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ON HYPO-JORDAN OPERATORS

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Abstract. In this paper, we show that if $T = S + N$, where S is similar to a hyponormal operator, S and N commute and N is a nilpotent operator of order m (i.e., $N^m = 0$), then T is a subscalar operator of order $2m$. As a corollary, we get that such a T has a nontrivial invariant subspace if its spectrum $\sigma(T)$ has the property that there exists some non-empty open set U such that $\sigma(T) \cap U$ is dominating for U.

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1. Introduction. Let H be a separable, complex Hilbert space and let $\mathcal{L}(H,K)$ denote the space of all bounded linear operators from H to K. If $H = K$, we write $\mathcal{L}(H)$ in place of $\mathcal{L}(H,K)$. Recall that $S \in \mathcal{L}(H)$ is called a *hyponormal operator* if $SS^* \leq S^*S$, or equivalently, if $\|S^*h\| \leq \|Sh\|$ for every $h \in H$ and $N \in \mathcal{L}(H)$ is called a *nilpotent operator of order m* if $N^m = 0$ for some positive integer m. An operator $T \in \mathcal{L}(H)$ is said to be *hypo-Jordan of order m* if $T = S + N$ where S is similar to a hyponormal operator, S and N commute and N is a nilpotent operator of order m.

A bounded linear operator R on H is called *scalar of order m* if it possesses a spectral distribution of order m; i.e., if there is a continuous unital morphism of topological algebras

$$\Phi : C_0^m(\mathbf{C}) \to \mathcal{L}(H),$$

such that $\Phi(z) = R$, where z stands for the identity function on $\mathbf{C}$, and $C_0^m(\mathbf{C})$ stands for the space of compactly supported functions on $\mathbf{C}$, continuously differentiable of order m $(0 \leq m \leq \infty)$. An operator is *subscalar* if it is similar to the restriction of a scalar operator to an invariant subspace. As the weaker form of a subscalar operator, we introduce the following: an operator $T \in \mathcal{L}(H)$ is *quiasisubscalar* if there exists a one-to-one $V \in \mathcal{L}(H, K)$ such that $VT = RV$ where R $(= \Phi(z)$ in the above definition) is a scalar operator. There are examples of quasisubscalar operators in [**1**].

An operator $T \in \mathcal{L}(H)$ is said to satisfy the *single valued extension property* if for any open subset U in $\mathbf{C}$, the function

$$z - T : \mathcal{O}(U, H) \to \mathcal{O}(U, H)$$

defined by the obvious pointwise multiplication is one-to-one, where $\mathcal{O}(U, H)$ denotes the space of H-valued analytic functions on U. If, in addition, the above function $z - T$ has closed range on $\mathcal{O}(U, H)$, then T satisfies *Bishop's conditions* (β).

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In 1984 M. Putinar proved that any hyponormal operator is subscalar. His theorem was used to show that hyponormal operators with thick spectra have invariant subspaces, a result due to Scott W. Brown [**2**]. In this paper, we show that if $T = S + N$, where S is similar to a hyponormal operator, S and N commute and N is a nilpotent operator of order m, then T is a subscaler operator of order $2m$. As a corollary, we get that such a T has a nontrivial invariant subspace if its spectrum $\sigma(T)$ has the property that there exists some non-empty open set U such that $\sigma(T) \cap U$ is dominating for U.

The paper is organized as follows. In Section 2, we give some preliminary facts. In Section 3, we characterize hypo-Jordan operators and deal with applications of the main result.

2. Preliminaries. Let z be the coordinate in the complex plane **C** and $d\mu(z)$ denote the planar Lebesgue measure. Fix a complex (separable) Hilbert space H and a bounded (connected) open subset U of **C**. We shall denote by $L^2(U, H)$ the Hilbert space of measurable functions $f: U \to H$, such that

$$\|f\|_{2,U} = \left\{ \int_U \|f(z)\|^2 d\mu(z) \right\}^{\frac{1}{2}} < \infty.$$

The space of functions $f \in L^2(U, H)$ that are analytic on U (i.e. $\bar{\partial} f = 0$) is denoted by

$$A^2(U, H) = L^2(U, H) \cap \mathcal{O}(U, H).$$

Then $A^2(U, H)$ is called the *Bergman space for* U. It is known that $A^2(U, H)$ is complete.

