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Non-torsion algebraic cycles on the Jacobians of Fermat quotients

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Abstract. We study the Abel-Jacobi image of the Ceresa cycle $W_{k,e} - W_{k,e}^-$, where $W_{k,e}$ is the image of the k-th symmetric product of a curve X with a base point e on its Jacobian variety. For certain Fermat quotient curves of genus g, we prove that for any choice of the base point and $k \le g-2$, the Abel-Jacobi image of the Ceresa cycle is non-torsion. In particular, these cycles are non-torsion modulo rational equivalence.

1 Introduction

Let X be a smooth projective curve of genus g over $\mathbb{C}$ and $\mathrm{Jac}(X)$ be its Jacobian. Let $\mathrm{CH}_k(\mathrm{Jac}(X))_{\mathrm{hom}}$ be the Chow group of homologically trivial algebraic cycles of dimension k on $\mathrm{Jac}(X)$ modulo rational equivalence. To study this group, we consider the Abel-Jacobi map

$$\Phi_k\colon \mathrm{CH}_k(\mathrm{Jac}(X))_{\mathrm{hom}} \to J_k(\mathrm{Jac}(X)) \quad (k = 1, \ldots, g-1).$$

Here, $J_k(\mathrm{Jac}(X))$ is a complex torus, which is called the Griffiths intermediate Jacobian (see Section 3.1). It is well known that Φ_{g-1} is an isomorphism by the Abel-Jacobi theorem; however, for a general k, Φ_k is neither injective nor surjective. Fix a base point $e \in X$ and let ι_e be the embedding defined by

$$\iota_e\colon X \to \mathrm{Jac}(X); \quad x \mapsto [x] - [e].$$

Put $X_e = \iota_e(X)$. We denote X_e^- by the image of X_e under the inversion map. Since the inversion map acts trivially on the cohomology groups of even degree, we have

$$X_e - X_e^- \in \mathrm{CH}_1(\mathrm{Jac}(X))_{\mathrm{hom}}.$$

Let $W_{k,e}$ be the image of the k-th symmetric product of X on $\mathrm{Jac}(X)$. As in the case of $k = 1$, we have

$$W_{k,e} - W_{k,e}^- \in \mathrm{CH}_k(\mathrm{Jac}(X))_{\mathrm{hom}}.$$

These cycles are called the Ceresa cycles and for a generic curve X, Ceresa [4] proves that if $1 \le k \le g-2$, then $W_{k,e} - W_{k,e}^-$ is nontrivial modulo algebraic equivalence.

For a positive integer N and integers $a, b \in \{1, \ldots, N-1\}$, let $C_N^{a,b}$ be the smooth projective curve birational to the affine curve

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$$y^N = x^a(1-x)^b.$$

Let F_N be the Fermat curve of degree N. Then $C_N^{a,b}$ is a quotient of F_N by a cyclic group $G_N^{a,b}$ (see Section 2). Let g be the genus of $C_N^{a,b}$. The main theorem of this paper is as follows.

Theorem 1.1 *Suppose that N has a prime divisor $p > 7$ such that $p \nmid ab$ and $a^2 + ab + b^2 \equiv 0 \pmod{p}$. Then $\Phi_k(W_{k,e} - W_{k,e}^-) \in J_k(\mathrm{Jac}(C_N^{a,b}))$ is non-torsion for any choice of the base point $e \in C_N^{a,b}$ and $k = 1, \dots, g-2$.*

Remark 1.2 When N does not have a prime divisor $p > 7$, there exist some examples that the Abel-Jacobi image of the Ceresa cycle of $C_N^{a,b}$ is torsion. For example, $\Phi_1(X_e - X_e^-)$ is torsion for $X = C_9^{1,2}$, $C_{12}^{1,3}$, $C_{15}^{1,5}$ and $e = (0,0)$ ([1, §2 Theorem], [15, Theorem 3.2]).

