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USING IDEALS TO PROVIDE A UNIFIED APPROACH TO UNIQUELY CLEAN RINGS

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Abstract

In this article, we introduce the notion of the uniquely I-clean ring and show that, if R is a ring and I is an ideal of R then R is uniquely I-clean if and only if (R/I is Boolean and idempotents lift uniquely modulo I) if and only if (for each $a \in R$ there exists a central idempotent $e \in R$ such that $e - a \in I$ and I is idempotent-free). We examine when ideal extension is uniquely clean relative to an ideal. Also we obtain conditions on a ring R and an ideal I of R under which uniquely I-clean rings coincide with uniquely clean rings. Further we prove that a ring R is uniquely nil-clean if and only if ($N(R)$ is an ideal of R and R is uniquely $N(R)$-clean) if and only if R is both uniquely clean and nil-clean if and only if (R is an abelian exchange ring with $J(R)$ nil and every quasiregular element is uniquely clean). We also show that R is a uniquely clean ring such that every prime ideal of R is maximal if and only if R is uniquely nil-clean ring and $N(R) = \mathrm{Nil}_*(R)$.

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1. Introduction

Throughout this paper, by a ring R we mean an associative ring with identity. If R is a ring, then $U(R)$, $\mathrm{Id}(R)$ and $N(R)$ denote respectively the set of all units, the set of all idempotents and the set of all nilpotent elements of R. We denote the Jacobson radical, lower nil radical and center of R by $J(R)$, $\mathrm{Nil}_*(R)$ and $C(R)$ respectively and write $T_n(R)$ for the ring of all $n \times n$ upper triangular matrices over R.

A ring R is said to have *stable range one* if for any $a, b \in R$ with $aR + bR = R$ there exists $y \in R$ such that $a + by \in U(R)$. If in addition $y \in \mathrm{Id}(R)$ then R is said to *satisfy idempotent 1-stable range*. If I is a right (or left) ideal of a ring R then we say that idempotents *lift* (respectively, *lift uniquely*, *lift centrally)* modulo I if, for each $x \in R$ with $x - x^2 \in I$ there exists an idempotent (respectively, unique idempotent, central idempotent) $e \in R$ such that $e - x \in I$. Vaserstein [8] has shown that a ring R has stable

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range one if and only if $R/J(R)$ has stable range one, and in [4] author has proved that a ring R satisfies idempotent 1-stable range if and only if $R/J(R)$ satisfies idempotent 1-stable range and idempotents lift modulo $J(R)$.

A ring R is said to be an *exchange* ring if for every $a \in R$ there exists an idempotent $e \in R$ such that $e \in aR$ and $1 - e \in (1 - a)R$. A ring R is said to be *clean* (respectively, *uniquely clean*) if for every element a of R there exists an idempotent (respectively, a unique idempotent) $e \in R$ such that $a - e$ is a unit. The notion of the clean ring was introduced by Nicholson [6] as a sufficient condition for a ring R to be an exchange ring. The study of noncommutative uniquely clean rings was started by Nicholson and Zhou. In [7] the authors proved that every factor ring of a uniquely clean ring is uniquely clean. This implies that $R/M \cong \mathbb{Z}_2$ for all maximal ideals M of a uniquely clean ring R, and in [3] the author proved that a ring R is uniquely clean if and only if R is an abelian exchange ring such that $R/M \cong \mathbb{Z}_2$ for all maximal ideals M of R if and only if R is an exchange ring, $R/J^*(R)$ is Boolean and idempotents lift uniquely modulo $J^*(R)$ where $J^*(R) = \cap\{M | M$ is a maximal ideal of $R\}$, the *Brown–McCoy radical* of R. An ideal P of a ring R is said to be *prime* if $P \neq R$ and for ideals $U, B \subseteq R$, $U \cdot B \subseteq P$ implies that $U \subseteq P$ or $B \subseteq P$. It is proved in [3] that if R is a commutative ring then R is uniquely nil-clean if and only if R is uniquely clean and every prime ideal in it is maximal. Call a ring R *directly finite* if for $a, b \in R$, $ab = 1$ implies $ba = 1$. An idempotent e in a ring R is said to be a *full idempotent* if (e), the ideal generated by e, equals R, that is, $(e) = ReR = R$. A ring R is said to be *semipotent* if every right (respectively, left) ideal not contained in $J(R)$ contains a nonzero idempotent, and is said to be *potent* if in addition idempotents lift modulo $J(R)$. Following Alkan *et al.* [1], a left ideal I of a ring R is said to be *weakly enabling* if for any $a \in R$ with $a - 1 \in I$ there exists $f^2 = f \in Ra$ such that $a - f \in I$. An ideal I of a ring R is said to be *idempotent-free* if it contains no nonzero idempotent.

It is well known that uniquely clean $\Rightarrow$ clean $\Rightarrow$ exchange $\Rightarrow$ potent $\Rightarrow$ semipotent.

In this article we unify the notions of uniquely clean and uniquely nil-clean rings, using ideals. In [7] Nicholson and Zhou obtained equivalent conditions for a ring R to be uniquely clean. One of them is that for all $a \in R$ there exists a unique idempotent $e \in R$ such that $e - a \in J(R)$. Replacing $J(R)$ by an arbitrary ideal I, we introduce the notion of uniquely I-clean rings. First we obtain basic properties and characterization of such rings. We show that if R is a uniquely I-clean ring for some ideal I of R then I is prime if and only if I is maximal. It is shown that [8, Theorem 2.2] and [4, Theorem 9] continue to hold good if we replace $J(R)$ by any ideal contained in $J(R)$. Also we examine when an ideal extension is uniquely clean relative to an ideal and improve [7, Proposition 7]. In Section 3 we obtain conditions on a ring R and and an ideal I of R under which uniquely I-clean rings coincide with uniquely clean rings. In the last section we study uniquely nil-clean rings, characterizing them and improving [3, Theorem 4.1] and its corollary.

2. Uniquely I-clean rings

DEFINITION 2.1. If R is a ring and I is an ideal of R then we say that R is *uniquely I-clean* if for each $x \in R$ there exists unique idempotent $e \in R$ such that $x - e \in I$.

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Example 2.2. The ring $\mathbb{Z}$ of integers is uniquely $2\mathbb{Z}$-clean.

Proposition 2.3. *A ring R is uniquely clean if and only if R is uniquely $J(R)$-clean.*

Proof. To prove the 'only if' part, let $a \in R$. Since R is uniquely clean, there exist a unique $e \in \mathrm{Id}(R)$ and $u \in U(R)$ such that $-a = (1-e) + u$. This implies $e - a = 1 + u$.

Claim. $e - a \in J(R)$.

