v_0}}, \dots, Graph_{F_{v_k}}\}$. The following is a winning strategy for $G_m((f(\mathcal{G}), g_1, \dots, g_n), (f(\mathcal{H}), g_1, \dots, g_n))$ played in $\mathcal{R}$. Say that at turn i, the played substructures are G_i and H_i given by the partial isomorphism h_i. Assume we are on turn $i + 1$.

(1) If I plays an element c in the O-closure of $g_1, \dots, g_n$, then let $h_{i+1}(c) = c$.
(2) If I plays an element c in $f(\mathcal{G})$ not in the O-closure of G_i, say it is associated with (a, b) where none of a, b is in the O-closure of G_i, then pick vertices

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(a',b') in $f(\mathcal{H})$. If $c = a$ or b, let $h_{i+1}(c) = a'$, respectively, $h_{i+1}(c) = b'$. Otherwise start running a $G_m(\mathcal{S}_{(a,b)}, \mathcal{S}_{(a',b')})$ winning strategy $w_{(a,b)}^{(a',b')}$ and let $h_{i+1}(c) = w_{(a,b)}^{(a',b')}(c)$.

(3) If I plays an element c in $f(\mathcal{G})$ not in the O-closure of G_i but associated with (a,b) where either a or b is in G_i, then pick (a',b') such that a', respectively b', is the element corresponding to a, respectively b, in H_i and continue as in (2), mutatis mutandis.

(4) If I plays an element in $f(\mathcal{H})$ not in the O-closure of H_i, then as $f(\mathcal{G})$ is infinite, II can play as in the cases (2) and (3), mutatis mutandis.

(5) If I plays an element c in $f(\mathcal{G})$ that is in the O-closure of G_i but not in the O-closure of $g_1, \dots, g_n$, then it is associated with some (a,b) in $f(\mathcal{G})$ and by induction there is a winning strategy $w_{(a,b)}^{(a',b')}$ that has already been used. If $c = a$ or $c = b$, let $h_{i+1}(c) = a'$, respectively, $h_{i+1}(c) = b'$. Otherwise let $h_{i+1}(c) = w_{(a,b)}^{(a',b')}(c_1, \dots, c_k, c)$ where $c_1, \dots, c_k$ are the elements from the structures associated with (a,b) and (a',b') played by I so far.

(6) If I plays an element c in $f(\mathcal{H})$ that is in the O-closure of H_i but not in the O-closure of $g_1, \dots, g_n$, then play as in (5), mutatis mutandis.

Since at each turn we play according to winning strategies for games of the form $G_m(\mathcal{S}_i, \mathcal{S}_j)$ where $i, j \in \{0,1\}$ we obtain that h_m is a partial isomorphism between $(f(\mathcal{G}), g_1, \dots, g_n)$ and $(f(\mathcal{H}), g_1, \dots, g_n)$. We thus have given a winning strategy for $G_m((f(\mathcal{G}), g_1, \dots, g_n), (f(\mathcal{H}), g_1, \dots, g_n))$. ⊣

3.2. Graphs are complete for elementary embeddability. We will show that for every class of structures $\mathfrak{K}$, there is a computable reduction $\preccurlyeq_{\mathfrak{K}} \to \preccurlyeq_{\mathfrak{G}}$.

The result we are going to prove appeared in [20]. There, a proof sketch of the fact that the reduction preserves elementary bi-embeddability spectra was given. We will give a full proof of this fact in Section 4. Note that the coding used in the reduction is not new but was already used in [2] to show that graphs are universal for theory spectra. Let us first describe this coding.

We may assume without loss of generality that $\mathfrak{K}$ is a class of structures in relational language $\mathcal{L} = (R_1, \dots)$ where each R_i has arity i. Given $\mathcal{A} \in \mathfrak{K}$, the graph $g(\mathcal{A})$ has three vertices a, b, c where to a we connect the unique 3-cycle in the graph, to b the unique 5-cycle, and to c the unique 7-cycle. For each element $x \in A$ we add a vertex v_x and an edge $a \rightarrowtail v_x$. For every i tuple $x_1, \dots, x_i \in A$ we add chains of length $i + k$ for every k, $1 \leq k \leq i$, with common last elements y. We add an edge $v_{x_k} \rightarrowtail y_1$ only if y_1 is the first element of the chain of length of $i + k$. If $\mathcal{A} \models R_i(x_1, \dots, x_i)$ we add an edge $y \rightarrowtail b$ and otherwise add an edge $y \rightarrowtail c$. This finishes the construction. See Figure 1 for an example.

Let us fix the some notation for the following proofs. Given a structure $\mathcal{A}$ and $\bar{a} \in A^{<\omega}$ we let $\langle \bar{a} \rangle^{\mathcal{A}}$ be the substructure of $\mathcal{A}$ generated by $\bar{a}$.