The algebraical nontriviality of the Ceresa cycles of F_N ($N \le 1000$) and $C_p^{1,b}$ ($p \le 1000$ is a prime and $b^2 + b + 1 \equiv 0 \pmod{p}$) is proved by Harris [10], Bloch [2], Kimura [14], Tadokoro [18, 19, 20], and Otsubo [16]. Moreover, Otsubo [16] and Tadokoro [20] give a sufficient condition for the Ceresa cycles of these to be non-torsion modulo algebraic equivalence; however, it is impossible to confirm numerically these conditions. There are only two explicit examples of non-torsioness modulo algebraic equivalence for $k = 1$: F_4 by Bloch [2] and $C_7^{1,2}$ by Kimura [14]; they prove the non-torsioness of the l-adic Abel-Jacobi image.

Let N be a positive integer divisible by a prime $p > 7$. Eskandari-Murty [6, 7] prove that $\Phi_1(F_{N,e} - F_{N,e}^-)$ is non-torsion for any $e \in F_N$; in particular, $F_{N,e} - F_{N,e}^-$ is non-torsion modulo rational equivalence. Moreover, they conjecture that the same result holds for $C_p^{1,m}$ with $m \in \{1, \dots, p-2\}$ and $m \neq 1, (p-1)/2, p-2$ [7, Section 4, Remark (2)]. Theorem 1.1 partially but affirmatively answers their conjecture.

We briefly give a sketch of the proof. First, we reduce to the case $k = 1$ using a method of Otsubo [16] (see Proposition 3.1). The reduction to the case $N = p$ is easy. The rest of the proof is parallel to the method of Eskandari-Murty [6, 7]. First, the Abel-Jacobi image of the Ceresa cycle is described by an extension of mixed Hodge structures by Harris [10] and Pulte [17] (see Section 3.2). Second, we construct a 1-cycle Z on $C_p^{a,b} \times C_p^{a,b}$ and evaluate the extension of mixed Hodge structures at Z. Here, we use the assumptions on a and b so that an automorphism of F_p of order 3 descends to $C_p^{a,b}$. Then the extension class is expressed by a rational point $P_Z \in \mathrm{Jac}(C_N^{a,b})$ by formulas of Kaenders [13] and Darmon-Rotger-Sols [5] (see Sections 3.3, 3.4). Finally, since P_Z is non-torsion by a result of Gross-Rohrlich [8] (see Section 2), where we use the assumption $p > 7$, the theorem follows.

2 Fermat quotient curves

Let $N > 3$ be an integer, and for integers $a, b \in \{1, \dots, N-1\}$, let $C_N^{a,b}$ be the smooth projective curve birational to

$$y^N = x^a(1-x)^b.$$

The map

$$C_N^{a,b} \to \mathbb{P}^1; \quad (x,y) \mapsto x$$

is ramified at $x = 0, 1$ and ∞. Above 0 (resp. 1, ∞), there are $\gcd(N,a)$ (resp. $\gcd(N,b)$, $\gcd(N,a+b)$) branches and the ramification index is $N/\gcd(N,a)$ (resp. $N/\gcd(N,b)$, $N/\gcd(N,a+b)$). Therefore, by the Riemann-Hurwitz formula, the genus of $C_N^{a,b}$ is

$$\frac{1}{2}(N - (\gcd(N,a) + \gcd(N,b) + \gcd(N,a+b))) + 1.$$

We have an isomorphism

$$C_N^{a,b} \cong C_N^{b,a}$$

sending x to $1-x$. If two other integers $a', b' \in \{1, \dots, N-1\}$ satisfy the relation

$$(a', b') = (ha, hb) + (Ni, Nj)$$

for some integers h, i, j with $\gcd(N,h) = 1$, we have

$$C_N^{a,b} \cong C_N^{a',b'}; \quad (x,y) \mapsto (x, y^h x^i (1-x)^j).$$

Let F_N be the Fermat curve of degree N defined by

$$u^N + v^N = w^N.$$

Then there is a morphism

$$\pi_N^{a,b}: F_N \to C_N^{a,b}; \quad (u:v:w) \mapsto (x,y) = (u^N w^{-N}, u^a v^b w^{-a-b}).$$

