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ON THE MÖBIUS FUNCTION OF Hom(P, Q)

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A formula is given for the Möbius function of the poset $\text{Hom}(P, Q)$ of all order-preserving maps between two finite posets P and Q. Two applications of the formula are presented.

1. Introduction

Let P and Q be arbitrary finite partially ordered sets (posets) with *zeta* and *Möbius* functions ζ_P, ζ_Q and μ_P, μ_Q respectively. We give a formula for the Möbius function μ of the poset $\text{Hom}(P, Q)$ of all order-preserving maps $\varphi : P \to Q$ in terms of ζ_P, ζ_Q, μ_P and μ_Q ; see equation (1) below. An earlier result due to Rota ([6], p. 350) attacks the same problem, but the formula obtained there seems less explicit.

Two applications of our formula are given. The first rederives the well-known expression for the Möbius function of a finite distributive lattice L by using the anti-isomorphism of L with $\text{Hom}(P, \mathbf{2})$, where $P = P(L)$ is the poset of all join-irreducible elements of L and $\mathbf{2}$ is the 2-element chain. In our second application of (1) we obtain combinatorial generalisations on $\text{Hom}\big(P, \mathcal{P}(\underline{\underline{m}})\big)$ of the familiar descending powers $(n)_r = n(n-1) \ldots (n-r+1)$ which are related to ordinary powers n^r via the Möbius function of the lattices $\mathcal{P}(\underline{\underline{m}})$ of all partitions of the set $\underline{\underline{m}} = \{1, \ldots, m\}$. These results are required for some applications to

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statistics in which the orbits on m-tuples of the action of a generalised wreath product of symmetric groups play a prominent role; see Praeger *et al* [5].

2. The formula

PROPOSITION 1. *The Möbius function* μ *of* $\mathrm{Hom}(P, Q)$ *is given by*

$$(1) \qquad \mu(\varphi, \psi) = \prod_{p \in P} \left\{ \mu_Q\big(\varphi(p), \psi(p)\big) \prod_{p' > p} \zeta_Q\big(\psi(p), \varphi(p')\big) \right\}$$

where $\varphi, \psi \in \mathrm{Hom}(P, Q)$.

Proof. We begin by recalling the zeta function ζ of $\mathrm{Hom}(P, Q)$ which, since the ordering on $\mathrm{Hom}(P, Q)$ is componentwise, takes the form

$$(2) \qquad \zeta(\psi, \chi) = \prod_{p \in P} \zeta_Q\big(\psi(p), \chi(p)\big) .$$

Let us denote the right hand side of (1) by $\vartheta(\varphi, \psi)$; our aim is to take its product with (2) and sum over ψ , and if we can obtain the result $\delta(\varphi, \chi) = 1$ if $\varphi = \chi$, $\delta(\varphi, \chi) = 0$ otherwise, the result will be proved. If we regard the sum as being over $\psi(p) \in Q$ subject to the constraints $\varphi(p) \le \psi(p) \le \chi(p)$ and $\psi(p) \le \varphi(p')$ for all $p' > p$, as well as those imposed by the motonicity requirement $\psi(p) \le \psi(p')$ for all $p' > p$, and then sum over $p \in P$, the whole summation can be evaluated by working down from the maximal elements of P .

More precisely, let us suppose that p is a maximal element of P . Then we can sum over $\psi(p)$ in the product $\vartheta(\varphi, \psi)\zeta(\psi, \chi)$ subject only to the constraint $\varphi(p) \le \psi(p) \le \chi(p)$, and by the definition of μ_Q we obtain $\delta_Q\big(\varphi(p), \chi(p)\big)$, where δ_Q is the (Kronecker) delta function on Q : $\delta_Q(q, q') = 1$ if $q = q'$ and $\delta_Q(q, q') = 0$ otherwise. This argument can be used for all maximal elements of P .