Suppose the contrary. Since every uniquely clean ring is semipotent there exists $0 \neq f^2 = f \in (e-a)R$. Let $f = (e-a)r$ for some $r \in R$. Note that, by hypothesis and [7, Lemma 4], R is abelian. Now it is easy to see that $[(e-a)-(1-f)][(1-f)u^{-1} + fr] = 1$. Since R is abelian, it is directly finite. Therefore $v = [(e-a)-(1-f)] \in U(R)$. Thus $e - a = 1 + u = (1-f) + v$. Since R is uniquely clean we get $f = 0$, which is a contradiction. This proves the claim. Now suppose that $f \in \mathrm{Id}(R)$ such that $f - a \in J(R)$. This implies $w = 1 - (f-a) \in U(R)$, that is, $-a = (1-f) + (-w)$. Since R is uniquely clean we get $e = f$. Thus R is uniquely $J(R)$-clean.

Turning to the 'if' part, note that, by hypothesis, it follows that R is clean and $R/J(R)$ is Boolean. Thus for any $u \in U(R)$ we have $\bar{u} = \bar{1}$ in $R/J(R)$. Now let $a \in R$ and suppose that $a = e + u = f + v$ for some $e, f \in \mathrm{Id}(R)$ and $u, v \in U(R)$. This implies $\bar{e} - \bar{f} = \bar{v} - \bar{u} = \bar{1} - \bar{1} = \bar{0}$ which implies $e - f \in J(R)$. Thus by hypothesis $e = f$. Hence R is uniquely clean. □

Proposition 2.4. *If R is a uniquely I-clean ring for some ideal I of R then the following hold.*

(1) *I is idempotent-free.*
(2) *I contains all the quasiregular elements (in particular, $N(R) \subseteq I$ and $J(R) \subseteq I$).*
(3) *R is abelian and hence R is directly finite.*

Proof. (1) If $e^2 = e \in I$ then $1 + 0 = 1 = (1-e) + e$. Since R is uniquely I-clean we get $e = 0$.

(2) Let $a \in R$ be quasiregular. Since R is uniquely I-clean, $1 - a = 1 - e + y$ for some $e \in \mathrm{Id}(R)$ and $y \in I$ which implies $(1-a)e = ye \in I$. Since $1 - a$ is a unit, we get that $e \in I$. Thus by (1), we get $e = 0$ which implies $a \in I$.

(3) Let $e \in \mathrm{Id}(R)$. Now for any $r \in R$ we have $er - ere \in N(R) \subseteq I$, by (2). Clearly $x = e + er - ere \in \mathrm{Id}(R)$. Therefore $x = (e + er - ere) + 0 = e + (er - ere)$. Since R is uniquely I-clean, we get that that $er - ere = 0$, that is $er = ere$. Similarly, one can prove that $re = ere$. Thus R is abelian. The latter statement follows from the fact that every abelian ring is directly finite. □

Lemma 2.5. *Let R be a ring and $e = f + n$ where $n \in N(R)$, $e \in \mathrm{Id}(R)$ and $f^2 = f \in C(R)$. Then $n = 0$.*

Proof. Let $n \in N(R)$, $e \in \mathrm{Id}(R)$ and $f^2 = f \in C(R)$ with $e = f + n$. Note that e, f and n commute with each other. Therefore $(1-e)f = (1-e)(-n) \in N(R)$. Since $(1-e)f \in \mathrm{Id}(R)$ we get $(1-e)f = 0$, that is, $f = ef$. Similarly, $(1-f)e = (1-f)n \in N(R)$ which implies $e = fe$. Therefore $e = f$ and hence $n = 0$. □

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We now give examples of noncommutative rings R which are uniquely I-clean for some ideal I of R.

Example 2.6. If R is uniquely I-clean for some ideal I of R then $S = \{[a_{ij}] \in T_n(R) | a_{11} = a_{22} = \cdots = a_{nn}\}$ is uniquely I'-clean where $I' = \{[a_{ij}] \in S | a_{ii} \in I\}$.

Proof. Clearly I' is an ideal of S. Now note that $\mathrm{Id}(S) = \{eI_n | e \in \mathrm{Id}(R)$ and I_n is the $n \times n$ identity matrix$\}$. For let $F = [f_{ij}] \in \mathrm{Id}(S)$ with $f_{ii} = f$ for all $i = 1, \ldots, n$. Then $f \in \mathrm{Id}(R)$. Now $F = fI_n + (F - fI_n)$. Since R is uniquely I-clean we have by Proposition 2.4(3) that R is abelian. Thus f is a central idempotent and hence fI_n is a central idempotent in S. Also note that $(F - fI_n)$ is nilpotent (all the diagonal entries being zero). Therefore by Lemma 2.5, $F = fI_n$. The other part is easy to verify. Now let $A = [a_{ij}] \in S$ with $a_{ii} = a$ for all $i = 1, \ldots, n$. Since R is uniquely I-clean there exist unique $e \in \mathrm{Id}(R)$ and $y \in I$ such that $a = e + y$. This implies $A = eI_n + B$ where

$$B_{ij} = \begin{cases} y & \text{if } i = j, \\ a_{ij} & \text{otherwise.} \end{cases}$$

The uniqueness of this expression follows from the uniqueness in R. □

In [3] the author proved that for a uniquely clean ring R, $J^*(R) = J(R)$. Our next lemma generalizes this result.

Lemma 2.7. *For a semipotent ring R without full idempotents (for example, an abelian semipotent ring) $J^*(R) = J(R)$.*

Proof. Let R be a semipotent ring without full idempotents. Note that $J(R) \subseteq J^*(R)$ is always true. Now we prove that $J^*(R) \subseteq J(R)$. Suppose the contrary. Since R is semipotent, there exists a nonzero idempotent e in $J^*(R)$ which implies that $e \in M$ for every maximal ideal. Since by hypothesis R has no full idempotent, there exists a maximal ideal, say M_0, containing the ideal $((1 - e))$. Therefore M_0 contains both e and $(1 - e)$, which is a contradiction. Hence $J^*(R) = J(R)$. □

Lemma 2.8. *Let R be a ring and I be an ideal such that $N(R) \subseteq I$. Then the following are equivalent.*

(1) *Idempotents lift uniquely modulo I.*
(2) *Idempotents lift modulo I, R is abelian and I is idempotent-free.*
(3) *Idempotents lift centrally modulo I and I is idempotent-free.*

Proof. (1) ⇒ (2). Let $e \in \mathrm{Id}(R)$ and $r \in R$. Then $er - ere \in N(R) \subseteq I$, by hypothesis. Note that both e and $e + er - ere$ are idempotents such that $\bar{e} = \overline{e + er - ere}$ in R/I. Since idempotents lift uniquely modulo I, we get that $e = e + er - ere$ which implies $er - ere = 0$, that is, $er = ere$. A similar argument shows that $re = ere$. Hence R is abelian. Now let $e^2 = e \in I$. Then $\bar{e} = \bar{0}$ in R/I. Since idempotents lift uniquely modulo I we get that $e = 0$. Thus I is idempotent-free.

(2) ⇒ (3). Obvious.

