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MAPPING PROPERTIES OF A SCALE INVARIANT CASSINIAN METRIC AND A GROMOV HYPERBOLIC METRIC

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Abstract

We consider a scale invariant Cassinian metric and a Gromov hyperbolic metric. We discuss a distortion property of the scale invariant Cassinian metric under Möbius maps of a punctured ball onto another punctured ball. We obtain a modulus of continuity of the identity map from a domain equipped with the scale invariant Cassinian metric (or the Gromov hyperbolic metric) onto the same domain equipped with the Euclidean metric. Finally, we establish the quasi-invariance properties of both metrics under quasiconformal maps.

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1. Introduction and preliminaries

An integral part of geometric function theory is to study the behaviour of distances under well-known classes of maps such as Möbius maps, Lipschitz maps and quasiconformal maps. Some metrics, such as the hyperbolic metric, the Apollonian metric [3, 11, 15] and the Seittenranta metric [28], are Möbius invariant, whereas others, such as the quasihyperbolic metric [7, 8] and the distance ratio metric [31], are not. These metrics are also known as hyperbolic-type metrics in the literature. The quasi-invariance or distortion properties of the metrics that are not Möbius invariant and the quasi-invariance properties under quasiconformal maps are of recent interest (see [13, 23]). Note that the quasihyperbolic and the distance ratio metrics do satisfy the bilipschitz property with bilipschitz constant 2 under Möbius maps (see [32, page 36], [8, Corollary 2.5] and [7, Proof of Theorem 4]). Similar properties have also been studied recently for the Cassinian metric [16] under Möbius maps of the unit ball and of a punctured ball onto another punctured ball (see [19, 22]).

Unless otherwise stated, we denote by D, a proper subdomain of $\mathbb{R}^n$, that is, $D \subsetneq \mathbb{R}^n$. A scale-invariant Cassinian metric, recently introduced by Ibragimov [18], is defined by

$$\tilde{\tau}_D(x,y) = \log\Big(1 + \sup_{p\in\partial D} \frac{|x-y|}{\sqrt{|x-p|\,|p-y|}}\Big) \quad \text{for } x,y \in D \subsetneq \mathbb{R}^n.$$

Similar to the Apollonian metric, the (scale-invariant) Cassinian metric is described through ovals of Cassini [16, 18]. The comparisons of the $\tilde{\tau}_D$-metric with the hyperbolic-type metrics and their metric ball inclusion properties are studied in [26]. The hyperbolic-type metrics share a nice connection with the hyperbolic metric to characterise certain domains such as quasidisks, uniform domains and John domains (see, for instance, [6, 7, 20]). Characterisations of such domains in terms of the Cassinian metric or its scale invariant metric $\tilde{\tau}$ with other hyperbolic-type metrics are not known though a conjecture is stated in [16].

The $\tilde{\tau}_D$-metric is Möbius invariant in punctured spaces $\mathbb{R}^n\setminus\{p\}$, $p \in \mathbb{R}^n$ only [18, Lemma 2.1]. This leads us to study the quasi-invariance property of the $\tilde{\tau}_D$-metric under Möbius maps of domains other than the punctured spaces $\mathbb{R}^n\setminus\{p\}$, $p \in \mathbb{R}^n$. In this regard, we prove a distortion property of the $\tilde{\tau}_D$-metric under Möbius maps of the punctured ball $\mathbb{B}^n\setminus\{0\}$ onto another punctured ball $\mathbb{B}^n\setminus\{a\}$, $0 \neq a \in \mathbb{B}^n$. A distortion property of the $\tilde{\tau}$-metric under Möbius maps of the unit ball $\mathbb{B}^n := \{x \in \mathbb{R}^n : |x| < 1\}$ was recently established in [18]. Hence, we examine the quasi-invariance property of the $\tilde{\tau}_D$-metric under quasiconformal maps.

