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The topological structure of $\mathcal{D}$-classes

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Let S be a compact, topological semigroup with identity. Suppose D, L and R are the $\mathcal{D}$, $\mathcal{L}$ and $\mathcal{R}$ classes of some $x \in S$. Let $(L,\alpha,L/H)$, $(R,\beta,R/H)$, $(D,\gamma,D/H)$ and $(D,\delta,D/R)$ by the fibre spaces gotten where α, β, γ and δ are the natural maps. It is shown that $(D,\gamma,D/H)$ has topologically the same structure as the fibre space associated with $(L,\alpha,L/H)$ by R. Also if $(L,\alpha,L/H)$ is locally trivial (locally a cartesian product) then so is $(D,\delta,D/R)$ and if both $(L,\alpha,L/H)$ and $(R,\beta,R/H)$ are locally trivial then so is $(D,\gamma,D/H)$.

A compact simple semigroup is homeomorphic with the space $(E \cap R_e) \times (E \cap L_e) \times H_e$ where R_e, L_e and H_e are the $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{H}$ classes of any arbitrarily chosen idempotent e of the semigroup [6]. Since such a semigroup is a single $\mathcal{D}$-class it is natural to ask if all $\mathcal{D}$-classes of compact semigroups possess a similar topological structure. Hunter and Anderson in [1] showed that, in general, a $\mathcal{D}$-class cannot be represented as a cartesian product similar to the above. However, they showed that a regular $\mathcal{D}$-class (a $\mathcal{D}$-class possessing an idempotent) is a special type of fibre space. In this paper we shall show that this is true for arbitrary $\mathcal{D}$-classes. All undefined terms and unstated theorems are to be found in [3].

First, we recall several definitions and results which appear in [2]. A *fibre space* is a triple (X,p,B) where p is a continuous open map of X onto B such that, for b_1 and b_2 in B, $p^{-1}(b_1)$ is homeomorphic with $p^{-1}(b_2)$. B is called the *base* and $p^{-1}(b_1)$ is called the *fibre*. A fibre

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space is called *principal* if it is gotten by means of the action of a topological group. We say that (X,p,B) has a *cross-section* C if C is a closed subset of X such that p restricted to C is a homeomorphism onto B. (X,p,B) is trivial if it is homeomorphic to $B \times p^{-1}(b)$ for any $b \in B$. (X,p,B) is locally trivial at $b \in B$ if there is a closed neighborhood U of b such that $(p^{-1}(U),p,U)$ is a trivial fibre space. Suppose (X,p,B) is a principal fibre space with group G. Assume that G acts on X on the right and that G acts on a space F on the left. We define a fibre space called the fibre space associated with (X,p,B) by F to be $((X,F),p,B)$ where (X,F) is the orbit space of $X \times F$ under the action of G and where q is the natural map of (X,F) onto B. The fibres of this space are homeomorphic with F. We shall refer to this space as (X,F).

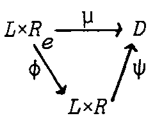
We shall assume in all that follows that S is a compact semigroup with identity. If S does not possess an identity, then an identity can be appended. This does not affect the $\mathcal{D}$-class structure of S.

We summarize several results in [5] to get:

LEMMA 1. *Let* $x \in S$. *Then there are idempotents* e *and* f *in* S *such that* $x = fx = xe$, $xR_e = R_x$, $L_f x = L_x$, *and* $L_f x R_e = D_x$.

Let $D = D_x$ be any $\mathcal{D}$-class of a compact semigroup S. To simplify notation let $H = H_x$, $L = L_x$ and $R = R_x$. Suppose e and f are idempotents satisfying Lemma 1. Define a map μ from $L \times R_e$ to D by $\mu(s,t) = st$; define a map η from R_e to R by $\eta(t) = xt$; and define a map ϕ from $L \times R_e$ to $L \times R$ by $\phi(s,t) = (s,\eta(t))$. Because multiplication is continuous in S all three maps are continuous and from Lemma 1 they are onto.

LEMMA 2. *There is a continuous map* ψ *from* $L \times R$ *onto* D *such that the diagram*



is commutative.

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Proof. Let (s_1,t_1) and (s_2,t_2) be in $L \times R_e$ and suppose that $\phi(s_1,t_1) = \phi(s_2,t_2)$. Then $s_1 = s_2$ and $xt_1 = xt_2$. Since $s_1 \in L$ there is $a \in S$ such that $s_1 = ax$. Then

$$\mu(s_1,t_1) = \mu(ax,t_1) = axt_1 = axt_2 = \mu(ax,t_2) = \mu(s_2,t_2) .$$

Thus there is a map ψ from $L \times R$ to D such that $\psi \circ \phi = \mu$. All spaces involved are compact so the map ψ is continuous. From Lemma 1 μ and ϕ are onto so ψ is onto. □