Let us define now a special Sobolev type space. Let U be again a bounded open subset of **C** and m a fixed non-negative integer. The *vector valued Sobolev space* $W^m(U, H)$ *with respect to* $\bar{\partial}$ *and of order* m will be the space of those functions $f \in L^2(U, H)$ whose derivatives $\bar{\partial} f, \cdots, \bar{\partial}^m f$ in the sense of distributions still belong to $L^2(U, H)$. Endowed with the norm

$$\|f\|^2_{W^m} = \sum_{i=0}^{m} \|\bar{\partial}^i f\|^2_{2,U},$$

$W^m(U, H)$ becomes a Hilbert space contained continuously in $L^2(U, H)$.

Let U be a (connected) bounded open subset of **C** and let m be a non-negative integer. The linear operator $M(= M_z)$ of multiplication by z on $W^m(U, H)$ is continuous and it has a spectral distribution of order m, defined by the functional calculus

$$\Phi_M : C_0^m(\mathrm{C}) \to \mathcal{L}(W^m(U, H)), \quad \Phi_M(f) = M_f.$$

Therefore, M is a scalar operator of order m.

3. Main results. In this section, it is shown that any hypo-Jordan operator of order m is subscalar. The starting point of this section deals with the basic inequality for the proof of the main result.

LEMMA 3.1. [**8**, Proposition 2.1]. *For a bounded open disk D in $\mathbf{C}$ there is a constant C_D such that, for an arbitrary operator $T \in \mathcal{L}(H)$ and $f \in W^2(D, H)$, we have*

$$\|(I-P)f\|_{2,D} \le C_D\Big(\|(zI-T)^*\bar{\partial}f\|_{2,D} + \|(zI-T)^*\bar{\partial}^2 f\|_{2,D}\Big),$$

where P denotes the orthogonal projection of $L^2(D,H)$ onto the Bergman space $A^2(D,H)$.

COROLLARY 3.2. [**8**, Corollary 2.2]. *If S is hyponormal, then*

$$\|(I-P)f\|_{2,D} \le C_D\Big(\|(z-S)\bar{\partial}f\|_{2,D} + \|(z-S)\bar{\partial}^2 f\|_{2,D}\Big),$$

where z denotes zI.

LEMMA 3.3. *Let $T \in \mathcal{L}(H)$ be an operator such that $T = S + N$, where S is similar to a hyponormal operator, S and N commute and N is a nilpotent operator of order m. Let D be a bounded disk which contains $\sigma(T)$. Then the operator $V : H \to H(D)$, defined by $Vh = 1 \otimes h + \overline{(z-T)W^{2m}(D,H)}(= \widetilde{1 \otimes h})$, is one-to-one and has closed range, where $H(D) = W^{2m}(D,H)/\overline{(z-T)W^{2m}(D,H)}$ for $m = 1, 2, \cdots$ and $1 \otimes h$ denotes the constant function sending any $z \in D$ to h.*

Proof. It suffices to prove the following assertion: if $h_n \in H$ and $f_n \in W^{2m}(D,H)$ are sequences such that

$$\lim_{n\to\infty} \|(z-T)f_n + 1 \otimes h_n\|_{W^{2m}} = 0, \tag{1}$$

then $\lim_{n\to\infty} h_n = 0$.

By the definition of the norm a Sobolev space, the assertion (1) implies that

$$\lim_{n\to\infty} \|(z-T)\bar{\partial}^i f_n\|_{2,D} = 0 \tag{2}$$

for $i = 1, 2, \cdots, 2m$. Since $T = S + N$, we have

$$\lim_{n\to\infty} \|(z-S)\bar{\partial}^i f_n - N\bar{\partial}^i f_n\|_{2,D} = 0 \tag{3}$$

for $i = 1, 2, \cdots, 2m$. From the equation (3) and $SN = NS$, we have

$$\lim_{n\to\infty} \|(z-S)\bar{\partial}^i (N^k f_n) - N^{k+1}\bar{\partial}^i f_n\|_{2,D} = 0 \tag{4}$$

for $i = 1, 2, \cdots, 2m$ and $k = 0, 1, \cdots, m-1$. If in particular $k = m-1$, then

$$\lim_{n\to\infty} \|(z-S)\bar{\partial}^i (N^{m-1} f_n)\|_{2,D} = 0 \tag{5}$$

for $i = 1, 2, \cdots, 2m$.