In 1987, Gromov [10] introduced the notion of an abstract metric space and it is natural to investigate whether or not a metric space is hyperbolic in the sense of Gromov. Ibragimov [17] introduced a metric,

$$u_Z(x,y) = 2\log\frac{d(x,y) + \max\{\mathrm{dist}(x,\partial Z), \mathrm{dist}(y,\partial Z)\}}{\sqrt{\mathrm{dist}(x,\partial Z)\,\mathrm{dist}(y,\partial Z)}} \quad \text{for } x,y \in Z,$$

which hyperbolises the locally compact noncomplete metric space (Z,d) without changing its quasiconformal geometry. For a domain $D \subsetneq \mathbb{R}^n$ equipped with the Euclidean metric, the *u_D-metric* is defined by

$$u_D(x,y) = 2\log\frac{|x-y| + \max\{\mathrm{dist}(x,\partial D), \mathrm{dist}(y,\partial D)\}}{\sqrt{\mathrm{dist}(x,\partial D)\,\mathrm{dist}(y,\partial D)}} \quad \text{for } x,y \in D.$$

As indicated in [26], the u_D-metric does not satisfy the domain monotonicity property and it coincides with the distance ratio metric in punctured spaces $\mathbb{R}^n\setminus\{p\}$, for $p \in \mathbb{R}^n$. Gromov hyperbolicity is preserved under rough quasi-isometries (see [30, Theorem 3.17], [12]). This motivates our study of the u_D-metric in the setting of hyperbolic-type metrics associated with quasiconformal maps. Although the Gromov hyperbolic metric u_D is compared with some of the hyperbolic-type metrics [17, 26], domain characterisations in terms of the u_D-metric are still open.

We turn now to uniform continuity. The importance and applications of uniform continuity can be seen in many areas of mathematics and physics (see, for instance, [24, 27]). A *modulus of continuity* is a function $\omega : [0,\infty] \to [0,\infty]$ used to measure quantitatively the uniform continuity of functions. Let (X_j, d_j), $j = 1,2$, be metric spaces. A function $f : X_1 \to X_2$ admits ω as *modulus of continuity* if and only

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if $d_2(f(x), f(y)) \le \omega(d_1(x, y))$ for all $x, y \in X_1$. We also call such functions *uniformly continuous with modulus of continuity* ω (or ω-*uniformly continuous*). As examples, for $k > 0$, the modulus $\omega(t) = kt$ describes the k-Lipschitz functions, and the moduli $\omega(t) = kt^\alpha$, $\alpha > 0$, describe Hölder continuity. To simplify matters, we always assume that $\omega(t)$ is an increasing homeomorphism.

We give some related examples. The hyperbolic metric, $\rho_{\mathbb{B}^n}$, of the unit ball $\mathbb{B}^n$ is

$$\rho_{\mathbb{B}^n}(x, y) = \inf_{\gamma} \int_{\gamma} \frac{2|dz|}{1 - |z|^2},$$

where the infimum is taken over all rectifiable curves $\gamma \subset \mathbb{B}^n$ joining x and y. If $X_1 = \mathbb{B}^n = X_2$ and $f : \mathbb{B}^n \to \mathbb{B}^n$ is quasiconformal, then the quasiconformal counterpart of the Schwarz lemma says that $f : (\mathbb{B}^n, \rho_{\mathbb{B}^n}) \to (\mathbb{B}^n, \rho_{\mathbb{B}^n})$ is uniformly continuous. If $X_1 = \mathbb{B}^2$, $X_2 = \mathbb{R}^2\backslash\{0, 1\}$, the Schottky theorem gives, in an explicit form, a growth estimate for $|f(z)|$ in terms of $|z|$ when $f : \mathbb{B}^2 \to \mathbb{R}^2\backslash\{0, 1\}$ is an analytic function [14, pages 685, 702]. In fact, Nevanlinna's principle of the hyperbolic metric [14, page 683] yields an estimate for the modulus of continuity of $f : (\mathbb{B}^2, \rho_{\mathbb{B}^2}) \to (X_2, d_2)$, where d_2 is the hyperbolic metric of the twice punctured plane X_2. If q is the chordal metric and $f : (\mathbb{B}^2, \rho_{\mathbb{B}^2}) \to (\overline{\mathbb{R}}^2, q)$ is a meromorphic function, then f is normal (in the sense of Lehto and Virtanen [25]) if and only if it is uniformly continuous. In the context of quasiregular maps, uniform continuity has been discussed in [31, 33]. For the uniform continuity of mappings with respect to the distance ratio metric and the quasihyperbolic metric in the unit ball, see [21]. In this connection, we consider the identity map $\mathrm{id} : (\mathbb{B}^n, m_{\mathbb{B}^n}) \to (\mathbb{B}^n, |\cdot|)$ and prove that it is uniformly continuous, where $m_{\mathbb{B}^n} \in \{\tilde{\tau}_{\mathbb{B}^n}, u_{\mathbb{B}^n}\}$. We also prove that the identity map $\mathrm{id} : (D, \tilde{\tau}_D) \to (D, |\cdot|)$ is uniformly continuous, where $D \subsetneq \mathbb{R}^n$ is bounded.