(3) ⇒ (1). Let $\bar{x}$ be an idempotent in R/I. By hypothesis there exists a central idempotent $e \in R$ such that $x - e \in I$. Suppose that $f \in \mathrm{Id}(R)$ such that $x - f \in I$. Then $e - f \in I$ and hence $(1 - f)e \in I$. Since e is central, $(1 - f)e \in \mathrm{Id}(R)$. This implies $(1 - f)e = 0$ because I is idempotent-free. Therefore $e = fe$. Similarly, one can prove that $f = ef$. Since e is central, $ef = fe$ and hence $e = f$. Therefore idempotents lift uniquely modulo I. □

Remark 2.9. The condition $N(R) \subseteq I$ in the above lemma cannot be dropped. For example, let R be a nonabelian regular ring (in the sense of von Neumann). Then $J(R) = 0$ and $N(R)$ is not contained in $J(R)$. Note that idempotents lift uniquely modulo $J(R)$ but R is not abelian.

Theorem 2.10. *Let R be a ring and I be an ideal. Then the following are equivalent.*

(1) *R is uniquely I-clean.*
(2) *R/I is Boolean and idempotents lift uniquely modulo I.*
(3) *R/I is Boolean, idempotents lift modulo I, R is abelian and I is idempotent-free.*

Proof. (1) ⇒ (2). Obvious.

(2) ⇒ (1). Let $x \in R$. Since R/I is Boolean, $\bar{x}$ is an idempotent in R/I. Since idempotents lift uniquely modulo I, there exists a unique $e \in \mathrm{Id}(R)$ such that $\bar{x} = \bar{e}$, that is, $x = e + y$ for a unique $e \in \mathrm{Id}(R)$ and $y \in I$. Therefore R is uniquely I-clean.

(2) ⇔ (3). Since R/I is Boolean, $N(R) \subseteq I$. Now the result follows from Lemma 2.8. □

Corollary 2.11. *Let R be a ring and I be an ideal of R. Then the the following are equivalent.*

(1) *R is uniquely I-clean.*
(2) *R/I is Boolean, idempotents lift centrally modulo I and I is idempotent-free.*
(3) *For each $a \in R$ there exists a central idempotent $e \in R$ such that $e - a \in I$ and I is idempotent-free.*

Proof. (1) ⇒ (2). Since by hypothesis R is uniquely I-clean, we get by Theorem 2.10 that R/I is Boolean, and idempotents lift uniquely modulo I. Therefore $N(R) \subseteq I$ and hence, by Lemma 2.8, idempotents lift centrally modulo I and I is idempotent-free.

(2) ⇒ (1). Since R/I is Boolean, $N(R) \subseteq I$. Since idempotents lift centrally modulo I and I is idempotent-free, we have by Lemma 2.8 that idempotents lift uniquely modulo I. Hence, by Theorem 2.10, R is uniquely I-clean.

(2) ⇔ (3). Obvious. □

As a consequence of Theorem 2.10, we deduce [7, Theorem 20] and [3, Theorem 2.1] as corollaries.

Corollary 2.12 [7, Theorem 20]. *The following are equivalent for a ring R.*

(1) *R is uniquely clean.*
(2) *$R/J(R)$ is Boolean and idempotents lift uniquely modulo $J(R)$.*

(3) *$R/J(R)$ is Boolean, idempotents lift modulo $J(R)$ and idempotents in R are central.*
(4) *For all $a \in R$ there exists a unique idempotent $e \in R$ such that $e - a \in J(R)$.*

Proof. (1) ⇒ (2). This follows from Proposition 2.3 and Theorem 2.10.

(2) ⇒ (3). This follows from Theorem 2.10.

(3) ⇒ (4). Since $J(R)$ is idempotent-free ideal we have, by Theorem 2.10, R is uniquely $J(R)$-clean and hence (4) follows.

(4) ⇒ (1). By hypothesis R is uniquely $J(R)$-clean and hence, by Proposition 2.3, R is uniquely clean. □

Corollary 2.13 [3, Theorem 2.1]. *A ring R is uniquely clean if and only if:*

(1) *R is an exchange ring with all idempotents central;*
(2) *for all maximal ideals M of R, $R/M \cong \mathbb{Z}_2$.*

Proof. To prove the 'only if' part, let R be uniquely clean. This implies, by Proposition 2.3, that R is uniquely $J(R)$-clean. Therefore, by Theorem 2.10, R is abelian and $R/J(R)$ is Boolean. As R is uniquely clean it is clean and hence, by [6, Proposition 1.8(1)], R is an exchange ring. Note that for every maximal ideal M of R, $J(R) \subseteq M$. Therefore $R/M \cong (R/J(R))/M/J(R)$. Since $R/J(R)$ is Boolean, R/M is Boolean. As M is a maximal ideal, R/M is a simple ring and hence $R/M \cong \mathbb{Z}_2$.

We now prove the 'if' part. Since by hypothesis R is an abelian exchange ring, it is an abelian semipotent ring. Therefore, by Lemma 2.7, $J(R) = J^*(R) = \cap\{M | M$ is a maximal ideal of $R\}$. Now by hypothesis (2), for any $a \in R$ we have $a^2 - a = a(a-1) \in M$ for every maximal ideal M. Therefore $a^2 - a \in J(R)$ for all $a \in R$ and hence $R/J(R)$ is Boolean. Further, by hypothesis (1), R is an exchange ring which implies that idempotents lift modulo $J(R)$. Thus, by Theorem 2.10, R is uniquely $J(R)$-clean and hence, by Proposition 2.3, R is uniquely clean. □

Proposition 2.14. *For a ring R and an ideal I of R, the following are equivalent.*

(1) *I is prime and R is uniquely I-clean.*
(2) *R is uniquely I-clean and 0,1 are the only idempotents in R.*
(3) *$R/I \cong \mathbb{Z}_2$ and I is idempotent-free.*
(4) *I is maximal and R is uniquely I-clean.*

Proof. (1) ⇒ (2). Let $e \in \mathrm{Id}(R)$. Since R is uniquely I-clean, we get by Proposition 2.4(3) that R is abelian. Therefore $eR(1-e) = 0 \in I$. Since I is prime, either $e \in I$ or $1 - e \in I$. Note that by Proposition 2.4(1), I is idempotent-free. Therefore either $e = 0$ or $e = 1$.

(2) ⇒ (3). Let $\bar{x} \in R/I$. Now by hypothesis, $x \in I$ or $x - 1 \in I$. Therefore $\bar{x} = \bar{0}$ or $\bar{x} = \bar{1}$ in R/I. Thus $R/I \cong \mathbb{Z}_2$.

(3) ⇒ (4). Since $R/I \cong \mathbb{Z}_2$, I is maximal. Further, for any $x \in R$, $\bar{x} = \bar{0}$ or $\bar{x} = \bar{1}$ in R/I. Therefore, by Corollary 2.11, R is uniquely I-clean.

(4) ⇒ (1). Very clear. □

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As a consequence of the above proposition we deduce [7, Theorem 15] as a corollary.