Lemma 3.3. *For $\mathcal{A}, \mathcal{B} \in \mathfrak{K}$, $\mathcal{A} \preccurlyeq \mathcal{B}$ if and only if $g(\mathcal{A}) \preccurlyeq g(\mathcal{B})$.*

Proof. $(\Rightarrow)$. Assume that $\mathcal{A} \preccurlyeq \mathcal{B}$ and that $\mathcal{A}$ is an elementary substructure of $\mathcal{B}$. We may also assume without loss of generality that $g(\mathcal{A}) \subseteq g(\mathcal{B})$. We will show that for all $n \in \omega$ and any $\overline{a} \in g(\mathcal{A})^{<\omega}$ Player II has a winning strategy for the n turn

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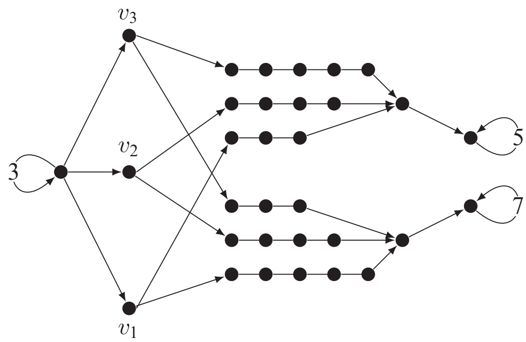

FIGURE 1. Part of the graph $F(\mathcal{A})$ coding that $\mathcal{A} \not\models R_3(3,2,1)$ and $\mathcal{A} \models R_3(1,2,3)$.

Ehrenfeucht–Fraïssé game $G_n((g(\mathcal{A}),\overline{a}),(g(\mathcal{B}),\overline{a}))$. Assume that n is the least such that Player II has no winning strategy for $G_n((g(\mathcal{A}),\overline{a}),(g(\mathcal{B}),\overline{a}))$. Consider the set of partial isomorphisms from $(g(\mathcal{A}),\overline{a})$ to $(g(\mathcal{B}),\overline{a})$. This set cannot have the back-and-forth property. In particular, the back-and-forth property fails already if we only consider partial isomorphisms with domain of size $n+|\overline{a}|$. Otherwise there would be a winning strategy for $G_n((g(\mathcal{A}),\overline{a}),(g(\mathcal{B}),\overline{a}))$. So, either there is $\overline{v} \in g(\mathcal{A})^n$ such that for all $\overline{u} \in g(\mathcal{B})^n$, $\langle \overline{av}\rangle^{g(\mathcal{A})} \ncong \langle \overline{au}\rangle^{g(\mathcal{B})}$ or there is $\overline{u} \in g(\mathcal{B})^n$ such that for all $\overline{v} \in g(\mathcal{B})^n$, $\langle \overline{au}\rangle^{g(\mathcal{B})} \ncong \langle \overline{av}\rangle^{g(\mathcal{A})}$. We will derive a contradiction assuming the second case. Deriving one from the first case can be done in a similar fashion.

Notice that $\overline{au}$ is in a substructure of $g(\mathcal{B})$ coding a finite substructure of $\mathcal{B}$ in a finite part $\mathcal{L}_1$ of the language of $\mathcal{B}$. Extend $\langle \overline{au}\rangle^{g(\mathcal{B})}$ so that it codes such a substructure $\mathcal{B}_1$ of $\mathcal{B}$. Consider the conjunction φ of atomic formulas, or negations thereof, true of $\mathcal{B}_1$ in $\mathcal{L}_1$. Let $\overline{a}'$ be the elements in $B_1 \cap A$ and $\overline{u}'$ the elements in $B_1 \setminus A$. Then $\mathcal{B} \models \varphi(\overline{a}'\overline{u}')$ and the Tarski–Vaught test gives us elements $\overline{v}'$ in $\mathcal{A}$ such that $\mathcal{A} \models \varphi(\overline{a}'\overline{v}')$. It follows that we have a partial isomorphism between $\langle \overline{a}'\overline{u}'\rangle^{\mathcal{B}}$ and $\langle \overline{a}'\overline{v}'\rangle^{\mathcal{A}}$ in $\mathcal{L}_1$. This induces an isomorphism between the subgraph coding $\mathcal{B}_1$ and the subgraph coding $\langle \overline{a}'\overline{v}'\rangle^{\mathcal{A}}$. But $\langle \overline{au}\rangle^{g(\mathcal{B})}$ is a subgraph of the graph coding $\mathcal{B}_1$ and thus it is isomorphic to a substructure $\langle \overline{av}\rangle^{g(\mathcal{A})}$ of the structure coding $\langle \overline{a}'\overline{v}'\rangle^{\mathcal{A}}$, a contradiction.

($\Leftarrow$). An easy induction on the quantifier depth of formulas in $\mathcal{L}$ shows that for every $\mathcal{A} \in \mathfrak{K}$ and $\mathcal{L}$-formula φ with n-free variables the set

$$D_\varphi^{\mathcal{A}} = \{(v_{a_1},\dots,v_{a_n}) : (\mathcal{A}, a_1,\dots,a_n) \models \varphi(a_1,\dots,a_n)\}$$

is definable in $g(\mathcal{A})$. Now, assume that $g(\mathcal{A}) \preccurlyeq g(\mathcal{B})$ and without loss of generality that $g(\mathcal{A})$ is an elementary substructure of $g(\mathcal{B})$. Let $g_{\mathcal{B}} : \mathcal{B} \to g(\mathcal{B})$ be defined by $g_{\mathcal{B}} : b \mapsto v_b$. Notice that the map $a \mapsto g_{\mathcal{B}}^{-1}(v_a)$ is an embedding of $\mathcal{A}$ in $\mathcal{B}$. To see that this embedding is elementary assume that $(\mathcal{A},\overline{a}) \models \varphi$, then $\bar{v}_{\bar{a}} \in D_\varphi^{\mathcal{A}}$ and by elementarity $\bar{v}_{\bar{a}} \in D_\varphi^{\mathcal{B}}$. So, $(\mathcal{B}, g_{\mathcal{B}}^{-1}(\bar{v}_{\bar{a}})) \models \varphi(g_{\mathcal{B}}^{-1}(\bar{v}_{\bar{a}}))$. $\dashv$