Finally, using the bilipschitz relation between the u_D-metric and the $\tilde{\tau}_D$-metric discussed in [26, Theorem 3.5], we study the quasi-invariance property of these two metrics under quasiconformal maps of $\mathbb{R}^n$. If a metric is not invariant under certain classes of mappings, it is of interest to study its quasi-invariance properties (also called distortion properties). Distortion properties were the basis for several classical theorems in univalent function theory of one and several complex variables (see [5, 9]). Distortion properties also feature in converting a sphere to a flat surface when it is projected through mappings and the distortion constant indicates the amount of stretching involved. In this paper, one of our main objectives is to deal with distance properties subject to distortion. A surprising fact, in this context, is that though some metrics are not invariant under quasiconformal mappings, domains that are characterised through such metric inequalities are invariant under quasiconformal mappings of $\mathbb{R}^n$ (see [7]) and geometric properties of the image of such domains remain unchanged.

2. Distortion of the $\tilde{\tau}$-metric under Möbius maps of a punctured ball

Our objective in this section is to study the distortion property of the $\tilde{\tau}$-metric under Möbius maps from a punctured ball onto another punctured ball. Distortion properties of the $\tilde{\tau}_{\mathbb{B}^n}$-metric under Möbius maps of the unit ball $\mathbb{B}^n$ were recently studied in [18].

Theorem 2.1. *Let $a \in \mathbb{B}^n$ and $f : \mathbb{B}^n\backslash\{0\} \to \mathbb{B}^n\backslash\{a\}$ be a Möbius map with $f(0) = a$. Then, for $x, y \in \mathbb{B}^n\backslash\{0\}$,*

$$\frac{1-|a|}{1+|a|}\tilde{\tau}_{\mathbb{B}^n\backslash\{0\}}(x,y) \le \tilde{\tau}_{\mathbb{B}^n\backslash\{a\}}(f(x),f(y)) \le \frac{1+|a|}{1-|a|}\tilde{\tau}_{\mathbb{B}^n\backslash\{0\}}(x,y).$$

Proof. If $a = 0$, the proof is given in [18]. Now, assume that $a \neq 0$. Let σ be the inversion in the sphere $\mathbb{S}^{n-1}(a^\star, r)$, where

$$a^\star = \frac{a}{|a|^2} \quad \text{and} \quad r = \sqrt{|a^\star|^2 - 1} = \frac{\sqrt{1-|a|^2}}{|a|}.$$

Note that the sphere $\mathbb{S}^{n-1}(a^\star, r)$ is orthogonal to $\mathbb{S}^{n-1}$ and that $\sigma(a) = 0$. In particular, σ is a Möbius map with $\sigma(\mathbb{B}^n\backslash\{a\}) = \mathbb{B}^n\backslash\{0\}$. Recall from [2] that

$$\sigma(x) = a^\star + \left(\frac{r}{|x-a^\star|}\right)^2(x-a^\star).$$

Then $\sigma \circ f$ is an orthogonal matrix (see [2, Theorem 3.5.1(i)]). In particular,

$$|\sigma(f(x)) - \sigma(f(y))| = |x-y|. \tag{2.1}$$

We will need the following property of σ (see [2, page 26]):

$$|\sigma(x) - \sigma(y)| = \frac{r^2|x-y|}{|x-a^\star|\,|y-a^\star|}. \tag{2.2}$$

It follows from (2.1) and (2.2) that

$$|f(x) - f(y)| = \frac{|f(x)-a^\star|\,|f(y)-a^\star|}{|a^\star|^2-1}|x-y|. \tag{2.3}$$

Since $|f(z)| \le 1$ whenever $|z| \le 1$ and since $|a^\star| > 1$,

$$|a^\star| - 1 \le |f(z) - a^\star| \le |a^\star| + 1.$$

Write $P = \min\{\sqrt{|f(x)-a|\,|f(y)-a|}, \inf_{p\in\partial\mathbb{B}^n}\sqrt{|f(x)-p|\,|f(y)-p|}\}$. From the definition,

$$\tilde{\tau}_{\mathbb{B}^n\backslash\{a\}}(f(x),f(y)) = \log\left(1 + \frac{|f(x)-f(y)|}{P}\right),$$

and

$$\tilde{\tau}_{\mathbb{B}^n\backslash\{0\}}(x,y) = \log\left(1 + \frac{|x-y|}{\min\{\sqrt{|x|\,|y|}, \inf_{z\in\partial\mathbb{B}^n}\sqrt{|x-z|\,|y-z|}\}}\right).$$

Here we have two choices for P.