Define a finite group by

$$G_N = \mathbb{Z}/N\mathbb{Z} \oplus \mathbb{Z}/N\mathbb{Z}$$

and denote an element $(r,s) \in G_N$ by $g_N^{r,s}$. Fix a primitive N-th root of unity ζ_N and let G_N act on F_N by

$$g_N^{r,s}(u:v:w) = (\zeta_N^r u : \zeta_N^s v : w).$$

Let $G_N^{a,b}$ be a subgroup of G_N defined by

$$G_N^{a,b} = \{g_N^{r,s} \in G_N \mid ar + bs = 0\}.$$

If $\gcd(N,a,b) = 1$, F_N is generically Galois over $C_N^{a,b}$ and

$$\mathrm{Gal}(F_N/C_N^{a,b}) = G_N^{a,b} = \langle g_N^{b,-a} \rangle \simeq \mathbb{Z}/N\mathbb{Z}.$$

There is an automorphism α of F_N of order 2 defined by

$$\alpha((u:v:w)) = (v:u:w).$$

When N is odd, there is an automorphism β of F_N of order 3 defined by

$$\beta((u:v:w)) = (-v:w:u).$$

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Lemma 2.1 (cf. [12, Section 3.1]) *Suppose that* $\gcd(N, a, b) = 1$*. Then,*

(i) α *descends to* $C_N^{a,b}$ *if and only if* $a^2 \equiv b^2 \pmod{N}$.

(ii) *Suppose that* N *is odd. Then* β *descends to* $C_N^{a,b}$ *if and only if* $a^2 + ab + b^2 \equiv 0 \pmod{N}$*. We denote this automorphism by* $\widetilde{\beta}$*.*

Proof We only prove (ii) since we use the morphism $\widetilde{\beta}$ to prove Theorem 1.1 and (i) is similarly proved. The automorphism β descends to $\widetilde{\beta}$ if and only if

$$\pi_N^{a,b}(\beta(g_N^{b,-a}(u : v : w))) = \pi_N^{a,b}(\beta(u : v : w));$$

that is, there exists an integer i such that

$$\left(-\zeta_N^{-a} v : w : \zeta_N^{b} u\right) = \left(-\zeta_N^{bi} v : \zeta_N^{-ai} w : u\right)$$

for all $(u : v : w) \in F_N$. This is equivalent to

$$a + b \equiv -bi \quad \text{and} \quad b \equiv ai \pmod{N}. \tag{2.1}$$

First, (2.1) implies $a^2 + ab + b^2 \equiv 0 \pmod{N}$. However, if $a^2 + ab + b^2 \equiv 0 \pmod{N}$, then we have $\gcd(N, a) = \gcd(N, b) = 1$ by the assumption $\gcd(N, a, b) = 1$. Therefore, there is an integer i such that $ai \equiv b \pmod{N}$, which satisfies (2.1). ■

Remark 2.2

(i) If N is a prime, the condition $a^2 + ab + b^2 \equiv 0 \pmod{N}$ implies that $N \equiv 1 \pmod{3}$.

(ii) When $N = a^2 + ab + b^2$, the curve $C_N^{a,b}$ is isomorphic to the Hurwitz curve ([12, Lemma 3.8]) which is the smooth projective curve birational to

$$X^b Y^{a+b} + Y^b Z^{a+b} + Z^b X^{a+b} = 0.$$

(iii) The condition $a^2 + ab + b^2 \equiv 0 \pmod{N}$ (for N prime) appears in Tadokoro [20]. He uses $\widetilde{\beta}$ to construct from a 1-form ω on $C_N^{a,b}$ two other 1-forms of the same Hodge type and evaluate the Abel-Jacobi image of the Ceresa cycle for $k = 1$ at $\omega \wedge \widetilde{\beta}^* \omega \wedge (\widetilde{\beta}^2)^* \omega$.

When $\gcd(N, 6) = 1$, the automorphism β of F_N has two fixed points

$$S = (\zeta_6 : \zeta_6^{-1} : 1), \quad \overline{S} = (\zeta_6^{-1} : \zeta_6 : 1),$$

and there is no other fixed point.