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Concatenating the reductions f and g and from the fact that $\hookrightarrow_{\mathfrak{G}}, \approx_{\mathfrak{G}}$ are complete $\mathbf{\Sigma}^1_1$ quasi-orders, respectively equivalence relations, we obtain Theorem 1.1.

To prove Theorem 1.2 notice that f and g are computable. Thus there is a Turing operator Φ such that $\Phi = g \circ f$. We can find a Turing machine φ_i such that $\varphi_i(j,k) = \Phi^{\mathcal{A}_j}(k)$ for all $k \in \omega$ if $\mathcal{A}_j$ is a total computable structure. Using the s-m-n theorem we can then get a computable function $j \mapsto u(i,j)$ where $u(i,j)$ is an index for $\Phi^{\mathcal{A}_j}$. Thus $\hookrightarrow_{\mathfrak{G}}$ is computably reducible to $\preccurlyeq_{\mathfrak{G}}$ as a quasi-order on ω. Fokina and Friedman [6] showed that $\hookrightarrow_{\mathfrak{G}}$ is Σ^1_1 complete. Thus, $\preccurlyeq_{\mathfrak{G}}$ is also Σ^1_1 complete and Theorem 1.2 follows.

§4. Degree spectra. In this section we finish the proof of Theorem 1.3. As noticed before the two reductions $f : \mathfrak{G} \to \mathfrak{C}$ and $g : \mathfrak{C} \to \mathfrak{G}$ are computable. We will see that the two functions induce an even stronger notion of reduction that allows us to relate the degree spectra realized by $\approx_{\mathfrak{G}}$ and $\approxeq_{\mathfrak{G}}$.

DEFINITION 4.1 (cf. [13, 17]). Let $\mathfrak{C}$ and $\mathfrak{D}$ be categories. A *computable functor* between $\mathfrak{C}$ and $\mathfrak{D}$ is a pair of computable operators (Φ, Φ_*) such that

(1) for all $\mathcal{A} \in \mathfrak{C}_1$, $F(\mathcal{A}) = \Phi^{\mathcal{A}}$, and
(2) for all $f : \mathcal{A} \to \mathcal{B} \in \mathfrak{C}_2$, $F(f) = \Phi_*^{\mathcal{A}\oplus f \oplus \mathcal{B}}$.

Computable functors preserve many computability theoretic properties. One example are degree spectra: Recall that for $X, Y \subseteq \mathcal{P}(\omega)$, X is *Medvedev reducible* to Y, $X \leq_s Y$, if there is a Turing operator Φ such that for all $y \in Y$, there is $x \in X$ such that $\Phi^y = x$. We have in particular that if $F : (\mathfrak{C}, \rightsquigarrow_1) \to (\mathfrak{D}, \rightsquigarrow_2)$ is a computable functor, and $\sim_i$ is the equivalence relation given by

$$\mathcal{A} \sim_i \mathcal{B} \Leftrightarrow \mathcal{A} \rightsquigarrow_i \mathcal{B} \wedge \mathcal{B} \rightsquigarrow_i \mathcal{A},$$

then for all $\mathcal{A} \in \mathfrak{C}$, $DgSp_{\sim_1}(F(\mathcal{A})) \leq_s DgSp_{\sim_2}(\mathcal{A})$.

It is an easy exercise to see that $g \circ f$ induces a computable functor $H : (\mathfrak{G}, \hookrightarrow) \to (\mathfrak{G}, \preccurlyeq)$ and thus for all $\mathcal{G} \in \mathfrak{G}$, $DgSp_{\approx}(H(\mathcal{G})) \leq_s DgSp_{\approxeq}(\mathcal{G})$.

To get that every degree spectrum realized in $\mathfrak{C}$ is also realized in $\mathfrak{D}$ we need a stronger notion of reducibility. To define this we need an effectivization of the category theoretic notion of a natural isomorphism between functors.

DEFINITION 4.2 [13]. A functor $F : \mathfrak{C} \to \mathfrak{D}$ is *effectively isomorphic* to $G : \mathfrak{C} \to \mathfrak{D}$ if there is a Turing operator Λ such that for every $\mathcal{A} \in \mathfrak{C}$, $\Lambda^{\mathcal{A}}$ is an isomorphism from $F(\mathcal{A})$ to $G(\mathcal{A})$, and the following diagram commutes for all $\mathcal{A}, \mathcal{B} \in \mathfrak{C}_1$ and every $\gamma : \mathcal{A} \to \mathcal{B} \in \mathfrak{C}_2$.

$$\begin{tikzcd} F(\mathcal{A}) \arrow[r, "\Lambda^{\mathcal{A}}"] \arrow[d, "F(\gamma)"] & G(\mathcal{A}) \arrow[d, "G(\gamma)"] \\ F(\mathcal{B}) \arrow[r, "\Lambda^{\mathcal{B}}"] & G(\mathcal{B}) \end{tikzcd}$$