Case 1. $P = \sqrt{|f(x)-a|\,|f(y)-a|}$. From (2.3), it is clear that

$$|f(x)-a| = \frac{|f(x)-a^\star|\,|a-a^\star|}{|a^\star|^2-1}|x| \quad \text{and} \quad |f(y)-a| = \frac{|f(y)-a^\star|\,|a-a^\star|}{|a^\star|^2-1}|y|.$$

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Now,

$$\begin{aligned}\tilde{\tau}_{\mathbb{B}^n\setminus\{a\}}(f(x),f(y)) &= \log\Big(1+\frac{|f(x)-f(y)|}{\sqrt{|f(x)-a|\,|f(y)-a|}}\Big)\\ &\le \log\Big(1+\frac{1+|a|}{1-|a|}\frac{|x-y|}{\sqrt{|x|}}\Big)\\ &\le \frac{1+|a|}{1-|a|}\log\Big(1+\frac{|x-y|}{\sqrt{|x|\,|y|}}\Big)\le \frac{1+|a|}{1-|a|}\tilde{\tau}_{\mathbb{B}^n\setminus\{0\}}(x,y),\end{aligned}$$

where the second inequality follows from the well-known Bernoulli's inequality

$$\log(1+ax)\le a\log(1+x),\quad \text{for } a\ge 1, x>0. \tag{2.4}$$

Similarly, by taking the inverse mapping, we can prove that

$$\frac{1-|a|}{1+|a|}\tilde{\tau}_{\mathbb{B}^n\setminus\{0\}}(x,y)\le \tilde{\tau}_{\mathbb{B}^n\setminus\{a\}}(f(x),f(y)).$$

Case 2. $P=\inf_{p\in\partial\mathbb{B}^n}\sqrt{|f(x)-p||f(y)-p|}$. This case follows from the proof of [18, Theorem 7.1]. This completes the proof of the theorem. □

3. Uniform continuity

We begin this section with the following proposition which gives the formula for the $\tilde{\tau}_{\mathbb{B}^n}$-metric in the special cases when $x=ty$ for real $t\neq 0$. We observe that for $t>0$ the points x and y lie on a radial segment whereas for $t<0$ they are diametrically opposite. We assume without loss of generality that $|x|\le|y|$.

Proposition 3.1. *Let $x,y\in\mathbb{B}^n$ with $x=ty, 0\neq t\in\mathbb{R}$ and $|x|\le|y|$.*

(1) *If $t>0$, then*

$$\tilde{\tau}_{\mathbb{B}^n}(x,y)=\log\Big(1+\frac{|x-y|}{\sqrt{(1-|x|)(1-|y|)}}\Big);$$

(2) *if $t<0$, then*

$$\tilde{\tau}_{\mathbb{B}^n}(x,y)=\log\Big(1+\frac{|x-y|}{\sqrt{(1+|x|)(1-|y|)}}\Big).$$

Proof. The proof follows easily from the definition of the $\tilde{\tau}$-metric. Indeed, for the proof of (1), the maximal Cassinian oval touches the nearest boundary point p of $\mathbb{B}^n$ in the direction of the radial segment. It follows that $|x-p|\,|y-p|=(1-|x|)(1-|y|)$. This gives the desired formula.

For the proof of (2), the maximal Cassinian oval touches the nearest boundary point p close to y. This yields $|x-p|\,|y-p|=(1+|x|)(1-|y|)$. Now, the required formula follows from the definition of the $\tilde{\tau}$-metric. □

We now discuss the uniform continuity of the identity map $\mathrm{id} : (D, m_D) \to (D, |\cdot|)$, where m_D is either the u_D-metric or the $\tilde{\tau}_D$-metric. More precisely, we aim to find a bound, as sharp as possible, for the modulus of continuity of the identity map

$$\mathrm{id} : (D, m_D) \to (D, |\cdot|), \tag{3.1}$$

where $D \subsetneq \mathbb{R}^n$ is a bounded domain. First, we obtain the modulus of continuity of the id map (3.1) when $D = \mathbb{B}^n$.