Now suppose that p is an arbitrary element of P and that we have summed over $\psi(p')$ for all $p' > p$, respecting the constraints noted above, and that in each case the result was $\delta\big(\varphi(p'), \chi(p')\big)$. Then we may sum over $\psi(p)$ subject only to the constraint $\varphi(p) \le \psi(p) \le \chi(p)$, since the remaining constraints $\psi(p) \le \varphi(p')$ and $\psi(p) \le \psi(p')$ for all $p' > p$ are automatically satisfied when $\varphi(p') = \chi(p')$ for all $p' > p$, and this

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is true by our inductive hypothesis. Thus we obtain the term $\delta(\varphi(p), \psi(p))$ once more and our inductive proof is complete. □

It is well known, see for example Birkhoff [3], that every finite distributive lattice L is anti-isomorphic to the lattice $\mathrm{Hom}(P, 2)$ where $P = P(L)$ is the set of all join-irreducibles of L and 2 is the 2-element chain. Using this fact we can re-derive the following known result.

COROLLARY. *The Möbius function* μ_L *of a finite distributive lattice* L *is given by*

$$\mu_L(a, b) = \begin{cases} 1 & \text{if } a = b , \\ (-1)^m & \text{if } b \text{ is the join of } m \text{ elements covering } a , \\ 0 & \text{otherwise.} \end{cases}$$

Proof. We represent L as the dual of $\mathrm{Hom}(P, 2)$ where $P = P(L)$ is the poset of join-irreducibles of L via the map $a \to \varphi_a$ where $\varphi_a : P \to 2$ is given by

$$\varphi_a(p) = \begin{cases} 0 & \text{if } p \le a , \\ 1 & \text{otherwise.} \end{cases}$$

Clearly we must have $\mu_L(a, b) = \mu(\varphi_b, \varphi_a)$, where μ is the Möbius function of $\mathrm{Hom}(P, 2)$ and we use (1) to find the expression above for μ_L .

Suppose that $a < b$. Then $\mu(\varphi_b, \varphi_a)$ given by (1) is the product of terms taking the form

$$(3) \qquad \mu_2(\varphi_b(p), \varphi_a(p)) \prod_{p'>p} \zeta_2(\varphi_a(p), \varphi_b(p')) , \quad p \in P .$$

Define the sets $P_1 = \{p \in P : \varphi_a(p) = 0\}$, $P_2 = \{p \in P : \varphi_b(p) = 1\}$ and $P_3 = \{p \in P : \varphi_b(p) = 0, \varphi_a(p) = 1\}$. Then it is clear that if $p \in P_1$, the expression (3) takes the value 1 ; similarly if $p \in P_2$. Finally, if $p \in P_3$, then (3) takes the value -1 as long as there is no

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$p' > p$ also belonging to P_3 ; otherwise (3) takes the value 0 . Thus $\mu(\varphi_b, \varphi_a) = (-1)^m$ if $|P_3| = m$ and no pair of elements in P_3 is comparable, $|P_3| = 0$ otherwise. In the former case b is the sup of the m elements $\{a \vee p : p \in P_3\}$ which cover a , and the corollary is proved.

3. Hom$(P, P(\underline{\underline{m}}))$

When $Q = P(\underline{\underline{m}})$, the lattice of all partitions of the set $\underline{\underline{m}} = \{1, \ldots, m\}$, we can obtain natural extensions of some standard formulae. These extensions are required for some statistical applications which build upon the main result proved in Praeger *et al* [5] namely, that the orbits of the action of a generalised wreath product group on m-tuples of elements of the basic set are labelled by $L = \text{Hom}(P, P(\underline{\underline{m}}))$. These applications required not only the Möbius function μ_L of L but also some natural generalisations of ascending and descending powers.