Lemma 2.3 *Suppose that* $\gcd(N, a, b) = \gcd(N, 6) = 1$ *and* $a^2 + ab + b^2 \equiv 0 \pmod{N}$*. Then the fixed points of the automorphism* $\widetilde{\beta}$ *of* $C_N^{a,b}$ *are* $\pi_N^{a,b}(S)$ *and* $\pi_N^{a,b}(\overline{S})$*, which are distinct.*

Proof We regard a, b as elements in $(\mathbb{Z}/N\mathbb{Z})^*$. Put $\gamma = g_N^{b,-a}$. Then we have

$$\beta\gamma = g_N^{-a-b,-b}\beta = \gamma^{a^{-1}b}\beta$$

since $-a-b = a^{-1}b^2$ by the assumption $a^2+ab+b^2=0$ in $\mathbb{Z}/N\mathbb{Z}$. For $P \in C_N^{a,b}$, suppose that $\widetilde{\beta}(P) = P$ and take any $Q \in F_N$ such that $\pi_N^{a,b}(Q) = P$. Then

$$\beta(Q) = \gamma^k Q$$

for some $k \in \mathbb{Z}/N\mathbb{Z}$. Since $(a-b)^2 = 3ab \in (\mathbb{Z}/N\mathbb{Z})^*$, we have $a-b \in (\mathbb{Z}/N\mathbb{Z})^*$. We take $i = a(a-b)^{-1}k$. Then we have

$$\beta(\gamma^i Q) = \gamma^{a^{-1}bi}\beta(Q) = \gamma^{a^{-1}bi+k}Q = \gamma^i Q,$$

which means that $\gamma^i Q = S$ or $\overline{S}$; hence, $P = \pi_N^{a,b}(S)$ or $\pi_N^{a,b}(\overline{S})$.

We are to show that $\pi_N^{a,b}(S) \neq \pi_N^{a,b}(\overline{S})$. Suppose that $\pi_N^{a,b}(S) = \pi_N^{a,b}(\overline{S})$; that is, there exists an integer i such that

$$\zeta_6 = \zeta_N^{bi}\zeta_6^{-1}, \quad \zeta_6^{-1} = \zeta_N^{-ai}\zeta_6.$$

Then we have $\zeta_6^{2N} = 1$, which contradicts the assumption $\gcd(N,6) = 1$. ∎

Put $P_0 = (0:1:1) \in F_N$ and let $F_N \to \mathrm{Jac}(F_N)$ be the map defined by $Q \mapsto [Q] - [P_0]$. Similarly, we define a map $C_N^{a,b} \to \mathrm{Jac}(C_N^{a,b})$ by sending Q' to $[Q'] - [\pi_N^{a,b}(P_0)]$. Then we have a commutative diagram

$$\begin{array}{ccc} F_N & \longrightarrow & \mathrm{Jac}(F_N) \\ {\scriptstyle \pi_N^{a,b}}\downarrow & & \downarrow{\scriptstyle (\pi_N^{a,b})_*} \\ C_N^{a,b} & \longrightarrow & \mathrm{Jac}(C_N^{a,b}). \end{array}$$

The following result of Gross and Rohrlich is one of the key ingredients to the proof of Theorem 1.1.

Theorem 2.4 [8, Theorem 2.1] *Let N be an integer such that* $\gcd(N,6) = 1$ *and N is divisible by a prime $p > 7$. If $a-b, a+2b, 2a+b \not\equiv 0 \pmod{p}$, then the point $(\pi_N^{a,b})_*([S] + [\overline{S}] - 2[P_0])$ on* $\mathrm{Jac}(C_N^{a,b})$ *is non-torsion.*

3 Algebraic cycles and Hodge theory of quadratic iterated integrals

3.1 Extension of mixed Hodge structures

Let $R = \mathbb{Z}$ or $\mathbb{Q}$. An R-mixed Hodge structure H is an R-module H_R of finite rank equipped with an increasing weight filtration $W_\bullet$ on $H_\mathbb{Q} := H_R \otimes_R \mathbb{Q}$ and a decreasing Hodge filtration $F^\bullet$ on $H_\mathbb{C} := H_R \otimes_R \mathbb{C}$ such that for each k, $\mathrm{Gr}_k^W(H_\mathbb{Q})$ with the induced filtration $F^\bullet$ is a pure $\mathbb{Q}$-Hodge structure of weight k. Let $R(n)$ be the Tate object of pure weight $-2n$ and put $H(n) = H \otimes_R R(n)$. Let $H^\vee$ be the dual R-mixed Hodge structure of H.