DEFINITION 4.3 (cf. [13]). We say that $(\mathfrak{C}, \rightsquigarrow_1)$ is *CBF-reducible* to $(\mathfrak{D}, \rightsquigarrow_2)$, $(\mathfrak{C}, \rightsquigarrow_1) \leq_{CBF} (\mathfrak{D}, \rightsquigarrow_2)$ if

(1) there is a computable functor $F : \mathfrak{C} \to \mathfrak{D}$ and a computable functor $G : \mathfrak{D} \supseteq \hat{\mathfrak{D}} \to \mathfrak{C}$ where $\hat{\mathfrak{D}}$ is the $\sim_2$-closure of $F(\mathfrak{C})$,

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(2) $F \circ G$ is effectively isomorphic to $Id_{\hat{\mathfrak{D}}}$, $G \circ F$ is effectively isomorphic to $Id_{\mathfrak{C}}$, and
(3) if $\Lambda_{\mathfrak{C}}$, $\Lambda_{\mathfrak{D}}$ are the operators witnessing the effective isomorphism between $G \circ F$ and $Id_{\mathfrak{C}}$, respectively, $F \circ G$ and $Id_{\hat{\mathfrak{D}}}$, then for every $\mathcal{A} \in \mathfrak{C}$, $F(\Lambda_{\mathfrak{C}}^{\mathcal{A}}) = \Lambda_{\mathfrak{D}}^{F(\mathcal{A})} : F(\mathcal{A}) \to F(G(F(\mathcal{A})))$ and every $\mathcal{B} \in \hat{\mathfrak{D}}$, $G(\Lambda_{\mathfrak{D}}^{\mathcal{B}}) = \Lambda_{\mathfrak{C}}^{G(\mathcal{B})} : G(\mathcal{B}) \to G(F(G(\mathcal{B})))$.

Consider two structures $\mathcal{A}$ and $\mathcal{B}$ and a morphism $f : \mathcal{A} \cong \mathcal{B}$. Then, clearly $\mathcal{A} \leq_T \mathcal{B} \oplus f$; after all, we have that $R^{\mathcal{A}}(a_1, \dots, a_n)$ if and only if $R^{\mathcal{B}}(f(a_1), \dots, f(a_n))$. The following definition generalizes this observation.

Definition 4.4. A category $\mathfrak{C}$ is *degree invariant* if for every $\mathcal{A}, \mathcal{B} \in \mathfrak{C}_1$ and every $f : \mathcal{A} \to \mathcal{B} \in \mathfrak{C}_2$, $f \equiv_T f^{-1}$ and $\mathcal{A} \leq_T \mathcal{B} \oplus f$.

Proposition 4.5. *If $\mathfrak{C}$ and $\mathfrak{D}$ are degree invariant and $\mathfrak{C} \leq_{CBF} \mathfrak{D}$, then every set realized as a $\sim_1$-spectrum in $\mathfrak{C}$ is realized as a $\sim_2$-spectrum in $\mathfrak{D}$.*

Proof. Say $F : \mathfrak{C} \to \mathfrak{D}$ and $G : \mathfrak{D} \to \mathfrak{C}$ witness that $\mathfrak{C} \leq_{CBF} \mathfrak{D}$. Fix $\mathcal{A} \in \mathfrak{C}$ and let Λ be the Turing operator witnessing that $G \circ F$ is effectively isomorphic to the identity functor on $\mathfrak{C}$. Then, for $\hat{\mathcal{A}} \sim_1 \mathcal{A}$, $\hat{\mathcal{A}} \geq_T F(\hat{\mathcal{A}}) \geq_T G(F(\hat{\mathcal{A}}))$ and by degree invariance $\hat{\mathcal{A}} \leq_T \Lambda^{\mathcal{A}} \oplus G(F(\hat{\mathcal{A}})) \equiv_T G(F(\hat{\mathcal{A}})) \leq_T F(\hat{\mathcal{A}})$. Thus, $DgSp_{\sim_1}(\mathcal{A}) \subseteq DgSp_{\sim_2}(F(\hat{\mathcal{A}}))$. The proof that $DgSp_{\sim_1}(\mathcal{A}) \supseteq DgSp_{\sim_2}(F(\hat{\mathcal{A}}))$ is similar. So, if X is a $\sim_1$ spectrum realized by $\mathcal{A}$ in $\mathfrak{C}$, then it is realized as a $\sim_2$ spectrum in $\mathfrak{D}$. ⊣

Notice that if $\mathfrak{K}$ is a class of relational structures, then whether $(\mathfrak{K}, \rightsquigarrow)$ is degree invariant only depends on $\rightsquigarrow$. Thus we might say that a relation on structures is degree invariant.

Definition 4.6. A class of structures $\mathfrak{C}$ is *CBF-complete* with respect to a degree invariant relation $\rightsquigarrow$, if for every class $\mathfrak{K}$, $(\mathfrak{K}, \rightsquigarrow) \leq_{CBF} (\mathfrak{C}, \rightsquigarrow)$.