Let us review the results on $P(\underline{\underline{m}})$ (the case P a singleton) which we wish to generalise. When σ is a partition of $\underline{\underline{m}}$ into $b = b(\sigma)$ blocks, write $n^\sigma := n^{b(\sigma)}$ and $(n)_\sigma = n(n-1) \ldots (n-b(\sigma)+1)$, $n \in \mathbb{N}$. Then the following formulae are well known, see Aigner [1]:

(4a) $$n^\sigma = \sum_\tau \zeta(\sigma, \tau)(n)_\tau ,$$

(4b) $$(n)_\sigma = \sum_\tau \mu(\sigma, \tau)n^\tau ,$$

where ζ and μ are the zeta and Möbius functions of $P(\underline{\underline{m}})$. A related number is $(n)_{(\rho,\tau)}$ defined by

(5) $$(n)_{(\rho,\tau)} = \sum_\pi \mu(\rho, \pi)\zeta(\pi, \tau)n^\pi .$$

The number $(n)_\sigma$ can be viewed as the number of maps $h : \underline{\underline{n}} \to \underline{\underline{m}}$ whose kernel equivalence $\ker h = \sigma$. The corresponding result for $(n)_{(\rho,\tau)}$ is the following lemma.

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LEMMA. *For arbitrary elements* $\rho, \tau \in P(\underline{\underline{m}})$ *with* $\rho \le \tau$,

(6) $$(n)_{(\rho,\tau)} = |\{h \in \underline{\underline{n}}^{\underline{\underline{m}}} : \ker h \wedge \tau = \rho\}| .$$

Proof.

$$\sum_{\pi} \mu(\rho, \pi)\zeta(\pi, \tau)n^{\pi} = \sum_{\pi} \mu(\rho, \pi)\zeta(\pi, \tau) \sum_{\sigma} \zeta(\pi, \sigma)(n)_{\sigma} \quad \text{by (4a)}$$

$$= \sum_{\sigma} \left\{\sum_{\pi} \mu(\rho, \pi)\zeta(\pi, \tau \wedge \sigma)\right\}(n)_{\sigma}$$

$$= \sum_{\sigma} \delta(\rho, \tau \wedge \sigma)(n)_{\sigma}$$

and the result follows from the remark preceding the lemma. □

It is clear that $(n)_{(\rho,\rho)} = n^{\rho}$, $(n)_{(\rho,\underline{\underline{m}})} = (n)_{\rho}$ where $\underline{\underline{m}}$ is the single block partition of $\underline{\underline{m}}$. Partitions π such that $\pi \wedge \tau = 0$ and hence $(n)_{(0,\tau)}$ also play a role in certain combinatorial matters, see Doubilet [4]; here 0 denotes the partition $0 = 1|2|\ldots|m$ of $\underline{\underline{m}}$ into singletons. There is no simple general expression for $(n)_{(\rho,\tau)}$, $\rho, \tau \in P(\underline{\underline{m}})$.

Another view of the numbers $(n)_{\sigma}$, n^{σ} and $(n)_{(\rho,\tau)}$ follows from the fact that the orbits of the symmetric group S_n acting on ordered m-tuples $\underline{\underline{n}}^{\underline{\underline{m}}}$ of elements from $\underline{\underline{n}} = \{1, \ldots, n\}$ are naturally labelled by partitions $\sigma \in P(\underline{\underline{m}})$. Indeed if we denote them by $\{O_{\sigma} : \sigma \in P(\underline{\underline{m}})\}$, then $|O_{\sigma}| = (n)_{\sigma}$, $\left|\bigcup_{\tau \ge \sigma} O_{\tau}\right| = n^{\sigma}$ and, more generally,

$$\left|\bigcup_{\sigma \wedge \tau = \rho} O_{\sigma}\right| = (n)_{(\rho,\tau)} .$$

Our desired extensions of these results concern group actions $(S_{n_p}, \underline{\underline{n}}_p)$, $p \in P$, labelled by a poset P , where S_{n_p} is the symmetric group on n_p elements and $\underline{\underline{n}}_p = \{1, \ldots, n_p\}$, and their generalised wreath product $(G, \underline{\underline{n}}_P)$ where $\underline{\underline{n}}_P = \prod_{p \in P} \underline{\underline{n}}_p$. This product is defined and

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studied in Bailey *et al* [2], and further in Praeger *et al* [5] where it is proved that the orbits of G acting on $\underline{\underline{n}}_P^{\underline{\underline{m}}}$ take the form

$$O_\sigma = \left\{h \in \underline{\underline{n}}_P^{\underline{\underline{m}}} : \varphi^h = \varphi\right\}, \quad \varphi \in \mathrm{Hom}\big(P, P(\underline{\underline{m}})\big),$$

where $\varphi^h : P \to P(\underline{\underline{m}})$ is given by $\varphi^h(p) = \wedge\{\ker h_{p'} : p' \geq p\}$.