Theorem 3.2. *Let $x, y \in \mathbb{B}^n$ be arbitrary and $w = |x-y|e_1/2$, where $e_1 = (1, 0, \ldots, 0) \in \mathbb{R}^n$. Then*

$$\tilde{\tau}_{\mathbb{B}^n}(x, y) \geq \tilde{\tau}_{\mathbb{B}^n}(-w, w) = \log\left(1 + \frac{2|x-y|}{\sqrt{4-|x-y|^2}}\right) \geq c\,|x-y|,$$

where $c\,(\approx 0.76)$ is the solution of the equation

$$(4-t^2)(2t + \sqrt{4-t^2}) \log\left(1 + \frac{2t}{\sqrt{4-t^2}}\right) - 8t = 0.$$

The first inequality becomes an equality when $y = -x$.

Proof. Let $x, y \in \mathbb{B}^n$ with $|x| \leq |y|$. Here we consider two cases.

Case 1. Suppose x and y lie on a diameter of $\mathbb{B}^n$ with $0 \in [x, y]$. From Proposition 3.1(2),

$$\tilde{\tau}_{\mathbb{B}^n}(x, y) = \log\left(1 + \frac{|x-y|}{\sqrt{(1+|x|)(1-|y|)}}\right)$$

and hence

$$\tilde{\tau}_{\mathbb{B}^n}(-w, w) = \log\left(1 + \frac{2|w|}{\sqrt{1-|w|^2}}\right) = \log\left(1 + \frac{2|x-y|}{\sqrt{4-|x-y|^2}}\right).$$

With a suitable rotation and translation, it can easily be seen geometrically that the maximal Cassinian oval with foci at x and y will lie inside the maximal Cassinian oval with foci at $-w$ and w. Expressed analytically,

$$\inf_{p \in \partial\mathbb{B}^n} \sqrt{|x-p|\,|p-y|} \leq \inf_{p \in \partial\mathbb{B}^n} \sqrt{|w+p|\,|p-w|},$$

that is,

$$\sqrt{(1+|x|)(1-|y|)} \leq \sqrt{1 - \frac{|x-y|^2}{4}},$$

which is true. Hence, $\tilde{\tau}_{\mathbb{B}^n}(x, y) \geq \tilde{\tau}_{\mathbb{B}^n}(-w, w)$.

If $x \in [0, y]$, then by Proposition 3.1(1),

$$\tilde{\tau}_{\mathbb{B}^n}(x, y) = \log\left(1 + \frac{|x-y|}{\sqrt{(1+|x|)(1-|y|)}}\right)$$

and the proof follows similarly.

For the second inequality, we need to find the minimum of the function

$$\frac{1}{t}\log\Big(1+\frac{2t}{\sqrt{4-t^2}}\Big) \quad (\text{where } t=|x-y|).$$

By the derivative test, it can be seen that the minimum is attained at the point $t \approx 1.16$ and the minimum value is approximately 0.76.

Case 2. Let $x, y \in \mathbb{B}^n$ be arbitrary. Choose $y' \in \mathbb{B}^n$ such that $|x-y| = |x-y'|$ with x and y' on a diameter of $\mathbb{B}^n$. With the same argument as in *Case 1*, we can show that

$$\tilde{\tau}_{\mathbb{B}^n}(x,y) \geq \tilde{\tau}_{\mathbb{B}^n}(x,y') \geq \tilde{\tau}_{\mathbb{B}^n}(-w,w).$$

This completes the proof. □

Now, we obtain the modulus of continuity of the id map (3.1) when D is a bounded proper subdomain of $\mathbb{R}^n$.

Theorem 3.3. *Let $D \subsetneq \mathbb{R}^n$ be a domain with* $\operatorname{diam} D < \infty$ *and* $r = \sqrt{n/(2n+2)}\operatorname{diam} D$. *Then*

$$\tilde{\tau}_D(x,y) \geq \log\Big(1+\frac{2|x-y|}{\sqrt{4r^2-|x-y|^2}}\Big) \geq c\,\frac{|x-y|}{r}$$

for distinct $x, y \in D$ with equality in the first inequality when $D = B^n(z,r)$ and $z = (x+y)/2$. Here c is the number defined in Theorem 3.2.