LEMMA 3. *$\psi(s,t) \in R_s \cap L_t$ for all $(s,t) \in L \times R$. Also if $\psi(s_1,t_1) = \psi(s_2,t_2)$ for (s_1,t_1) and (s_2,t_2) in $L \times R$, then $s_1 H s_2$ and $t_1 H t_2$.*

Proof. Let $(s,t) \in L \times R$. We may write $s = ax$ and $t = xb$ for some $a,b \in S$. Left multiplication by a maps R onto R_s and right multiplication by b maps L onto L_t [3].
$\psi(s,t) = \psi(ax,t) = (ax)\eta^{-1}(t) = at = axb$. Since $xb \in R$ we have $a(xb) \in R_{ax} = R_s$ and since $ax \in L$ we have $(ax)b \in L_{xb} = L_t$. Therefore $\psi(s,t) \in R_s \cap L_t$.

Now, suppose that $\psi(s_1,t_1) = \psi(s_2,t_2)$. Since $\psi(s_i,t_i) \in R_{s_i} \cap L_{t_i}$ for $i = 1,2$ we must have $R_{s_1} \cap L_{t_1} = R_{s_2} \cap L_{t_2}$. This implies that $R_{s_1} = R_{s_2}$ and $L_{t_1} = L_{t_2}$. But we already have $s_1 L s_2$ and $t_1 R t_2$. Thus $s_1 H s_2$ and $t_1 H t_2$. □

Let Γ and Γ' be the right and left Schützenberger groups associated with H. Γ acts on L on the right by defining $s\gamma = a(x\gamma)$ for $s \in L$ and $\gamma \in \Gamma$ where a is any element of S such that $s = ax$. Define a map θ from Γ onto Γ' by $\theta(\gamma) = \nu$ if $x\gamma = \nu x$. θ is an iseomorphism of Γ onto Γ'. By means of θ, Γ can be made to act on R on the left by defining $\gamma r = \theta(\gamma)^{-1} r$ for $\gamma \in \Gamma$ and $r \in R$. Moreover, Γ acts on $L \times R$ by defining $(s,t)\gamma = (s\gamma, \theta(\gamma)^{-1}t)$ for $\gamma \in \Gamma$ and $(s,t) \in L \times R$. The orbit space of L by Γ is homeomorphic with L/H and the orbit space of $L \times R$ by Γ is (L,R), the fibre space associated with L by R having base L/H and fibre homeomorphic with R. Let σ be the map of (L,R) onto L/H.

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It is known that for $\gamma \in \Gamma$ and $\nu \in \Gamma'$ $\nu x\gamma = \nu(x\gamma) = (\nu x)\gamma$. Also for $a,b \in S$ we have $(ax)\gamma = a(x\gamma)$ and $\nu(xb) = (\nu x)b$.

LEMMA 4. *The decompositions induced on* $L \times R$ *by the map* ψ *and the action of* Γ *are the same.*

Proof. To establish this result we shall show that $\psi^{-1}(\psi(s,t)) = (s,t)\Gamma$ for any $(s,t) \in L \times R$. Suppose $(s,t) \in L \times R$ with $s = ax$ and $t = xb$. Then

$$\psi(s,t) = \mu(\phi^{-1}(s,t)) = \mu(ax,\eta^{-1}(t)) = ax\eta^{-1}(t) = at$$

since $\eta^{-1}(t) = \{r \in R_e \; ; \; xr = t\}$. If $(s_1,t_1) = (s,t)\gamma$ for some $\gamma \in \Gamma$, then we have

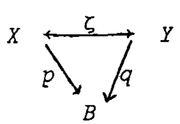
$$\begin{aligned} \psi(s_1,t_1) &= \psi(s\gamma,\theta(\gamma)^{-1}t) = (s\gamma)\eta^{-1}(\theta(\gamma)^{-1}t) \\ &= a(x\gamma)\eta^{-1}(\theta(\gamma)^{-1}t) = a(\theta(\gamma)x)\eta^{-1}(\theta(\gamma)^{-1}t) \\ &= a(\theta(\gamma)x\eta^{-1}(\theta(\gamma)^{-1}t)) = a(\theta(\gamma)\theta(\gamma)^{-1}t) \\ &= at = \psi(s,t) \; . \end{aligned}$$

Now, if $\psi(s,t) = \psi(s_1,t_1)$, we have sHs_1 and tHt_1 by Lemma 3. We may write $s_1 = s\gamma$ and $t_1 = \nu t$ for some $\gamma \in \Gamma$ and $\nu \in \Gamma'$. Then

$$\begin{aligned} at = \psi(s,t) &= \psi(s_1,t_1) = \psi(s\gamma,\nu t) \\ &= (s\gamma)\eta^{-1}(\nu t) = a(\theta(\gamma)x\eta^{-1}(\nu t)) \\ &= a\theta(\gamma)\nu t \; . \end{aligned}$$

Left multiplication by a is a one-to-one map of H_t onto H_{at} , so we must have $\theta(\gamma)\nu t = t$. This implies that $\nu = \theta(\gamma)^{-1}$, that is, $(s_1,t_1) = (s,t)\gamma$.