Let $\mathrm{MHS}(R)$ be the category of R-mixed Hodge structures. For R-mixed Hodge structures A, B, let $\mathrm{Ext}_{\mathrm{MHS}(R)}(A,B)$ denote the set of equivalence classes of extensions of R-mixed Hodge structures (i.e., exact sequences

$$0 \to B \to E \to A \to 0$$

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of R-mixed Hodge structures up to natural equivalence relation). There is a natural operation called the Baer sum which makes $\mathrm{Ext}_{\mathrm{MHS}(R)}(A,B)$ an abelian group. If X is a smooth projective variety over $\mathbb{C}$, the cohomology group $H^n(X,\mathbb{Z})$ underlies a pure $\mathbb{Z}$-Hodge structure of weight n, which we denote by $H^n(X)$.

For a pure $\mathbb{Z}$-Hodge structure H of weight -1, the intermediate Jacobian is defined by

$$JH = H_{\mathbb{C}}/(F^0 H_{\mathbb{C}} + H_{\mathbb{Z}}),$$

which is a complex torus. We have Carlson's isomorphism [3]

$$JH \cong \mathrm{Ext}_{\mathrm{MHS}(\mathbb{Z})}(H^{\vee}, \mathbb{Z}(0)).$$

For a smooth projective variety X over $\mathbb{C}$, $H_{2k+1}(X)(-k)$ is a pure $\mathbb{Z}$-Hodge structure of weight -1, and

$$J_k(X) := JH_{2k+1}(X)(-k) \cong (F^{k+1}H^{2k+1}(X,\mathbb{C}))^{\vee}/H_{2k+1}(X,\mathbb{Z})$$

is the k-th intermediate Jacobian of Griffiths. The Carlson isomorphism is written as

$$J_k(X) \cong \mathrm{Ext}_{\mathrm{MHS}(\mathbb{Z})}(H^{2k+1}(X)(k), \mathbb{Z}(0)).$$

Let $\mathrm{CH}_k(X)$ be the Chow group of k-dimensional algebraic cycles on X modulo rational equivalence, and $\mathrm{CH}_k(X)_{\mathrm{hom}}$ be the subgroup of homologically trivial cycles. Then we have the Abel-Jacobi map

$$\Phi_k \colon \mathrm{CH}_k(X)_{\mathrm{hom}} \to J_k(X); \quad Z \mapsto \left(\eta \mapsto \int_{\Gamma} \eta\right)$$

for any $\eta \in F^{k+1}H^{2k+1}(X,\mathbb{C})$, where Γ is a topological $(2k+1)$-chain such that $\partial\Gamma = Z$.

From now on, let X be a smooth projective curve of genus $g \geq 3$ over $\mathbb{C}$. Let

$$\langle\ \rangle \colon H^1(X) \otimes H^1(X) \to H^2(X) = \mathbb{Z}(-1)$$

be the cup product $\varphi \otimes \varphi' \mapsto \int_X \varphi \wedge \varphi'$. Choosing a base point $e \in X$, X is embedded into $\mathrm{Jac}(X)$ sending e to zero. It induces isomorphisms

$$H_1(X) \xrightarrow{\simeq} H_1(\mathrm{Jac}(X)), \quad H^1(\mathrm{Jac}(X)) \xrightarrow{\simeq} H^1(X),$$

which do not depend on the choice of e. We identify these and denote them by H_1 and H^1, respectively. Recall that the cup product induces an isomorphism

$$\wedge^n H^1 \xrightarrow{\simeq} H^n(\mathrm{Jac}(X)).$$

For $e \in X$, let $\iota_e \colon X \to \mathrm{Jac}(X)$ be the map defined by $P \mapsto [P] - [e]$. Let X^k (resp. $\mathrm{Jac}(X)^k$) be the k-fold product of X (resp. $\mathrm{Jac}(X)$) and $\mu \colon \mathrm{Jac}(X)^k \to \mathrm{Jac}(X)$ be the addition. We put

$$W_{k,e} = (\mu \circ (\iota_e)^k)(X^k) \quad (1 \leq k \leq g).$$

Then $W_{k,e}$ defines an algebraic k-cycle on $\mathrm{Jac}(X)$, and $W_{k,e} - W_{k,e}^{-}$ defines an element of $\mathrm{CH}_k(\mathrm{Jac}(X))_{\mathrm{hom}}$.