Proof. By the well-known Jung's theorem [4, Theorem 11.5.8], there exists $z \in \mathbb{R}^n$ with $D \subset B(z,r)$, where $r = \sqrt{n/(2n+1)}\operatorname{diam} D$. By the monotone property of $\tilde{\tau}_D$,

$$\tilde{\tau}_D(x,y) \geq \tilde{\tau}_{B(z,r)}(x,y).$$

Without loss of generality assume that $z = 0$. Choose $u, v \in B(0,r)$ in such a way that $|u-v| = 2|u| = |x-y|$. By Theorem 3.2,

$$\tilde{\tau}_D(x,y) \geq \tilde{\tau}_{B(z,r)}(x,y) \geq \tilde{\tau}_B(-u,u) = \log\Big(1+\frac{2|x-y|}{\sqrt{4r^2-|x-y|^2}}\Big).$$

This completes the proof. □

Next, we obtain the modulus of continuity of the identity map:

$$\mathrm{id} : (\mathbb{B}^n, u_{\mathbb{B}^n}) \to (\mathbb{B}^n, |\cdot|).$$

Theorem 3.4. *If $x, y \in \mathbb{B}^n$ are arbitrary and $w = |x-y|\,e_1/2$, then*

$$u_{\mathbb{B}^n}(x,y) \geq u_{\mathbb{B}^n}(-w,w) = 2\log\Big(\frac{2+|x-y|}{2-|x-y|}\Big) \geq |x-y|,$$

where the first inequality becomes an equality when $y = -x$.

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Proof. Let $x, y \in \mathbb{B}^n$ with $|x| \le |y|$. We consider two cases.

Case 1. Suppose that x and y lie on a diameter of $\mathbb{B}^n$. We have two possibilities. If $0 \in [x, y]$, then, by the assumption that $\operatorname{dist}(x, \partial\mathbb{B}^n) = 1 - |x| \ge 1 - |y| = \operatorname{dist}(y, \partial\mathbb{B}^n)$,

$$u_{\mathbb{B}^n}(x, y) = 2\log\Big(\frac{|x - y| + 1 - |x|}{\sqrt{(1 - |x|)(1 - |y|)}}\Big)$$

and hence

$$u_{\mathbb{B}^n}(-w, w) = 2\log\Big(\frac{1 + |w|}{1 - |w|}\Big) = 2\log\Big(\frac{2 + |x - y|}{2 - |x - y|}\Big).$$

By the arithmetic mean-geometric mean inequality,

$$\frac{1}{\sqrt{(1 - |x|)(1 - |y|)}} \ge \frac{2}{2 - |x| - |y|}. \tag{3.2}$$

To prove our claim, it is enough to show

$$\frac{|x - y| + 1 - |x|}{\sqrt{(1 - |x|)(1 - |y|)}} \ge \frac{2 + |x - y|}{2 - |x - y|}.$$

Since $|x - y| = |x| + |y|$ and $|x| \le |y|$,

$$\frac{|x - y| + 1 - |x|}{\sqrt{(1 - |x|)(1 - |y|)}} \ge \frac{2(1 + |y|)}{2 - |x| - |y|} \ge \frac{2 + |x| + |y|}{2 - |x| - |y|} = \frac{2 + |x - y|}{2 - |x - y|},$$

where the first inequality follows from (3.2).

If $x \in [0, y]$, then $\operatorname{dist}(x, \partial\mathbb{B}^n) = 1 - |x| \ge 1 - |y| = \operatorname{dist}(y, \partial\mathbb{B}^n)$ and $|x - y| = |y| - |x|$. By the definition of u-metric,

$$u_{\mathbb{B}^n}(x, y) = 2\log\Big(\frac{1 + |y| - 2|x|}{\sqrt{(1 - |x|)(1 - |y|)}}\Big)$$

and hence

$$u_{\mathbb{B}^n}(-w, w) = 2\log\Big(\frac{1 + |w|}{1 - |w|}\Big) = 2\log\Big(\frac{2 + |y| - |x|}{2 - |y| + |x|}\Big).$$

To show $u_{\mathbb{B}^n}(x, y) \ge u_{\mathbb{B}^n}(-w, w)$, it is enough to show that

$$\frac{1 + |y| - 2|x|}{\sqrt{(1 - |x|)(1 - |y|)}} \ge \frac{2 + |y| - |x|}{2 - |y| + |x|}.$$