We showed in Section 3.2 that for any class $\mathfrak{K}$ equipped with the elementary embeddability relation there is a computable reduction g from $(\mathfrak{K}, \preccurlyeq)$ to $(\mathfrak{G}, \preccurlyeq)$. We can now show that these reductions induce CBF-reductions $(\mathfrak{K}, \preccurlyeq) \leq_{CBF} (\mathfrak{G} \preccurlyeq)$ and that thus graphs are CBF-complete for elementary embeddability. Verifying the conditions of Definition 4.3 is quite technical, but the core ideas of the proof should not be too difficult.

Theorem 4.7. *The class of graphs is CBF-complete for elementary embeddability.*

Proof. Fix a class $\mathfrak{K}$. It is clear from the construction that g induces a computable functor $F : (\mathfrak{K}, \approxeq) \to (\mathfrak{G}, \approxeq)$. We have to show that $F(\mathfrak{K})$ is closed under elementary bi-embeddability, that there is a functor $G : F(\mathfrak{K}) \to \mathfrak{K}$ such that $F \circ G$ and $G \circ F$ are effectively isomorphic to the identity on $\mathfrak{K}$, respectively, $F(\mathfrak{K})$ and that the witnesses of these effective isomorphisms agree.

Let $\mathcal{G} \approxeq F(\mathcal{A})$ for some $\mathcal{A} \in \mathfrak{K}$. We may assume without loss of generality that $\mathcal{G}$ is an elementary substructure of $F(\mathcal{A})$. For every $\bar{a} \in \mathcal{G}^{<\omega}$, $tp_{\mathcal{G}}(\bar{a}) = tp_{\mathcal{A}}(\bar{a})$. Thus $\mathcal{G}$ must contain the elements a, b, c of $F(\mathcal{A})$ with unique 3-cycles, respectively, 5-cycles and 7-cycles connected to them. Furthermore, say $\bar{a} \in \mathcal{G}$ codes elements of A in $F(\mathcal{A})$ such that $\mathcal{A} \models R_i(\bar{a})$, then this information must also be coded in $\mathcal{G}$ as

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it is preserved in the type of $\bar{a}$. We can compute a structure $G(\mathcal{G})$ as follows. Fix a $\mathcal{G}$ computable injective enumeration f of the set $\{x : a \rightarrowtail x\}$. Notice that this can be done uniformly since the set $\{x : a \rightarrowtail x\}$ is uniformly computable in all structures in $F(\mathfrak{K})$. Let the universe of $G(\mathcal{G})$ be the pull-back along f. Then for all $a_1, \dots, a_i = \bar{a} \in \omega^i$, $G(\mathcal{G}) \models R_i(\bar{a})$ if for every a_j, $j < i$, there is a chain of $i + j$ connected elements $y_1, \dots, y_{i+j}$ with $f(a_j) \rightarrowtail y_1$, all j chains share the same last element y and $y \rightarrowtail b$. Likewise, $G(\mathcal{G}) \models \neg R_i(\bar{a})$ if there are chains satisfying the above conditions with $y \rightarrowtail c$. This finishes the construction of $G(\mathcal{G})$.

Let $\mathcal{G}, \hat{\mathcal{G}} \in F(\mathfrak{K})$ and $g : \mathcal{G} \preccurlyeq \hat{\mathcal{G}}$. As both graphs are elementary bi-embeddable with images of structures in $\mathfrak{K}$, they have unique vertices a, respectively, $\hat{a}$ with 3-cycles connected to them. Computably in $\mathcal{G}$ and $\hat{\mathcal{G}}$ find the vertices and enumerate the sets $\{x : a \rightarrowtail x\}$, and $\{x : \hat{a} \rightarrowtail x\}$ using the same procedure as in the construction of $G(\mathcal{G})$ above. Let f, respectively, $\hat{f}$ be these enumerations. Now let $G(g) = \hat{f}^{-1} \circ g \circ f$. By construction $G(g) : G(\mathcal{G}) \hookrightarrow G(\hat{\mathcal{G}})$ and $G(g)$ is uniformly computable in $\mathcal{G} \oplus g \oplus \hat{\mathcal{G}}$. To see that $G(g)$ is elementary, assume towards a contradiction that it is not. Then there is $\overline{a} \in G(\mathcal{G})$ and φ such that $G(\mathcal{G}) \models \varphi(\overline{a})$ but $G(\hat{\mathcal{G}}) \not\models \varphi(G(g)(\overline{a}))$. Recall that the atomic diagram of the tuple $\overline{a}$ is coded in the type of $f(\overline{a})$ in $\mathcal{G}$ and similarly, the atomic diagram of $G(g)(\overline{a})$ is coded in the type of $g(f(\overline{a}))$ in $\hat{\mathcal{G}}$. So, g could not be elementary, a contradiction.