Corollary 2.15 [7, Theorem 15]. *The following are equivalent for a ring $R \neq 0$.*

(1) *R is local and uniquely clean.*
(2) *R is uniquely clean and the only idempotents in R are 0 and 1.*
(3) *$R/J(R) \cong \mathbb{Z}_2$.*

Proof. This result follows from Propositions 2.3, 2.14 and the fact that $J(R)$ is idempotent-free. □

The following result is proved in [4, Theorem 9]; we state it as a lemma for reference purposes.

Lemma 2.16. *If a ring R satisfies idempotent 1-stable range then R is clean.*

Proposition 2.17. *If R is a ring and I is an ideal of R, then the following are equivalent.*

(1) *For $x \in R$, $\bar{x}$ is left invertible in R/I implies x is left invertible in R.*
(1′) *For $x \in R$, $\bar{x}$ is right invertible in R/I implies x is right invertible in R.*
(1″) *For $x \in R$, $\bar{x}$ is invertible in R/I implies x is invertible in R.*
(2) *$I \subseteq J(R)$.*
(3) *I is weakly enabling and idempotent-free.*

Proof. (1) ⇒ (1″). Let $\bar{x}$ be a unit in R/I. Then, by hypothesis, x is left invertible in R. Therefore there exists $y \in R$ such that $yx = 1$ which implies $\bar{y}\bar{x} = \bar{1}$. Since $\bar{x}$ is a unit we get that $\bar{y}$ is a unit in R/I. Therefore, by hypothesis, y is left invertible and hence x is invertible in R.

(1″) ⇒ (1). Let $\bar{x}$ be left invertible in R/I. Then there exists $\bar{y} \in R/I$ such that $\bar{y}\bar{x} = \bar{1}$. Since $\overline{yx}$ is a unit in R/I we get by hypothesis that $yx \in U(R)$. Therefore x is left invertible in R.

(1′) ⇔ (1″). This is similar to (1) ⇔ (1″).

(1) ⇒ (2). Let $y \in I$. By [5, Lemma 4.1], it is enough to show that $1 - xy$ is left invertible for all $x \in R$. This follows by hypothesis and the fact that $\overline{1 - xy} = \bar{1}$ in R/I for all $x \in R$.

(2) ⇒ (3). Let $x \in R$ such that $x - 1 \in I$. Since $I \subseteq J(R)$, we get that $1 - x = 1 - u$ for some $u \in U(R)$. Thus $x \in U(R)$ and hence $1 \in Rx$.

(3) ⇒ (1). Let $x \in R$ such that $\bar{x}$ is left invertible in R/I. Therefore there exists $\bar{y}$ in R/I such that $\bar{y}\bar{x} = \bar{1}$ which implies $yx - 1 \in I$. Since I is weakly enabling, there exists $e^2 = e \in Ryx$ such that $yx - e \in I$. This implies that $1 - e \in I$. Since I is idempotent-free we get that $e = 1$. Therefore x is left invertible, which proves (1). □

Proposition 2.18. *If I is an ideal of a ring R such that $I \subseteq J(R)$ then:*

(1) *R satisfies idempotent 1-stable range if and only if R/I satisfies idempotent 1-stable range and idempotents lift modulo I;*
(2) *R has stable range one if and only if R/I has stable range one.*

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Proof. (1) Let R satisfy idempotent 1-stable range. Then by Lemma 2.16, R is clean. Thus by [6, Proposition 1.8(1)], R is an exchange ring. Therefore by [6, Corollary 1.3], idempotents lift modulo I. Now let $\bar{a}, \bar{b} \in \bar{R} = R/I$ be such that $\bar{a}\bar{R} + \bar{b}\bar{R} = \bar{R}$. This implies $\bar{a}\bar{x} + \bar{b}\bar{y} = \bar{1}$ for some $\bar{x}, \bar{y} \in \bar{R}$. Therefore, by Proposition 2.17, $ax + by \in U(R)$. Thus $aR + bR = R$. Therefore by hypothesis there exists $e \in \mathrm{Id}(R)$ such that $a + be$ is a unit in R. Thus $\bar{a} + \bar{b}\bar{e}$ is a unit in R/I and hence R/I satisfies idempotent 1-stable range.

Conversely, suppose that $\bar{R} = R/I$ satisfies idempotent 1-stable range and idempotents lift modulo I. Let $a, b \in R$ such that $aR + bR = R$. Then clearly, $\bar{a}\bar{R} + \bar{b}\bar{R} = \bar{R}$. Therefore there exists $\bar{e} \in \mathrm{Id}(\bar{R})$ such that $\bar{a} + \bar{b}\bar{e}$ is a unit in R/I. Since idempotents lift modulo I, we may assume that $e \in \mathrm{Id}(R)$. Now the result follows from Proposition 2.17.

(2) The proof is similar to that given above. □

As a consequence of the above proposition we deduce [4, Theorem 9] and [8, Theorem 2.2] as corollaries.

Corollary 2.19 [4, Theorem 9]. *The following are equivalent for a ring R.*

(1) *R satisfies idempotent 1-stable range.*
(2) *$R/J(R)$ satisfies idempotent 1-stable range and idempotents can be lifted modulo $J(R)$.*

Corollary 2.20 [8, Theorem 2.2]. *A ring R has stable range one if and only if $R/J(R)$ has stable range one.*

Remark 2.21. If R is a uniquely I-clean ring for some ideal I of R, then R need not have stable range one (see Example 2.2). The same example shows that the condition $I \subseteq J(R)$ in Proposition 2.18 cannot be dropped.

Let R be a ring and let V be an R-R-bimodule which is also a ring, not necessarily with 1, in which $(rv)w = r(vw)$, $(vr)w = v(rw)$ and $(vw)r = v(wr)$ hold for all $v, w \in V$ and $r \in R$. Then the *ideal extension* of R by V, denoted by $I(R; V)$, is defined to be the additive abelian group $I(R; V) = R \oplus V$ with multiplication: $(r, v)(s, w) = (rs, rw + vs + vw)$ for all $v, w \in V$ and $r, s \in R$.

We end this section by examining when an ideal extension is uniquely clean relative to an ideal.

Lemma 2.22. *Let R be a ring, V be an R-R-bimodule which is also a ring (not necessarily with 1) and let $S = I(R; V)$ be the ideal extension. If V is idempotent-free and for each $e \in \mathrm{Id}(R)$, $ev = ve$ for all $v \in V$, then $\mathrm{Id}(S) = \{(e, 0) | e \in \mathrm{Id}(R)\}$.*

Proof. Let $(e, v) \in \mathrm{Id}(S)$. This implies $e \in \mathrm{Id}(R)$ and $ev + ve + v^2 = v$. Therefore, by hypothesis, $2ev + v^2 = v$. This implies $v^2 = (1 - 2e)v$. Since e and v commute and $(1 - 2e)^2 = 1$, we get $v^2 \in \mathrm{Id}(V)$. Since by hypothesis V is idempotent-free we

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have $v^2 = 0$ which implies that $(1 - 2e)v = 0$ which in turn implies that $v = 0$. This completes the proof. □

Lemma 2.23. *Let R be a ring, V be an R-R-bimodule which is also a ring (not necessarily with 1) and let $S = I(R; V)$ be the ideal extension. Then the following are equivalent.*

(1) *For every $v \in V$ there exists $w \in V$ such that $v + w + vw = 0$.*
(2) *$(1, v) \in U(S)$ for all $v \in V$.*
(3) *$U(S) = \{(u, v) \in S \mid u \in U(R)\}$.*
(4) *$J(S) = J(R) \times V$.*

Further if any of four equivalent conditions hold then $J(R) = \{r \in R \mid (r, 0) \in J(S)\}$.