From (3.2),

$$\frac{1 + |y| - 2|x|}{\sqrt{(1 - |x|)(1 - |y|)}} \ge \frac{2(1 + |y| - 2|x|)}{2 - |x| - |y|}.$$

Now our aim is to show

$$\frac{2(1 + |y| - 2|x|)}{2 - |x| - |y|} \ge \frac{2 + |y| - |x|}{2 - |y| + |x|},$$

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or, equivalently,

$$2|y| - 2|x| - |y|^2 + 6|x||y| - 5|x|^2 \geq 0.$$

Since, $|x| \leq |y|$,

$$\begin{aligned} 2|y| - 2|x| - |y|^2 + 6|x||y| - 5|x|^2 &\geq 2|y| - 2|x| - |y|^2 + |x|^2 \\ &= (1 - |x|)^2 - (1 - |y|)^2 \geq 0. \end{aligned}$$

Finally, since the function

$$f(t) = 2\log\Big(\frac{2+t}{2-t}\Big) - t$$

is increasing in t, the conclusion follows.

Case 2. Let $x, y \in \mathbb{B}^n$ be arbitrary. Choose $y' \in \mathbb{B}^n$ such that $|x - y| = |x - y'|$ and $x, 0$ and y' are co-linear. Geometrically, it is clear that $|y'| \leq |y|$. Hence,

$$u_{\mathbb{B}^n}(x, y) \geq u_{\mathbb{B}^n}(x, y') \geq u_{\mathbb{B}^n}(-w, w),$$

where the first inequality follows from the definition and the second inequality follows from *Case 1*. The proof is complete. □

REMARK 3.5. It is remarkable that the domain monotonicity property of the $\tilde{\tau}_D$-metric plays a crucial role in the proof of Theorem 3.3. Since the u_D-metric does not satisfy the domain monotonicity property, it is not easy to extend Theorem 3.4 to arbitrary bounded domains of $\mathbb{R}^n$ in a similar manner.

4. Quasi-invariance property of the u_D-metric and the $\tilde{\tau}_D$-metric under quasiconformal maps

Quasiconformal mappings are natural generalisations of conformal mappings. There are several equivalent definitions of quasiconformal mappings in the literature. We adopt the metric definition of quasiconformality introduced by Väisälä in [29]. See also [1, 32] for further developments in quasiconformal theory.

Let $D \subsetneq \mathbb{R}^n$ be a domain and let $f : D \to f(D) \subsetneq \mathbb{R}^n$ be a homeomorphism. The function f is said to be K-quasiconformal ($1 \leq K < \infty$), if the linear dilatation of f at $x \in D$, defined by

$$H(f, x) = \limsup_{r \to 0} \frac{\sup\{|f(x) - f(y)| : |x - y| = r\}}{\inf\{|f(x) - f(y)| : |x - y| = r\}}, \tag{4.1}$$

is bounded by K. If f is K-quasiconformal then $\sup_{x\in D} H(f, x) \leq c(n, K) < \infty$.

As an example of a quasiconformal mapping, consider an L-bilipschitz map of $\mathbb{R}^n$, that is, $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies

$$\frac{1}{L}|x - y| \leq |f(x) - f(y)| \leq L|x - y| \quad \text{for } x, y \in \mathbb{R}^n.$$

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It is easy to verify from (4.1) that an L-bilipschitz map f is K-quasiconformal with $K = L^2$. Further, an L-bilipschitz map f is also L^2-bilipschitz with respect to the $\tilde{\tau}_D$-metric, that is,

$$\frac{1}{L^2}\tilde{\tau}_D(x,y) \le \tilde{\tau}_{f(D)}(f(x), f(y)) \le L^2\tilde{\tau}_D(x,y) \quad \text{for } x,y \in D.$$

Indeed,

$$\begin{aligned}\tilde{\tau}_{f(D)}(f(x), f(y)) &= \log\Big(1 + \sup_{f(p)\in\partial f(D)} \frac{|f(x)-f(y)|}{\sqrt{|f(x)-f(p)|\,|f(p)-f(y)|}}\Big)\\ &\le \log\Big(1 + \sup_{p\in\partial D} \frac{L^2|x-y|}{\sqrt{|x-p|\,|p-y|}}\Big)\\ &\le L^2\log\Big(1 + \sup_{p\in\partial D} \frac{|x-y|}{\sqrt{|x-p|\,|p-y|}}\Big) = L^2\tilde{\tau}_D(x,y),\end{aligned}$$

where the first inequality follows from the bilipschitz condition on f and the second from Bernoulli's inequality (2.4). Observe that the distortion constant L^2 is independent of the dimension of the space. Now we ask whether we can distort the $\tilde{\tau}_D$-metric if we replace the L-bilipschitz map by an arbitrary K-quasiconformal map. The answer is 'yes' and the distortion constant will depend upon n (the dimension of the space) and K. In general, the problem is as follows.