Proposition 3.1 *If $\Phi_1(X_e - X_e^-)$ is non-torsion, then $\Phi_k(W_{k,e} - W_{k,e}^-)$ is non-torsion for any $k = 2, \dots, g-2$.*

Proof Let $S = \{e_i, f_i \mid 1 \le i \le g\}$ be a symplectic basis of $H^1_{\mathbb{Z}}$ (i.e., $\langle e_i, e_j \rangle = \langle f_i, f_j \rangle = 0$, $\langle e_i, f_j \rangle = \delta_{ij}$). Under the identification

$$J_k(\mathrm{Jac}(X)) \cong \mathrm{Hom}(\wedge^{2k+1} H^1_{\mathbb{Z}}, \mathbb{R}/\mathbb{Z}),$$

if $\Phi_1(X_e - X_e^-)$ is non-torsion, there exists elements $\varphi_1, \varphi_2, \varphi_3 \in S$ such that

$$\Phi_1(X_e - X_e^-)(\varphi_1 \wedge \varphi_2 \wedge \varphi_3)$$

is non-torsion. By renumbering, we may assume that $\varphi_1, \varphi_2, \varphi_3 \in \{e_i, f_i \mid 1 \le i \le 3\}$. For $i = 1, \dots, k-1$, we put

$$\varphi_{2i+2} = e_{i+3}, \quad \varphi_{2i+3} = f_{i+3}.$$

Note that $i + 3 \le g$ by the assumption. Put $\varphi = \varphi_1 \wedge \cdots \wedge \varphi_{2k+1}$. Then, by [16, Proposition 3.7], we have

$$\begin{aligned} & k! \cdot \Phi_k(W_{k,e} - W_{k,e}^-)(\varphi) \\ &= k! \cdot \sum_{\sigma} \Phi_1(X_e - X_e^-)(\varphi_{\sigma(1)} \wedge \varphi_{\sigma(2)} \wedge \varphi_{\sigma(3)}) \prod_{i=1}^{k-1} \langle \varphi_{\sigma(2i+2)}, \varphi_{\sigma(2i+3)} \rangle \\ &= k! \cdot \Phi_1(X_e - X_e^-)(\varphi_1 \wedge \varphi_2 \wedge \varphi_3), \end{aligned}$$

where σ runs through the elements of the symmetric group S_{2k+1} such that $\sigma(1) < \sigma(2) < \sigma(3)$, $\sigma(2i+2) < \sigma(2i+3)$ for $1 \le i \le k-1$, and $\sigma(2i+2) < \sigma(2i+4)$ for $1 \le i \le k-2$. Therefore, $\Phi_k(W_{k,e} - W_{k,e}^-)$ is non-torsion. ■

Corollary 3.2 *Let N be an integer which has a prime divisor $p > 7$ and $X = F_N$ be the Fermat curve of degree N. Then $\Phi_k(W_{k,e} - W_{k,e}^-)$ is non-torsion for any $e \in F_N$ and $k = 1, \dots, g-2$.*

Proof By Proposition 3.1, we are reduced to the case $k = 1$, which is a theorem of Eskandari and Murty [6, Theorem 1.1]. ■

3.2 Harris-Pulte formula

In this subsection, we recall the Harris-Pulte formula, which is a relation between the Abel-Jacobi image of the Ceresa cycle and an extension class of mixed Hodge structures on the space of quadratic iterated integrals on the curve X.

We put

$$(H^1 \otimes H^1)' = \mathrm{Ker}(\cup: H^1 \otimes H^1 \to H^2(\mathrm{Jac}(X))).$$

Then the map

$$\phi: H^1 \otimes (H^1 \otimes H^1)' \to \wedge^3 H^1,$$

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which is obtained by restricting the natural quotient map $(H^1)^{\otimes 3} \to \wedge^3 H^1$, is surjective ([17, Lemma 4.7]), and induces the injective map

$$\phi^*: \mathrm{Ext}_{\mathrm{MHS}(\mathbb{Z})}(\wedge^3 H^1, \mathbb{Z}(-1)) \to \mathrm{Ext}_{\mathrm{MHS}(\mathbb{Z})}(H^1 \otimes (H^1 \otimes H^1)', \mathbb{Z}(-1)).$$