To see that $G \circ F$ and $F \circ G$ are effectively isomorphic to the identities on $\mathfrak{K}$ and $F(\mathfrak{K})$, respectively, first note that $G(F(\mathcal{A})) \cong \mathcal{A}$. There is a canonic isomorphism given by the composition of the maps $a \mapsto v_a$ and the enumeration f of the set $\{x : a \rightarrowtail x\}$, i.e., the isomorphism is defined by $a \mapsto f^{-1}(v_a)$. It is clearly uniformly computable, say by $\Lambda_{\mathfrak{K}}$. On the other hand let $\mathcal{G} \in F(\mathfrak{K})$, then we can compute an isomorphism between $F(G(\mathcal{G}))$ and $\mathcal{G}$ by doing the following. Every $v \in G$ either defines a relation R_i on some tuple, codes an element, or is used to define a, b, c. One can computably determine which of the three cases holds. In the second case simply map v to $v_{f^{-1}(v)}$, in the third case one can computably determine whether v is used to define a, b, c, and, using F and G, computably find the corresponding element in $F(G(\mathcal{G}))$. In the first case, we have to find the tuple $\overline{w}$ such that v is involved in the coding of the relation R_i on $\bar{w}$. We then map v to the corresponding element in the coding of R_i on the tuple $v_{f^{-1}(\overline{w})}$. It is easy to see that one can define a Turing operator $\Lambda_{\mathfrak{F}(\mathfrak{K})}$ computing this isomorphism. The Turing operators $\Lambda_{\mathfrak{F}(\mathfrak{K})}$ and $\Lambda_{\mathfrak{K}}$ will witness the effective isomorphism between $F \circ G$ and the identity on $F(\mathfrak{K})$, respectively, the effective isomorphism between $G \circ F$ and the identity on $\mathfrak{G}$.

It remains to show that the diagrams of Definition 4.2 commute and that for all $\mathcal{A} \in \mathfrak{K}$ and $\mathcal{G} \in F(\mathfrak{K})$, $F(\Lambda_{\mathfrak{K}}^{\mathcal{A}}) = \Lambda_{F(\mathfrak{K})}^{F(\mathcal{A})}$ and $G(\Lambda_{F(\mathfrak{K})}^{\mathcal{G}}) = \Lambda_{\mathfrak{K}}^{G(\mathcal{G})}$. For the commutation of the diagrams, say first that $\mathcal{A}, \hat{\mathcal{A}} \in \mathfrak{K}$ with $\imath : \mathcal{A} \preccurlyeq \hat{\mathcal{A}}$. Let $h : \mathcal{A} \to F(\mathcal{A})$ and $\hat{h} : \hat{\mathcal{A}} \to F(\mathcal{A})$ given by $h, \hat{h} : a \mapsto v_a$. We have not given an explicit definition of $F(\imath)$ yet. But notice that $F(\imath)$ is uniquely determined by the way it maps the elements v_a. In particular, if $v(x) = F(\imath)(x)$ on the elements with $a \rightarrowtail x$, then $v = F(\imath)$. Thus we have that

$$G(F(\imath)) = \hat{f}^{-1} \circ F(\imath) \circ f = \hat{f}^{-1} \circ \hat{h} \circ \imath \circ h^{-1} \circ f,$$

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and $\Lambda^{\mathcal{A}}_{\mathfrak{K}} = f^{-1} \circ h$, so

$$\Lambda^{\hat{\mathcal{A}}}_{\mathfrak{K}} \circ \imath = \hat{f}^{-1} \circ \hat{h} \circ \imath = G(F(\imath)) \circ \Lambda^{\mathcal{A}}_{\mathfrak{K}},$$

and thus $G \circ F$ is effectively isomorphic to $id_{\mathfrak{K}}$. Now, say $\mathcal{G}, \hat{\mathcal{G}} \in F(\mathfrak{K})$ with $\eta : \mathcal{G} \preccurlyeq \hat{\mathcal{G}}$. First let $x \in \mathcal{G}$ with $a \rightarrowtail x$. Let h, and $\hat{h}$ be as above, then $(\Lambda^{\hat{\mathcal{G}}}_{F(\mathfrak{K})} \circ \eta)(x) = (\hat{h} \circ \hat{f}^{-1} \circ \eta)(x)$ and $F(\hat{f}^{-1} \circ \eta \circ f) = \hat{h} \circ \hat{f}^{-1} \circ \eta \circ f \circ h^{-1}$, so

$$(F(G(\eta)) \circ \Lambda^{\mathcal{G}}_{F(\mathfrak{K})})(x) = (F(\hat{f}^{-1} \circ \eta \circ f) \circ h \circ f^{-1})(x) = (\hat{h} \circ \hat{f}^{-1} \circ \eta)(x).$$

Having established that the diagram commutes on the restricted universes we use the fact that any embedding is determined by these parts of the universes to obtain that $F \circ G$ is effectively isomorphic to $id_{F(\mathfrak{K})}$.

To verify the last condition in Definition 4.3 let $\mathcal{A} \in \mathfrak{K}$, then on $\{x : x \rightarrowtail a\}$

$$F(\Lambda^{\mathcal{A}}_{\mathfrak{K}})(x) = (\hat{h} \circ f^{-1} \circ h \circ h^{-1})(x) = (\hat{h} \circ f^{-1})(x) = \Lambda^{F(\mathcal{A})}_{F(\mathfrak{K})}(x)$$

and as there is a unique extension of this to a mapping $F(\mathcal{A}) \to F(G(F(\mathcal{A})))$ $F(\Lambda^{\mathcal{A}}_{\mathfrak{K}}) = \Lambda^{F(\mathcal{A})}_{F(\mathfrak{K})}$. At last, let $\mathcal{G} \in F(\mathfrak{K})$, then

$$G(\Lambda^{\mathcal{G}}_{F(\mathfrak{K})}) = \hat{f}^{-1} \circ h \circ f^{-1} \circ f = \hat{f}^{-1} \circ h = \Lambda^{G(\mathcal{G})}_{\mathfrak{K}},$$

where f is the enumeration of $\{x : x \rightarrowtail a\}$ in $\mathcal{G}$ and $\hat{f}$ the one in $G(F(\mathcal{G}))$. ⊣

The following is a direct consequence of Proposition 4.5 and Theorem 4.7.