Claim. $\lim_{n\to\infty} \|(z-S)\bar{\partial}^i (N^{m-j} f_n)\|_{2,D} = 0$ for $i = 1, 2, \cdots, 2(m+1-j)$ and $j = 1, 2, \cdots, m$.

We prove this claim by induction. If $j = 1$, it is clear from the equation (5). We assume that the above claim holds for some given $j = 1, 2, \cdots, m-1$. Indeed,

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$$\lim_{n\to\infty}\|(z-S)\bar{\partial}^i(N^{m-j}f_n)\|_{2,D}=0 \tag{6}$$

for $i=1,2,\cdots,2(m+1-j)$ and $j=1,2,\cdots,m-1$.

We only need to verify that

$$\lim_{n\to\infty}\|(z-S)\bar{\partial}^i(N^{m-(j+1)}f_n)\|_{2,D}=0$$

for $i=1,2,\cdots,2(m+1-(j+1))$ and $j=1,2,\cdots,m-1$.

Since S is similar to a hyponormal operator B, there exists an invertible operator R such that $RS=BR$. From the equation (6) we have

$$\lim_{n\to\infty}\|R(z-S)\bar{\partial}^i(N^{m-j}f_n)\|_{2,D}=0 \tag{7}$$

for $i=1,2,\cdots,2(m+1-j)$ and $j=1,2,\cdots,m-1$. From the equation (7) and $RS=BR$ we get

$$\lim_{n\to\infty}\|(z-B)R\bar{\partial}^i(N^{m-j}f_n\|_{2,D}=0 \tag{8}$$

for $i=1,2,\cdots,2(m+1-j)$ and $j=1,2,\cdots,m-1$. By Corollary 3.2,

$$\|(I-P)\bar{\partial}^i(RN^{m-j}f_n)\|_{2,D}\leq C_D(\|(z-B)\bar{\partial}^{i+1}(RN^{m-j}f_n)\|_{2,D}+\|(z-B)\bar{\partial}^{i+2}(RN^{m-j}f_n)\|_{2,D})$$

for $i=1,2,\cdots,2(m+1-(j+1))$ and $j=1,2,\cdots,m-1$. From the equation (8),

$$\lim_{n\to\infty}\|(I-P)\bar{\partial}^i(RN^{m-j}f_n)\|_{2,D}=0 \tag{9}$$

for $i=1,2,\cdots,2(m+1-(j+1))$ and $j=1,2,\cdots,m-1$. By the equations (8) and (9) we see that

$$\lim_{n\to\infty}\|(z-B)P[\bar{\partial}^i(RN^{m-j}f_n)]\|_{2,D}=0 \tag{10}$$

for $i=1,2,\cdots,2(m+1-(j+1))$ and $j=1,2,\cdots,m-1$. Since every hyponormal operator has the property (β) (see [**7**, Theorem 5.5]), we get that for $i=1,2,\cdots,2(m+1-(j+1))$ and $j=1,2,\cdots,m-1$,

$$P(\bar{\partial}^iRN^{m-j}f_n)\to 0$$

uniformly on compact subsets of D. Therefore, it is easy to show that

$$\lim_{n\to\infty}\|P[\bar{\partial}^i(RN^{m-j}f_n)]\|_{2,D}=0 \tag{11}$$

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for $i = 1, 2, \cdots, 2(m+1-(j+1))$ and $j = 1, 2, \cdots, m-1$. From the equations (9) and (11), we have

$$\lim_{n\to\infty} \|\bar{\partial}^i(RN^{m-j}f_n)\|_{2,D} = 0 \tag{12}$$

for $i = 1, 2, \cdots, 2(m+1-(j+1))$ and $j = 1, 2, \cdots, m-1$. Since R is invertible, we get from (12) that