Proof. (1) ⇒ (2). Let $v \in V$. Then, by hypothesis, there exists $w \in V$ such that $v + w + vw = 0$. Thus $(1, v)(1, w) = (1, 0)$ which implies $(1, v)$ is right invertible in S. Similarly, we can prove that $(1, w)$ is right invertible in S. Therefore $(1, v) \in U(S)$.

(2) ⇒ (1). Easy verification.

(2) ⇒ (3). Let $(u, v) \in S$ such that $u \in U(R)$. Clearly $(u, v) = (u, 0)(1, u^{-1}v)$. Now by hypothesis and the fact that $(u, 0) \in U(S)$, we conclude that $(u, v) \in U(S)$. The other part is obvious.

(3) ⇒ (4). First note that $J(S) \subseteq J(R) \times V$ is always true. For let $(r, v) \in J(S)$. We show that $r \in J(R)$. Let $a, b \in R$. Then by [5, Lemma 4.3], $(1, 0) - (a, 0)(r, v)(b, 0) \in U(S)$. This implies $1 - arb \in U(R)$. Therefore, by [5, Lemma 4.3], $r \in J(R)$. Thus $J(S) \subseteq J(R) \times V$. Now let $(x, v) \in J(R) \times V$. By [5, Lemma 4.3] it suffices to show that $(1, 0) - (y, v')(x, v)(z, v'') \in U(S)$ for all $(y, v'), (z, v'') \in S$. Consider $(1, 0) - (y, v')(x, v)(z, v'') = (1 - yxz, w)$ for some $w \in V$. Since $x \in J(R)$ we have by [5, Lemma 4.3] that $1 - yxz \in U(R)$. Therefore, by hypothesis, $(1 - yxz, w) = (1, 0) - (y, v')(x, v)(z, v'') \in U(S)$.

(4) ⇒ (2). Let $v \in V$. Then, by hypothesis, $(0, -v) \in J(S)$. Thus $(1, v) = (1, 0) - (0, -v) \in U(S)$.

The final statement follows from (4). □

Proposition 2.24. *Let R be a ring and let V be an R-R-bimodule which is also an idempotent-free ring not necessarily with 1. Let $S = I(R; V)$ be the ideal extension of R by V. Then the following are equivalent.*

(1) *$S = I(R; V)$ is uniquely I'-clean for some ideal I' of S.*
(2) (a) *R is uniquely I-clean for some ideal I of R.*
    (b) *If $e \in \mathrm{Id}(R)$ then $ev = ve$ for all $v \in V$.*

Proof. (1) ⇒ (2). First we prove (2(b)). Let $e \in \mathrm{Id}(R)$. Then clearly $(e, 0) \in \mathrm{Id}(S)$. Since S is uniquely I'-clean, by Proposition 2.4(3) it is abelian which implies $(e, 0)(0, v) = (0, v)(e, 0)$ for all $v \in V$, that is, $ev = ve$ for all $v \in V$. This proves (2(b)).

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Now we prove (2(a)). Note that by Lemma 2.22,

$$\mathrm{Id}(S) = \{(e, 0) \mid e \in \mathrm{Id}(R)\}. \tag{$\star$}$$

Let $I = \{r \in R \mid (r, 0) \in I'\}$. It is easy to verify that I is an ideal of R.

Claim. R is uniquely I-clean.

Let $r \in R$. Since S is uniquely I'-clean, by Corollary 2.11 and ($\star$) we have $(r, 0) = (e, 0) + (y, 0)$ where $(e, 0)$ is a central idempotent and $(y, 0) \in I'$. Thus by definition of I we get $y \in I$. Thus $r - e = y \in I$. Note that e is a central idempotent in R. Again by Corollary 2.11 it suffices to show that I is idempotent-free. Let $e^2 = e \in I$. This implies $(e, 0) \in I'$. Since S is uniquely I'-clean we get by Proposition 2.4(1) that I' is idempotent-free. Therefore $e = 0$ and hence I is idempotent-free.

(2) $\Rightarrow$ (1). First note that $I' = I \times V$ is an ideal of S. Let $(r, v) \in S$. Since, by hypothesis (2(a)), R is uniquely I-clean there exists a central idempotent $e \in R$ such that $r - e \in I$. Thus $(r, v) = (e, 0) + (r - e, v)$, that is, $(r, v) - (e, 0) \in I'$. Note that by hypothesis (2(b)) and the fact that e is central, $(e, 0)$ is a central idempotent in S. Now by Corollary 2.11, it suffices to show that I' is idempotent-free. Note that by hypothesis and Lemma 2.22 we have $\mathrm{Id}(S) = \{(e, 0) \mid e \in \mathrm{Id}(R)\}$. Now if $(e, 0) \in I'$ then by definition of I' we have $e \in I$. Since by hypothesis R is uniquely I-clean we have by Proposition 2.4(1) that I is idempotent-free. Thus $e = 0$ and hence I' is idempotent-free. □

Our next corollary improves [7, Proposition 7].

Corollary 2.25. *An ideal extension $S = I(R; V)$ is uniquely clean and V is idempotent-free if and only if the following conditions are satisfied.*

(a) *R is uniquely clean.*
(b) *If $e^2 = e \in R$ then $ev = ve$ for all $v \in V$.*
(c) *If $v \in V$ then $v + w + vw = 0$ for some $w \in V$.*

Proof. We first prove the 'if' part. First note that by (c), V is idempotent-free. For let $v \in \mathrm{Id}(V)$; then by (c) there exists $w \in V$ such that $(-v) + w + (-v)w = 0$. Multiplying by v from the left we get $-v = 0$ and hence $v = 0$. Now by Proposition 2.3, R is uniquely $J(R)$-clean, and by Lemma 2.23, $J(S) = J(R) \times V$. Therefore by the proof of Proposition 2.24, S is uniquely $J(S)$-clean. Thus by Proposition 2.3, S is uniquely clean.