Problem 4.1. For $n \ge 1$ and $K \ge 1$, does there exist a constant c depending only on n and K with the following property: if f is a K-quasiconformal map of (D,d) onto (D',d') and $\alpha = K^{1/(1-n)}$, then

$$d'_{D'}(f(x), f(y)) \le c\max\{d_D(x,y), d_D(x,y)^\alpha\} \quad \text{for all } x,y \in D.$$

This problem is studied in different contexts for different metrics. One question is whether $c \to 1$ as $K \to 1$. Obtaining such a distortion constant under a K-quasiconformal map of $\mathbb{R}^n$ for hyperbolic-type metrics in general is a challenging problem. Gehring and Osgood [7, Theorem 3] proved the quasi-invariance of the quasihyperbolic metric under quasiconformal maps of $\mathbb{R}^n$, but the distortion constant does not tend to 1 as $K \to 1$. However, the distortion constant for the Seittenranta metric [28, Theorem 1.2], and hence for the hyperbolic metric [23, Corollary 2.10], approaches 1 as $K \to 1$. We investigate the quasi-invariance of the $\tilde{\tau}_D$-metric and the u_D-metric under quasiconformal maps of $\mathbb{R}^n$. In the proof we need the quasi-invariance property of the distance ratio metric, $\tilde{j}_D$, defined by

$$\tilde{j}_D(x,y) = \log\Big(1 + \frac{|x-y|}{\min\{\mathrm{dist}(x,\partial D), \mathrm{dist}(y,\partial D)\}}\Big), \quad \text{for } x,y \in D.$$

Theorem 4.2. *For $n \ge 1$ and $K \ge 1$, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a K-quasiconformal mapping which maps $D \subsetneq \mathbb{R}^n$ onto $D' \subsetneq \mathbb{R}^n$ then there exists a constant C_1 depending only on n and K such that*

$$\tilde{\tau}_{D'}(f(x), f(y)) \le C_1\max\{\tilde{\tau}_D(x,y), \tilde{\tau}_D(x,y)^\alpha\}$$

for all $x,y \in D$, where $\alpha = K^{1/(1-n)}$.

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PROOF. For all $x, y \in D$,

$$\begin{aligned}\tilde{\tau}_{D'}(f(x), f(y)) &\le \tilde{j}_D(f(x), f(y)) \le C \max\{\tilde{j}_D(x, y), \tilde{j}_D(x, y)^\alpha\} \\ &\le C \max\{2\tilde{\tau}_D(x, y), 2^\alpha \tilde{\tau}_D(x, y)^\alpha\} \le C_1 \max\{\tilde{\tau}_D(x, y), \tilde{\tau}_D(x, y)^\alpha\},\end{aligned}$$

where the first and third inequalities follow from [18, Theorems 4.2, 4.3], the second from [13, Lemma 2.3], and the constant C_1 depends only on n and K. □

THEOREM 4.3. *For $n \ge 1$ and $K \ge 1$, if $f : \mathbb{R}^n \to \mathbb{R}^n$ is a K-quasiconformal mapping which maps $D \subsetneq \mathbb{R}^n$ onto $D' \subsetneq \mathbb{R}^n$ then there exists a constant C_2 depending only on n and K such that*

$$u_{D'}(f(x), f(y)) \le C_2 \max\{u_D(x, y), u_D(x, y)^\alpha\}$$

for all $x, y \in D$, where $\alpha = K^{1/(1-n)}$.

PROOF. Recall that the u_D-metric and the $\tilde{\tau}_D$-metric are bilipschitz equivalent. Indeed,

$$2\tilde{\tau}_D(x, y) \le u_D(x, y) \le 4\tilde{\tau}_D(x, y)$$

(see [26, Theorem 3.5]). The result now follows from Theorem 4.2. □

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