$$\lim_{n\to\infty} \|\bar{\partial}^i(N^{m-j}f_n)\|_{2,D} = 0 \tag{13}$$

for $i = 1, 2, \cdots, 2(m+1-(j+1))$ and $j = 1, 2, \cdots, m-1$. From the equations (4) and (13),

$$\lim_{n\to\infty} \|(z-S)\bar{\partial}^i(N^{m-(j+1)}f_n)\|_{2,D} = 0$$

for $i = 1, 2, \cdots, 2(m+1-(j+1))$ and $j = 1, 2, \cdots, m-1$, and so this completes the proof of the claim stated above.

Let us come back now to the proof of Lemma 3.3. By the claim, the following equation holds:

$$\lim_{n\to\infty} \|(z-S)\bar{\partial}^i f_n\|_{2,D} = 0 \tag{14}$$

for $i = 1, 2$. Since R is bounded,

$$\lim_{n\to\infty} \|R(z-S)\bar{\partial}^i f_n\|_{2,D} = 0 \tag{15}$$

for $i = 1, 2$. Since $RS = BR$, from the equation (15) we have

$$\lim_{n\to\infty} \|(z-B)\bar{\partial}^i(Rf_n)\|_{2,D} = 0 \tag{16}$$

for $i = 1, 2$. By Corollary 3.2 and the equation (16), we get

$$\lim_{n\to\infty} \|(I-P)Rf_n\|_{2,D} = 0. \tag{17}$$

Set $g_n = R^{-1}P[Rf_n]$. The $g_n \in A^2(D, H)$. Since

$$\|f_n - g_n\|_{2,D} \leq \|R^{-1}\| \, \|Rf_n - P[Rf_n]\|_{2,D,}$$

the equation (17) implies that

$$\lim_{n\to\infty} \|f_n - g_n\|_{2,D} = 0. \tag{18}$$

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Now from (1) and (18) we obtain the following equation.

$$\lim_{n\to\infty}\|(z-T)g_n + 1\otimes h_n\|_{2,D} = 0.$$

Let Γ be in a circle in D surrounding $\sigma(T)$. Then for $z \in \Gamma$

$$\lim_{n\to\infty}\|g_n(z) - (z-T)^{-1}(1\otimes h_n)\| = 0$$

uniformly. Hence, by the Riesz functional calculus,

$$\lim_{n\to\infty}\left\|\frac{1}{2\pi i}\int_\Gamma g_n(z)dz + h_n\right\| = 0,$$

where it is assumed that Γ is described once counterclockwise.

But $\int_\Gamma g_n(z)dz = 0$ by Cauchy's theorem. Hence $\lim_{n\to\infty} h_n = 0$. Thus V is one-to-one and has closed range. □

Now we state and prove the main theorem.

THEOREM 3.4. *If T is any operator such that $T = S + N$, where S is similar to a hyponormal operator, S and N commute and N is a nilpotent operator of order m (i.e. T is any hypo-Jordan operator of order m), then T is a subscalar operator of order $2m$.*

Proof. Consider an arbitrary bounded open disk D in the complex plane $\mathbf{C}$ that contains $\sigma(T)$ and the quotient space

$$H(D) = W^{2m}(D,H)/\overline{(z-T)W^{2m}(D,H)}$$

endowed with the Hilbert space norm. Let $M(=M_z)$ be the operator of multiplication by z on $W^{2m}(D,H)$. Then M is a scalar operator of order $2m$ and its spectral distribution is given by

$$\Phi : C_0^{2m}(\mathrm{C}) \to \mathcal{L}(W^{2m}(D,H)), \quad \Phi(f) = M_f,$$

where M_f is the operator of multiplication by f. Since M commutes with $z - T$, $\tilde{M}$ on $H(D)$ is still a scalar operator of order $2m$, with $\tilde{\Phi}$ as a spectral distribution.