By analogy with $P(\underline{\underline{m}})$ we make the following definitions, noting that our use of n is now symbolic, being an abbreviation for $\{n_p, p \in P\}$,

$$n^\varphi = \prod_{p \in P} n_p^{\varphi(p)},$$

$$(n)_\varphi = \prod_{p \in P} (n_p)_{(\varphi(p), \wedge\{\varphi(p') : p' > p\})},$$

and

$$(n)_{(\varphi,\chi)} = \sum_\psi \mu_L(\varphi, \psi)\zeta_L(\psi, \chi)n^\psi,$$

where $\varphi, \chi \in L = \mathrm{Hom}\big(P, P(\underline{\underline{m}})\big)$. With these definitions we have complete analogues of the results for $G = S_n$ acting on $\underline{\underline{n}}^{\underline{\underline{m}}}$.

PROPOSITION 2. *For every pair* $\varphi, \chi \in \mathrm{Hom}\big(P, P(\underline{\underline{m}})\big)$ *we have*

(7a) $$n^\varphi = \sum_\psi \zeta_L(\varphi, \psi)(n)_\psi,$$

(7b) $$(n)_\varphi = \sum_\psi \mu_L(\varphi, \psi)n^\psi,$$

(8) $$(n)_{(\varphi,\chi)} = \left|\{h \in \underline{\underline{n}}^{\underline{\underline{m}}} : \varphi^h \wedge \chi = \varphi\}\right|.$$

Proof. It suffices to prove (7b) as (7a) follows by Möbius inversion and (8) is proved in the same way as (6). Substituting the expression (1) for μ_L into the right-hand side of (7b) we find that we must simplify

$$\sum_\psi \prod_{p \in P} \left\{\mu\big(\varphi(p), \psi(p)\big) \prod_{p' > p} \zeta\big(\psi(p), \varphi(p')\big)n_p^{\varphi(p)}\right\}.$$

The result then follows by summing over $\psi \in \mathrm{Hom}\big(P, P(\underline{\underline{m}})\big)$ in the same way as we did in the proof of (1), that is, by first summing over $\psi(p)$ for p

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maximal, and then only summing over $\psi(p)$, p non maximal, after having summed over all $\psi(p')$, $p' > p$. □

As was the case in our earlier discussion these formulae also give the number of elements in orbits; in particular

$$|O_\varphi| = \left|\left\{h \in \underline{\underline{n}}_P^{\underline{m}} : \varphi^h = \varphi\right\}\right| = (n)_\varphi \ , \quad \varphi \in \mathrm{Hom}\left(P, \ P(\underline{\underline{m}})\right) \ .$$

EXAMPLE. Suppose that P is the 2-element chain (with $2 < 1$) and $m = 2$. Then for $\varphi = (1|2, 1|2)$ say,

$$(n)_\varphi = \left(n_1\right)_{\varphi(1)}\left(n_2\right)_{\left(\varphi(2),\varphi(1)\right)} = n_1\left(n_1-1\right)n_2^2 \ .$$

Similarly if $m = 3$ and $\varphi = (1|23, 1|2|3)$, then

$$(n)_\varphi = n_1\left(n_1-1\right)n_2^2\left(n_2-1\right) \ . \qquad \square$$

The preceding results enable a theory of functions which are symmetric under the generalised wreath product groups to be developed in a manner similar to that adopted by Doubilet [4] in his approach to the classical symmetric functions. These ideas, and their applications to statistics, will be reported elsewhere.

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