Turning to the 'only if' part, suppose that S is uniquely clean and V is idempotent-free. First note that (b) follows from hypothesis, Propositions 2.3 and 2.24. Now we prove (c). Let $v \in V$. Since S is uniquely clean there exist $(e, x) \in \mathrm{Id}(S)$ and $(u, y) \in U(S)$ such that $(1, v) = (e, x) + (u, y)$ which implies $1 - e = u$. Clearly $e \in \mathrm{Id}(R)$ and $u \in U(R)$. Therefore $e = 0$. Since V is idempotent-free we get $x = 0$. Thus $(1, v) \in U(S)$. Therefore, by Lemma 2.23, $v + w + vw = 0$ for some $w \in V$. Finally, we prove (a). By hypothesis and Proposition 2.3, S is uniquely $J(S)$-clean, and by Lemma 2.23, $J(R) = \{r \in R \mid (r, 0) \in J(S)\}$. Therefore by the proof of Proposition 2.24 we get that R is uniquely $J(R)$-clean and hence, by Proposition 2.3, R is uniquely clean. □

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3. Uniquely clean rings

In this section we obtain conditions on the ring R and ideal I of R under which uniquely I-clean rings coincide with uniquely clean rings. From this we obtain [3, Theorems 2.5 and 3.2] as corollaries.

PROPOSITION 3.1. *If R is a uniquely I-clean ring for some ideal I of R, then the following are equivalent.*

(1) *R is a semipotent ring.*
(2) *$I = J(R)$.*
(3) *I is weakly enabling.*
(4) *R satisfies idempotent 1-stable range.*

PROOF. (1) ⇒ (2). Since by hypothesis R is uniquely I-clean, we have by Proposition 2.4(2) that $J(R) \subseteq I$. Further, by hypothesis and Proposition 2.4(1), I is idempotent-free. Since R is a semipotent ring, every idempotent-free one-sided ideal is contained in $J(R)$. Therefore $I \subseteq J(R)$ and hence $I = J(R)$.

(2) ⇒ (3). This follows by hypothesis and Proposition 2.17.

(3) ⇒ (4). Since R is uniquely I-clean, by Proposition 2.4(1), I is idempotent-free, and by Theorem 2.10, idempotents lift modulo I. Thus, by Proposition 2.17, $I \subseteq J(R)$. Therefore, by Proposition 2.18, it suffices to show that R/I satisfies idempotent 1-stable range. Since R is uniquely I-clean, by Theorem 2.10, R/I is Boolean and hence R/I satisfies idempotent 1-stable range. This proves (4).

(4) ⇒ (1). Since by hypothesis R satisfies idempotent 1-stable range, it is clean. Therefore by [6, Propositions 1.8(1) and 1.9], we conclude that R is semipotent. □

As a consequence of the above proposition we deduce [3, Theorems 2.5 and 3.2] as corollaries.

COROLLARY 3.2 [3, Theorem 2.5]. *For a ring R the following are equivalent.*

(1) *R is uniquely clean.*
(2) (a) *R is an exchange ring.*
    (b) *$R/J^*(R)$ is Boolean and idempotents lift uniquely modulo $J^*(R)$.*

PROOF. (1) ⇒ (2). Let R be uniquely clean. This implies that R is clean and hence R is an exchange ring. Now by Proposition 2.3, R is uniquely $J(R)$-clean which implies, by Proposition 2.4(3), that R is abelian. Therefore R is an abelian semipotent ring and hence, by Lemma 2.7, $J^*(R) = J(R)$. Now the result follows from Theorem 2.10.

(2) ⇒ (1). By hypothesis (b) and Theorem 2.10, R is uniquely $J^*(R)$-clean. Since, by hypothesis (a), R is an exchange ring, we have by [6, Proposition 1.9] that R is a potent ring and hence R is semipotent. Therefore, by Proposition 3.1, $J^*(R) = J(R)$. Thus, by Proposition 2.3, R is uniquely clean. □

COROLLARY 3.3 [3, Theorem 3.2]. *For a ring R the following are equivalent.*

(1) *R is uniquely clean.*

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(2) *For any* $a \in R$ *there exists a central idempotent* $e \in R$ *such that* $e - a \in (a - a^2)R \subseteq J^*(R)$.
(3) $R/J^*(R)$ *is Boolean and for any* $a \in R$ *there exists a central idempotent* $e \in R$ *and* $u \in U(R)$ *such that* $a = e + u$.
(4) $R/J^*(R)$ *is Boolean and* $aR + bR = R$ *with* $a, b \in R$ *implies that there exists a central idempotent* $e \in R$ *such that* $a + be \in U(R)$.

Proof. (1) ⇒ (4). By Corollary 3.2, $R/J^*(R)$ is Boolean. Note that every uniquely clean ring is semipotent and, by Proposition 2.3, R is uniquely $J(R)$-clean. Therefore, by Proposition 3.1, R satisfies idempotent 1-stable range. Since, by Proposition 2.4(3), R is abelian we get (4).

(4) ⇒ (3). Let $a \in R$. Clearly $aR + (-1)R = R$. Therefore by hypothesis there exists a central idempotent, say, e such that $a - e = u \in U(R)$. This proves (3).

(3) ⇒ (2). Let $a \in R$, then by hypothesis there exist a central idempotent $e \in R$ and $u \in U(R)$ such that $a = e + u$. It is easy to see that $(1 - e) - a = (a - a^2)u^{-1} \in (a - a^2)R$. Since $R/J^*(R)$ is Boolean, $(a - a^2)R \subseteq J^*(R)$. This proves (2).

(2) ⇒ (1). By hypothesis and [6, Proposition 1.1] it follows that R is an exchange ring. Further, R is abelian because, if $f \in \mathrm{Id}(R)$ then by hypothesis there exists a central idempotent $e \in R$ such that $e - f \in (f - f^2)R = 0$ which implies $f = e$. Since every exchange ring is semipotent, by Lemma 2.7, $J^*(R) = J(R)$. Therefore, by Corollary 2.11, R is uniquely $J(R)$-clean. Hence, by Proposition 2.3, R is uniquely clean. □

Corollary 3.4. *For a ring* R *the following are equivalent.*

(1) R *is uniquely clean.*
(2) $R/J^*(R)$ *is Boolean, idempotents lift centrally modulo* $J^*(R)$, $J^*(R)$ *is weakly enabling and idempotent-free.*
(3) $R/J^*(R)$ *is Boolean, idempotents lift uniquely modulo* $J^*(R)$ *and* $J^*(R)$ *is weakly enabling.*

Proof. (1) ⇒ (2). By hypothesis and Proposition 2.3, R is uniquely $J(R)$-clean. Thus, by Proposition 3.1, R is semipotent, and by Proposition 2.4(3), R is abelian. Therefore, by Lemma 2.7, $J^*(R) = J(R)$. Thus $J^*(R)$ is idempotent-free. Further by Corollary 2.11, $R/J^*(R)$ is Boolean and idempotents lift centrally modulo $J^*(R)$. Finally by Proposition 3.1, $J^*(R)$ is weakly enabling.

(2) ⇒ (3). Since $R/J^*(R)$ is Boolean we have $N(R) \subseteq J^*(R)$. Therefore by Lemma 2.8, idempotents lift uniquely modulo $J^*(R)$.