Let $\pi_1(X, e)$ be the fundamental group. Let I be the augmentation ideal of the group ring $\mathbb{Z}[\pi_1(X, e)]$ – that is, the kernel of the degree map

$$\mathbb{Z}[\pi_1(X, e)] \to \mathbb{Z}; \quad \sum n_i \gamma_i \mapsto \sum n_i.$$

By Chen's π_1-de Rham theorem, $\mathrm{Hom}(\mathbb{Z}[\pi_1(X, e)]/I^{s+1}, \mathbb{R})$ is generated by closed iterated integrals of length $\leq s$. Using this, Hain [9] defines a $\mathbb{Z}$-mixed Hodge structure on $\mathbb{Z}[\pi_1(X, e)]/I^s$ such that the natural map $\mathbb{Z}[\pi_1(X, e)]/I^s \to \mathbb{Z}[\pi_1(X, e)]/I^t$ for $s \geq t$ is a morphism of mixed Hodge structures. Consider the exact sequence of mixed Hodge structures

$$0 \to I^2/I^3 \to I/I^3 \to I/I^2 \to 0. \tag{3.1}$$

The map $\pi_1(X, e) \to I/I^2; \gamma \mapsto \gamma - 1$ is well-defined and induces an isomorphism

$$H_1(X, \mathbb{Z}) \xrightarrow{\simeq} I/I^2$$

of Hodge structures of weight -1. However, the multiplication $I/I^2 \otimes I/I^2 \to I^2/I^3$ induces an isomorphism

$$\mathrm{Hom}(I^2/I^3, \mathbb{Z}) \xrightarrow{\simeq} (H^1 \otimes H^1)'$$

of Hodge structures of weight 2. Taking the dual of (3.1), we have an exact sequence

$$0 \to H^1 \to L_2(X, e) \to (H^1 \otimes H^1)' \to 0,$$

where we put $L_2(X, e) = \mathrm{Hom}(I/I^3, \mathbb{Z})$.

Let $\infty \neq e$ be another point on X. Put $U = X - \{\infty\}$. We identify $H^1(U)$ and H^1 via the map induced by the inclusion $U \subset X$. Then we can obtain an exact sequence of mixed Hodge structures

$$0 \to H^1 \to L_2(U, e) \to H^1 \otimes H^1 \to 0$$

similarly as above. We have a commutative diagram

$$\begin{array}{ccccccccc} 0 & \longrightarrow & H^1 & \longrightarrow & L_2(X, e) & \longrightarrow & (H^1 \otimes H^1)' & \longrightarrow & 0 \\ & & \| & & \cap & & \cap & & \\ 0 & \longrightarrow & H^1 & \longrightarrow & L_2(U, e) & \longrightarrow & H^1 \otimes H^1 & \longrightarrow & 0. \end{array}$$

Let $\mathbb{E}_e$ (resp. $\mathbb{E}_e^\infty$) be an extension class of the top (resp. bottom) row. We regard $\mathbb{E}_e$ as an element of

$$\begin{aligned} \mathrm{Ext}_{\mathrm{MHS}(\mathbb{Z})}((H^1 \otimes H^1)', H^1) &\cong \mathrm{Ext}_{\mathrm{MHS}(\mathbb{Z})}((H^1)^\vee \otimes (H^1 \otimes H^1)', \mathbb{Z}(0)) \\ &\cong \mathrm{Ext}_{\mathrm{MHS}(\mathbb{Z})}(H^1 \otimes (H^1 \otimes H^1)', \mathbb{Z}(-1)), \end{aligned}$$

and $\mathbb{E}_e^\infty$ as an element of

$$\begin{aligned}\mathrm{Ext}_{\mathrm{MHS}(\mathbb{Z})}(H^1\otimes H^1, H^1) &\cong \mathrm{Ext}_{\mathrm{MHS}(\mathbb{Z})}((H^1)^\vee\otimes H^1\otimes H^1, \mathbb{Z}(0))\\ &\cong \mathrm{Ext}_{\mathrm{MHS}(\mathbb{Z})}(H^1\otimes H^1\otimes H^1, \mathbb{Z}(-1)).\end{aligned}$$