COROLLARY 4.8. *For every structure $\mathcal{A}$, there is a graph $\mathcal{G}_{\mathcal{A}}$ such that*

$$DgSp_{\approxeq}(\mathcal{A}) = DgSp_{\approxeq}(\mathcal{G}_{\mathcal{A}}).$$

Unfortunately, for the reduction from bi-embeddability on graphs to elementary bi-embeddability on $\mathfrak{C}$ given in Section 3.1 we cannot deduce that $(\mathfrak{G}, \hookrightarrow) \leq_{CBF} (\mathfrak{C}, \preccurlyeq)$. However, we can still establish a relationship between the degree spectra in these classes. Recall that $\mathcal{S}_0$ is the substructure of $\mathcal{S}$ generated by the constant string of 0's and $\mathcal{S}_1$ is the substructure generated by the constant string of 1's.

LEMMA 4.9. *Let X be $\Delta^0_2(Y)$ for some set Y. Then there exist a sequence of structures $(\mathcal{C}_i)_{i \in \omega}$, uniformly computable in Y, such that for all $i \in \omega$*

$$\mathcal{C}_i \cong \begin{cases} \mathcal{S}_0 & \text{if } i \in X, \\ \mathcal{S}_1 & \text{if } i \notin X. \end{cases}$$

PROOF. As X is Δ^0_2 there is an X-computable two-valued function f such that

$$\lim_{s \to \infty} f(i, s) = \begin{cases} 0 & \text{if } i \in X, \\ 1 & \text{if } i \notin X. \end{cases}$$

Define a structure $\mathcal{C}$ as follows. Fix an enumeration g of $2^{<\omega}$. At Stage 0 define $\mathcal{C}_0$ to be the partial structure containing one element a on which no relation holds and leave all function symbols undefined. Say we have defined the structure $\mathcal{C}_s$. At Stage $s + 1$ we look at $f(i, j)$ for $j < s$ and define $\mathcal{C}_{s+1}$ as if a was the finite string with $a(j) = f(j)$ for $j < s$. To be more precise:

(1) For all k, if $k \leq s$ and $|g(k)| \leq s$ then let $R_{g(k)}(a)$ if and only if $g(k) \preceq a$, and if $F_{g(k)}(a)$ has not been defined yet add a new element and set $F_{g(k)}(a)$.
(2) We may assume by induction that for all elements b in $\mathcal{C}_{s+1}$ there is $k \leq s$ such that $b = F_{g(k)}(a)$. We set $R_{g(l)}(b)$ respecting this equation for all $l \leq s$.

It is easy to see that this procedure yields a computable sequence of structures $\mathcal{C}_s$ with $\mathcal{C}_s \subseteq \mathcal{C}_{s+1}$ and a computable structure as its limit. We let $\mathcal{C}$ be this structure. $\mathcal{C}$ contains an element a such that $\mathcal{A} \models R_\sigma(a)$ if and only if $a \preceq f(i,-)$ and all other elements are equal to $F_\tau(a)$ for some $\tau \in 2^{<\omega}$. Thus, in particular if $\lim f(i,s) = 0$, then there is an element representing the constant string of 0's in $\mathcal{A}$ and otherwise there is an element representing the constant string of 1's in $\mathcal{A}$. Let $\mathcal{C}_i = \mathcal{C}$, then $\mathcal{C}_i \cong \mathcal{S}_0$ if and only if $i \in X$ and $\mathcal{C}_i \cong \mathcal{S}_1$ if and only if $i \notin X$ as required. $\dashv$

We use the usual category theoretic definition of *pseudo-inverse*. Two functors $F : \mathfrak{C} \to \mathfrak{D}$ and $G : \mathfrak{D} \to \mathfrak{C}$ are pseudo-inverses if $F \circ G$ is naturally isomorphic to $id_{\mathfrak{D}}$ and $G \circ F$ is naturally isomorphic to $id_{\mathfrak{C}}$.

Recall that a structure $\mathcal{A}$ is automorphically trivial if there is a finite set $D \subseteq A$ such that every permutation of A that fixes D pointwise is an automorphism. Knight [16] showed that isomorphism spectra of automorphically trivial structures contain only one Turing degree and that the isomorphism spectra of automorphically non-trivial structures are upwards closed in the Turing degrees. In [7] the authors showed that if $\mathcal{A}$ is automorphically trivial and $\mathcal{B} \approx \mathcal{A}$, then $\mathcal{B} \cong \mathcal{A}$. Thus, as every bi-embeddability and elementary bi-embeddability spectrum is a union of isomorphism spectra, Knight's result carries over to this setting.