(3) ⇒ (1). By hypothesis and Theorem 2.10, R is uniquely $J^*(R)$-clean. Since by hypothesis $J^*(R)$ is weakly enabling, we have by Proposition 3.1 that $J^*(R) = J(R)$. Hence, by Proposition 2.3, R is uniquely clean. □

4. Uniquely nil-clean rings

In this section we study uniquely nil-clean rings and prove that a ring R is uniquely nil-clean if and only if R is an abelian exchange ring with $J(R)$ nil and every quasiregular element is uniquely clean. Also we improve [3, Theorem 4.1] and its corollary.

We recall that an element a in a ring R is said to be *nil-clean* (respectively, *uniquely nil-clean*) if there exists an idempotent (respectively, unique idempotent) $e \in R$ such that $a - e \in N(R)$, and a ring R is said to be *nil-clean* (respectively, *uniquely nil-clean*) if every element of R is nil-clean (respectively, uniquely nil-clean).

Lemma 4.1. *Every uniquely nil-clean ring is abelian.*

Proof. The proof is similar to that of Proposition 2.4(3). □

Lemma 4.2. *If R is a ring, $a \in R$ and $a = e + n$ for some $e \in \mathrm{Id}(R)$, $n \in N(R)$ with $en = ne$ then $e \in aR \cap Ra$.*

Proof. By hypothesis it is clear that a, e and n commute with each other. Therefore $ae = e(1 + n)$ which implies $e = ae(1 + n)^{-1}$ and hence $e \in aR \cap Ra$. □

Lemma 4.3. *If R is a ring, $e \in \mathrm{Id}(R)$ and $n \in N(R)$ such that $e \in nR$ (or $e \in Rn$) and $en = ne$ then $e = 0$.*

Proof. Let R be a ring, $e \in \mathrm{Id}(R)$ and $n \in N(R)$ such that $e \in nR$ and $en = ne$. Let k be the least positive integer such that $n^k = 0$. Let $e = nr$ for some $r \in R$. Since $en = ne$ we have $e = ner$. Thus $e = n(e)r = n(ner)r = n^2er^2 = n^ker^k = 0$. □

Theorem 4.4. *A ring R is uniquely nil-clean if and only if $N(R)$ is an ideal of R and R is uniquely $N(R)$-clean.*

Proof. The 'if' part is very clear. For the 'only if' part it is sufficient to prove that $N(R)$ is an ideal. We give below an elementary proof of this fact. We first prove that:

$$\text{if } n \in N(R) \text{ then } nr \in N(R) \text{ for all } r \in R \tag{$*$}$$

Let $n \in N(R)$. Suppose that there exists $r_0 \in R$ such that nr_0 is not in $N(R)$. Since, by hypothesis, R is uniquely nil-clean we can write nr_0 uniquely as $nr_0 = e + m$ for $e \in \mathrm{Id}(R)$ and $m \in N(R)$. By our assumption, $e \neq 0$. Now, by Lemma 4.1, R is abelian. Therefore, by Lemma 4.2, $e \in nr_0R \subseteq nR$ and hence, by Lemma 4.3, $e = 0$, which is a contradiction. Thus $nr \in N(R)$ for all $n \in N(R)$ and $r \in R$.

Similarly, we can prove that $rn \in N(R)$ for all $n \in N(R)$ and $r \in R$.

Now we prove that $n_1, n_2 \in N(R) \Rightarrow n_1 - n_2 \in N(R)$. Let $n_1, n_2 \in N(R)$. By hypothesis, we can write $n_1 - n_2$ uniquely as $n_1 - n_2 = f + n_3$ for some $f \in \mathrm{Id}(R)$ and $n_3 \in N(R)$. Again by Lemmas 4.1 and 4.2, we get $f \in (n_1 - n_2)R$ which implies $f = (n_1 - n_2)r$ for some $r \in R$ which in turn implies $n_1r = f + n_2r$. Now by ($*$) above, n_1r and n_2r are nilpotent. Since R is uniquely nil-clean we get $f = 0$ and hence $n_1 - n_2 = n_3 \in N(R)$. This completes the proof. □

Corollary 4.5. *For a ring R, the following are equivalent.*

(1) *R is uniquely nil-clean.*
(2) *$N(R)$ is an ideal, $R/N(R)$ is Boolean and idempotents lift centrally modulo $N(R)$.*

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(3) *$N(R)$ is an ideal and for any $a \in R$ there exists a central $e \in \mathrm{Id}(R)$ such that $e - a \in N(R)$.*

Proof. This result follows from Theorem 4.4, Corollary 2.11 and the fact that $N(R)$ is idempotent-free. □

Following Borooah *et al.* [2], given $e \in \mathrm{Id}(R)$ and $a \in R$, we say that a is *e-clean* if $a - e \in U(R)$.

Proposition 4.6. *If R is an abelian semipotent ring then $J(R) = \{a \in R \mid a$ is quasiregular and uniquely clean$\}$.*

Proof. Let R be an abelian semipotent ring. Let $a \in J(R)$. Then a is quasiregular. Therefore $a = 1 + u$ for some $u \in U(R)$. Thus a is 1-clean. Suppose that a is e-clean for some $e \in \mathrm{Id}(R)$. This implies $a = e + v$ for some $v \in U(R)$. Since R is abelian we get $1 - e = a(1 - e)v^{-1} \in aR \subseteq J(R)$. Therefore $e = 1$ and hence a is uniquely clean. Now suppose that $a \in R$ is both quasiregular and uniquely clean. Let $a = 1 + u$ for some $u \in U(R)$. We show that $a \in J(R)$. Suppose the contrary. Since, by hypothesis, R is semipotent there exists $0 \neq e \in \mathrm{Id}(R)$ such that $e \in aR$. Let $e = ar = (1 + u)r$ for some $r \in R$. Since R is abelian we get $e = e(1 + u)r = (e + u)er \in bR$ where $b = e + u$. Clearly $1 - e = b(1 - e)u^{-1} \in bR$. Thus both e and $1 - e$ belong to bR and hence $b \in U(R)$. Therefore $a = 1 - e + b$ is a clean expression of a. Since a is uniquely clean with $a = 1 + u$ we get that $e = 0$, which is a contradiction. Hence $a \in J(R)$. This completes the proof. □

Theorem 4.7. *For a ring R the following are equivalent.*

(1) *R is uniquely nil-clean.*
(2) *R is uniquely clean and nil-clean.*
(3) (i) *R is an abelian exchange ring with $J(R)$ nil.*
    (ii) *Every quasiregular element is uniquely clean.*

Proof. (1) $\Rightarrow$ (2). Since by hypothesis R is uniquely nil-clean, by Theorem 4.4, we get that $N(R)$ is an ideal and R is uniquely $N(R)$-clean. It is easy to see that if $N(R)$ is an ideal then it is always weakly enabling. Therefore, by Proposition 3.1, $N(R) = J(R)$, and hence, by Proposition 2.3, R is uniquely clean.