Let V be the operator

$$Vh = 1\otimes h + \overline{(z-T)W^{2m}(D,H)} \quad (= 1\otimes h),$$

from H into $H(D)$, denoting by $1\otimes h$ the constant function h. Then $VT = \tilde{M}V$. By Lemma 3.3, V is one-to-one and has closed range. Therefore, ran V is a closed invariant subspace for the scalar operator $\tilde{M}$. Hence T is a subscalar operator of order $2m$. □

Recall that if U is a non-empty open set in $\mathbf{C}$ and if $\Omega \subset U$ has the property that

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$$\sup_{\lambda\in\Omega}|f(\lambda)| = \sup_{\beta\in U}|f(\beta)|$$

for every function f in $H^\infty(U)$ (i.e. for all f bounded and analytic on U), then Ω is said to be *dominating* for U.

Corollary 3.5. *Let $T \in \mathcal{L}(H)$ be any hypo-Jordan operator of order m. If $\sigma(T)$ has the property that there exists some non-empty open set U such that $\sigma(T) \cap U$ is dominating for U, then T has a nontrivial invariant subspace.*

Proof. The proof follows that Theorem 3.4 and [**4**]. □

Corollary 3.6. *Any hypo-Jordan operator has the property* (β).

Proof. Since every scalar operator has the property (β) (see [**8**]) and the property (β) is transmitted from an operator to its restrictions to closed invariant subspaces, it follows from Theorem 3.4 that any hypo-Jordan operator has the property (β). □

Corollary 3.7. *If $T = S + N$ is hypo-Jordan of order m and quasinilpotent, then T is a nilpotent operator of order m.*

Proof. Since $\sigma(S) = \sigma(T) = \{0\}$, an operator S is quasinilpotent and is similar to a hyponormal operator. Therefore, S is a zero operator. Hence $T = N$. □

Recall that an operator $X \in \mathcal{L}(H, K)$ is called a *quasiaffinity* if it has trivial kernel and dense range. An operator $A \in \mathcal{L}(H)$ is said to be a *quasiaffine* transform of an operator $T \in \mathcal{L}(K)$ if there is a quasiaffinity $X \in \mathcal{L}(H, K)$ such that $XA = TX$.

Corollary 3.8. *Let $T \in \mathcal{L}(H)$ be any hypo-Jordan operator. If A is any quasiaffine transform of T, then $\sigma(T) \subset \sigma(A)$.*

Proof. The proof follows from Corollary 3.6 and [**6**, Theorem 3.2]. □

Corollary 3.9. *Let $T \in \mathcal{L}(H)$ be any hypo-Jordan operator. If A is any quasiaffine transform of T, then A is quasisubscalar.*

Proof. Let $X \in \mathcal{L}(H, K)$ be a quasiaffinity such that $XA = TX$. Since V (in Theorem 3.4) and X are one-to-one, VX is one-to-one. Therefore, VX implements the quasisubscalar properties. Thus A is quasisubscalar. □

In the following theorem we establishes an analogue of the single valued extension property for the space $W^k(D, H)$.

Proposition 3.10. *If $T \in \mathcal{L}(H)$ is a hypo-Jordan operator of order m, then the operator*

$$z - T : W^{2m}(D, H) \to W^{2m}(D, H)$$

is one-to-one, for an arbitrary bounded disk D in **C**.

Proof. Let $f \in W^{2m}(D, H)$ be such that $(z - T)f = 0$. Then by a similar method as in the proof of Lemma 3.3, we can show that $Rf = PRf \in A^2(D, H)$ (c.f. (17)). Since T is subscalar, by Theorem 3.4, we know that T has the single valued extension property. Therefore, $PRf = 0$; i.e., $f = 0$. Thus $z - T$ is one-to-one. □

COROLLARY 3.11. *Let $T = S + N$ be such that $SN = NS$, where S is similar to a normal operator and N is quasinilpotent. Let $\sigma(T)$ lie in a C^1-Jordan curve. Suppose that there exists a constant M such that*

$$\|(z - T)^{-1}\| \leq M/\{\mathrm{dist}(z, \sigma(T))\}^m$$

for all $z \in \rho(T)$ with $|z| \leq \|T\| + 1$. Then T is subscalar of order $2(4m + 4)$.

Proof. We know that $N^{4m+4} = 0$ by [**9**, Corollary 1.10] and so it follows from Theorem 3.4 that T is subscalar of order $2(4m + 4)$. □

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