LEMMA 4.10. *For every automorphically non-trivial structure $\mathcal{G} \in \mathfrak{G}$ there is $\mathcal{A} \in \mathfrak{C}$ such that*

$$DgSp_{\approxeq}(\mathcal{A}) = \{X : X' \in DgSp_{\approx}(\mathcal{G})\}.$$

PROOF. Recall the reduction from embeddability on graphs to elementary embeddability on $\mathfrak{C}$ given in Section 3.1. It is easy to see that it induces a computable functor $F : (\mathfrak{G}, \approx) \to (\mathfrak{C}, \approxeq)$. We show that the functor has a pseudo-inverse G on the $\approxeq$-saturation of $F(\mathfrak{G})$ and then use Lemma 4.9 to obtain the lemma. The minimality of the submodels $\mathcal{S}$ used in the construction of F will play a crucial role in the proof.

Say $\mathcal{B} \approxeq F(\mathcal{G})$ for $\mathcal{G} \in \mathfrak{G}$, that x, y are vertices in $\mathcal{B}$ and $\mathcal{S}_{(x,y)}$ is the substructure on the elements satisfying $O(x, y, -)$ in the reduct to the language of $\mathcal{S}$. We have that either $\mathcal{S}_{(x,y)} \cong \mathcal{S}_0$ or $\mathcal{S}_{(x,y)} \cong \mathcal{S}_1$ since it elementary embeds into $\mathcal{S}_{(u,v)}$ for some $u, v \in F(\mathcal{G})$ and $\mathcal{S}_{(u,v)} \cong \mathcal{S}_0$ or $\mathcal{S}_{(u,v)} \cong \mathcal{S}_1$ by minimality. Thus, we get a graph $G(\mathcal{B})$ from $\mathcal{B}$ by defining an edge between two vertex variables x, y from $\mathcal{B}$ if and only if $\mathcal{S}_{(u,v)} \cong \mathcal{S}_0$. Clearly every elementary embedding of $\mathcal{B}$ in $F(\mathcal{G})$ yields an embedding of $G(\mathcal{B})$ in $\mathcal{G} \cong G(F(\mathcal{G}))$ and the analogous fact is true for every elementary embedding of $F(\mathcal{G})$ in $\mathcal{B}$. Likewise, we can argue that $F(G(\mathcal{A})) \cong \mathcal{A}$ for every $\mathcal{A} \in F(\mathfrak{G})$. Thus G and F are pseudo-inverses.

However, notice that G is not effective. Within one jump over the diagram of any $\mathcal{B} \in F(\mathfrak{G})$ we can compute $G(\mathcal{B})$ as the isomorphism types of $\mathcal{S}_1$ and $\mathcal{S}_0$ are definable by Σ^c_2 formulas in $\mathfrak{C}$. This implies that for all $\mathcal{A} \in \mathfrak{G}$, $F(\mathcal{A})' \geq_T G(F(\mathcal{A})) \cong \mathcal{A}$. So, in particular,

$$DgSp_{\approxeq}(F(\mathcal{A})) \geq_s \{X : X' \in DgSp_{\approx}(G(F(\mathcal{A}))) = DgSp_{\approx}(\mathcal{A})\}. \qquad (1)$$

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On the other hand, let $X \in DgSp_{\approx}(\mathcal{A})$ and $\hat{\mathcal{A}} \approx \mathcal{A}$ such that $\hat{\mathcal{A}} \equiv_T X$. Then by Lemma 4.9 for every Y with $Y' \geq_T X$, there is $\mathcal{B} \cong F(\hat{\mathcal{A}})$ with $\mathcal{B} \equiv_T Y$. This process is uniform in $\mathcal{B}$ and Y. Thus

$$DgSp_{\approxeq}(F(\mathcal{A})) \leq_s \{X : X' \in DgSp_{\approx}(\mathcal{A})\}. \tag{2}$$

As both bi-embeddability and elementary bi-embeddability spectra of automorphically non-trivial structures are upwards closed, Equations (1) and (2) imply that

$$DgSp_{\approxeq}(F(\mathcal{A})) = \{X : X' \in DgSp_{\approx}(\mathcal{A})\}. \qquad \dashv$$

Theorem 1.3 follows directly from Lemma 4.10 and Corollary 4.8.

We note that Theorem 1.3 may not be optimal. Using a different proof one might be able to get an even stronger relationship between the spectra realized by bi-embeddability on graphs and elementary bi-embeddability on graphs. We thus ask:

QUESTION 4.11. *Is every bi-embeddability spectrum of a graph the elementary bi-embeddability spectrum of a graph and vice versa*?

One way to answer this question positively is by showing that if X is an elementary bi-embeddability spectrum then so is $X' = \{x' : x \in X\}$. This is true for isomorphism spectra and usually shown by considering an appropriate definition for the jump of a structure. However, all known definitions do not preserve elementary embeddability (and not even elementary equivalence). We thus ask:

QUESTION 4.12. *Let X be the elementary bi-embeddability spectrum of a graph. Is X' the elementary bi-embeddability spectrum of a graph*?

QUESTION 4.13. *Let X be the theory spectrum of a graph. Is X' the theory spectrum of a graph*?

Also, while Theorem 4.7 shows that graphs are complete for elementary bi-embeddability spectra, it is unknown whether the same is true for bi-embeddability.

QUESTION 4.14. *Is every bi-embeddability spectrum of a structure realized as the bi-embeddability spectrum of a graph*?

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