(2) $\Rightarrow$ (3). Since by hypothesis R is uniquely clean, by [7, Lemma 4] it is abelian and by [6, Proposition 1.8(1)] it is an exchange ring. It remains to show that $J(R)$ is nil. Let $a \in J(R)$. Since R is nil-clean there exists $e \in \mathrm{Id}(R)$ such that $a - e = n \in N(R)$. Note that by [7, Theorem 20], $R/J(R)$ is Boolean and hence $N(R) \subseteq J(R)$. Therefore $e = a - n \in J(R)$ which implies $e = 0$. Thus $a \in N(R)$ and hence $J(R)$ is nil.

(3) $\Rightarrow$ (1). Since by hypothesis R is an exchange ring, it is semipotent. Thus R is an abelian semipotent ring. Therefore, by hypothesis (3(ii)) and Proposition 4.6, $J(R) = \{a \in R \mid a$ is quasiregular$\}$. Therefore $N(R) \subseteq J(R)$. Since by hypothesis $J(R)$

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is nil we get $J(R) = N(R)$. Now by hypothesis R is an abelian exchange ring, which implies by [6, Proposition 1.8(2)] that R is clean. Thus for each $a \in R$ there exists $e \in \mathrm{Id}(R)$ such that $(a-1) - e = u$ where $u \in U(R)$ which implies $e - a = -(1+u) \in J(R) = N(R)$. Hence, by Corollary 2.11 and Theorem 4.4, R is uniquely nil-clean. □

Proposition 4.8. *Let R be a ring and I be an ideal. Then the following are equivalent.*

(1) *R is uniquely I-clean.*
(2) *I is idempotent-free and $R = \mathrm{Id}(C(R)) + I$.*
(3) (i) *I is idempotent-free.*
  (ii) *$C(R)$ is uniquely $(C(R) \cap I)$-clean.*
  (iii) *$R = C(R) + I$.*

Proof. (1) $\Leftrightarrow$ (2). This follows from Corollary 2.11.

(2) $\Rightarrow$ (3). It is sufficient to prove that $C(R)$ is uniquely $(C(R) \cap I)$-clean. Clearly $C(R) \cap I$ is an idempotent-free ideal of $C(R)$. Let $x \in C(R) \subseteq R$. Since, by hypothesis, $R = \mathrm{Id}(C(R)) + I$ there exist $e^2 = e \in C(R)$ and $y \in I$ such that $x = e + y$. Since $x, e \in C(R)$, we get $y = x - e \in C(R)$. Thus $y \in C(R) \cap I$. Therefore, by Corollary 2.11, $C(R)$ is uniquely $(C(R) \cap I)$-clean.

(3) $\Rightarrow$ (2). Let $x \in R$. By (iii), $x = \alpha + y$ for some $\alpha \in C(R)$ and $y \in I$. By (ii), $\alpha = e + z$ for some $e \in \mathrm{Id}(C(R))$ and $z \in C(R) \cap I$. Therefore $x = e + (z + y)$, which proves (2). □

As a consequence of the above proposition we deduce [3, Proposition 4.7] as a corollary.

Corollary 4.9 [3, Proposition 4.7]. *R is uniquely nil-clean if and only if:*

(1) *$C(R)$ is uniquely nil-clean;*
(2) *every idempotent in R is central;*
(3) *$R = C(R) + N(R)$ and $N(R)$ is an ideal of R.*

Proof. This follows from Proposition 4.8, Theorem 4.4 and the fact that $N(C(R)) = C(R) \cap N(R)$. □

Remark 4.10. Condition (2) in the above result is not necessary as conditions (1) and (3) together give (2). For let $e \in \mathrm{Id}(R)$. Then, by hypothesis (3), $e = a + x$ for some $a \in C(R)$ and $x \in N(R)$. Now by hypothesis (1), $a = f + y$ for a unique $f \in \mathrm{Id}(C(R))$ and $y \in N(C(R)) = C(R) \cap N(R)$. Therefore $e = f + (y + x)$. Since, by hypothesis (3), $N(R)$ is an ideal we get $x + y \in N(R)$. Thus, by Lemma 2.5, $e = f \in C(R)$. Hence R is abelian.

Proposition 4.11. *If R is a ring such that every prime ideal of R is maximal then $J(R) = \mathrm{Nil}_*(R)$.*

Proof. We first note that $J(R) \subseteq M$ for every maximal ideal M of R. Therefore,

$$\begin{aligned} J(R) &\subseteq \cap\{M | M \text{ is a maximal ideal of } R\} \\ &= \cap\{M | M \text{ is a prime ideal of } R\} \text{ (because by hypothesis a prime ideal} \\ &\qquad \text{is maximal)} \\ &= \mathrm{Nil}_*(R). \end{aligned}$$

Note that $\mathrm{Nil}_*(R) \subseteq J(R)$ is always true. Hence $J(R) = \mathrm{Nil}_*(R)$. □

Our next result improves both [3, Theorem 4.1] and its corollary.

Corollary 4.12. *For a ring R the following are equivalent.*

(1) *R is a uniquely clean ring such that every prime ideal of R is maximal.*
(2) *R is a uniquely nil-clean ring and $N(R) = \mathrm{Nil}_*(R)$.*

Proof. (1) ⇒ (2). Since R is uniquely clean, we have by [7, Theorem 20] that $R/J(R)$ is Boolean. Therefore $N(R) \subseteq J(R)$. Since, by hypothesis, every prime ideal is maximal, we have by Proposition 4.11 that $J(R) = \mathrm{Nil}_*(R) \subseteq N(R)$. Thus $J(R) = N(R)$. Therefore, by Proposition 2.3 and Theorem 4.4, R is uniquely nil-clean. Further, by Proposition 4.11, $J(R) = \mathrm{Nil}_*(R)$. Therefore, $N(R) = \mathrm{Nil}_*(R)$.

(2) ⇒ (1). If R is uniquely nil-clean, then by Theorem 4.7 we get that R is uniquely clean. Now we show that every prime ideal is maximal. Let P be a prime ideal of R and let U be an ideal of R such that $P \subseteq U \subseteq R$. If $U = R$ then there is nothing to prove. So suppose that U is a proper ideal of R. We show that $U = P$. Let $a \in U$. Since R is uniquely nil-clean we have $a = e + n$ for some $e \in \mathrm{Id}(R)$ and $n \in N(R)$. By Lemma 4.1, we have $en = ne$. Therefore, by Lemma 4.2, we get $e \in aR \subseteq U$. Since U is a proper ideal, it does not contain $1 - e$. Since, by hypothesis and Lemma 4.1, R is abelian we have $eR(1 - e) = 0 \in P$. As P is a prime ideal and $P \subseteq U$ we get $e \in P$. Since, by hypothesis, $N(R) = \mathrm{Nil}_*(R)$, we get $n \in P$. Thus $a = e + n \in P$. Therefore $U = P$ and hence every prime ideal of R is maximal. □

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