Let (X,p,B) and (Y,q,B) be two fibre spaces with the same base B . We say that a homeomorphism ζ of X onto Y is a B-homeomorphism if the diagram



is commutative. We now show that D possesses the same topological structure as (L,R) .

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THEOREM 1. *There is a* $L/\mathcal{H}$*-homeomorphism* ζ *from* (L,R) *onto* D .

Proof. From Lemma 4 we know that ψ and Γ induce the same decompositions on D . Let ζ be the canonical homeomorphism of (L,R) onto D . $\psi(\{s\}xR) = R$ for all $s \in L$, so the map σ of (L,R) onto $L/\mathcal{H}$ induces the map of D onto $D/\mathcal{R}$. To complete the proof we identify $L/\mathcal{H}$ with $D/\mathcal{R}$ [1].

From [1] we know that $(L,\alpha,L/\mathcal{H})$, $(R,\beta,R/\mathcal{H})$, $(D,\gamma,D/\mathcal{H})$ and $(D,\delta,D/\mathcal{R})$ are fibre spaces where α , β , γ and δ are the canonical maps.

THEOREM 2. *If* $(L,\alpha,L/\mathcal{H})$ *is locally trivial (trivial), then the fibre space* $(D,\delta,D/\mathcal{R})$ *is also. If* $(L,\alpha,L/\mathcal{H})$ *and* $(R,\beta,R/\mathcal{H})$ *are locally trivial (trivial), then so is* $(D,\gamma,D/\mathcal{H})$.

Proof. We may identify $L/\mathcal{H}$ with $D/\mathcal{R}$ since the $\mathcal{R}$-relation on D restricted to L is the same as the $\mathcal{H}$-relation. Choose x_0 such that $(\alpha^{-1}(U),\alpha,U)$ is a trivial fibre space. $(\alpha^{-1}(U),\alpha,U)$ has a cross-section; call it C . $C \times R$ is a closed subset of $L \times R$. We claim that ψ restricted to $C \times R$ is a homeomorphism. Since $C \times R$ is compact and the map is continuous, we need only show that it is one-to-one. Suppose $c_1,c_2 \in C$ and $r_1,r_2 \in R$ with $\psi(c_1,r_1) = \psi(c_2,r_2)$. Then from Lemma 3 we have $c_1\mathcal{H}c_2$ and $r_1\mathcal{H}r_2$. C meets each $\mathcal{H}$-class of L at most once, so $c_1 = c_2$. We may write c_1 as ax for some $a \in S$. Then $\psi(c_i,r_i) = ax\eta^{-1}(r_i)$ for $i = 1,2$. This implies that $ar_1 = ar_2$. But left multiplication by a is a one-to-one map of H_{r_1} onto H_{ar_1} , so $r_1 = r_2$ and ψ restricted to $C \times R$ is a homeomorphism. By the identification of $L/\mathcal{H}$ with $D/\mathcal{R}$ we see that $\psi(C\times R) = \delta^{-1}(U)$ and hence $(\delta^{-1}(U),\delta,U)$ is a trivial fibre space.

Now, suppose $(L,\alpha,L/\mathcal{H})$ and $(R,\beta,R/\mathcal{H})$ are locally trivial at $\alpha(x)$ and $\beta(x)$, respectively. We may identify $L/\mathcal{H} \times R/\mathcal{H}$ with $D/\mathcal{H}$ [1]. Choose U a closed heighborhood of $\alpha(x)$ so that $(\alpha^{-1}(U),\alpha,U)$ is trivial with cross-section C_1 . Choose V a closed neighborhood of $\beta(x)$ such that $(\beta^{-1}(V),\beta,V)$ is trivial with cross-section C_2 . Let $W = \alpha^{-1}(U) \times \beta^{-1}(V)$. By the first part of the theorem $\gamma^{-1}(W)$ is

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homeomorphic with $C_1 \times \beta^{-1}(V)$. Now $\beta^{-1}(V)$ is homeomorphic with $C_2 \times H$, so $\gamma^{-1}(W)$ is homeomorphic with $C_1 \times H \times C_2$. We need only prove the result for x , since all the definitions used can be reformulated for any other element of D .

The same proof is used in the trivial case using spaces instead of neighborhoods.

Although in general $\mathcal{L}$, $\mathcal{R}$ or $\mathcal{D}$ classes are not trivial fibre spaces we have the following:

THEOREM 3. *Let $R_x(L_x)$ be an R-(L-) class of the compact metric semigroup S . If $R_x(L_x)$ is zero-dimensional, then $R_x \to R_x/H$ $(L_x \to L_x/H)$ has a cross-section. Moreover, if D_x is zero-dimensional then $D_x \to D_x/H$ has a cross-section.*

Proof. The first part follows from [5], page 317, and the second part follows from Theorem 2.

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