Here, we used the Poincaré duality $H^1(1)\cong (H^1)^\vee$. One sees that $\mathbb{E}_e$ is the restriction of $\mathbb{E}_e^\infty$ to $H^1\otimes(H^1\otimes H^1)'$. Then Harris's formula [10, Section 4], reworked by Pulte [17, Theorem 4.10], is

$$\phi^*\circ\Phi_1(X_e - X_e^-) = 2\mathbb{E}_e$$

under the identification $J_1(\mathrm{Jac}(X)) = \mathrm{Ext}_{\mathrm{MHS}(\mathbb{Z})}(\wedge^3 H^1, \mathbb{Z}(-1))$.

3.3 The decomposition of $(H^1)^{\otimes 3}$

In this subsection, for a $\mathbb{Z}$-mixed Hodge structure H, we consider the image of H under the forgetful functor $\mathrm{MHS}(\mathbb{Z})\to\mathrm{MHS}(\mathbb{Q})$, which we denote by the same letter. The Hodge structure $(H^1)^{\otimes 3}$ can be decomposed in $\mathrm{MHS}(\mathbb{Q})$ as follows. Let $\xi_\Delta\in H^1\otimes H^1$ be the Künneth component of the Hodge class of the diagonal of X in $H^2(X\times X)$. Then we have a decomposition

$$H^1\otimes H^1\otimes H^1 = (H^1\otimes\langle\xi_\Delta\rangle)\oplus(H^1\otimes(H^1\otimes H^1)').$$

Since the Mumford-Tate group of H^1 is reductive, the map ϕ admits a section σ in $\mathrm{MHS}(\mathbb{Q})$, and we have

$$H^1\otimes(H^1\otimes H^1)' = \ker(\phi)\oplus\sigma(\wedge^3 H^1).$$

Let $\overline{\xi}_\Delta$ be the image of ξ_Δ in $\wedge^2 H^1$. Then we have a decomposition in $\mathrm{MHS}(\mathbb{Q})$

$$H^1\otimes H^1\otimes H^1 = (H^1\otimes\langle\xi_\Delta\rangle)\oplus\ker(\phi)\oplus\sigma(H^1\wedge\langle\overline{\xi}_\Delta\rangle)\oplus\sigma((\wedge^3 H^1)_{\mathrm{prim}}),$$

where the last summand (primitive part) is the kernel of the map $\wedge^3 H^1\to\wedge^{2g-1}H^1$ given by wedging by $\overline{\xi}_\Delta^{g-2}$ (cf. [7, Section 4.2]). We put $\mathbb{E} := \mathbb{E}_e|_{\sigma((\wedge^3 H^1)_{\mathrm{prim}})}$. Then $\mathbb{E}$ is independent of the choice of e ([17, Theorem 3.9] and [10]).

Proposition 3.3

(i) *Suppose that* $-2g[\infty]+2[e]+K=0$. *Then* $\mathbb{E}_e^\infty=0$ *if and only if* $\mathbb{E}_e=0$.
(ii) *Suppose that* $(2g-2)[e]-K=0$. *Then* $\mathbb{E}_e=0$ *if and only if* $\mathbb{E}=0$.

Proof (i) The statement follows from that $\mathbb{E}_e^\infty|_{H^1\otimes(H^1\otimes H^1)'}=\mathbb{E}_e$ and a result of Kaenders [13, Theorem 1.2] that

$$\mathbb{E}_e^\infty|_{H^1\otimes\langle\xi_\Delta\rangle} = -2g[\infty]+2[e]+K$$

under the identification (cf. [7, Section 4.3.1])

$$\mathrm{Ext}_{\mathrm{MHS}(\mathbb{Q})}(H^1\otimes\langle\xi_\Delta\rangle, \mathbb{Q}(-1))\cong \mathrm{CH}_0(X)_{\mathrm{hom}}\otimes\mathbb{Q}.$$

(ii) The statement follows from that results of Harris [10, Section 3] and Pulte [17, Theorem 4.10]] that $\mathbb{E}_e|_{\ker(\phi)}\in\mathrm{Ext}_{\mathrm{MHS}(\mathbb{Q})}(\ker(\phi),\mathbb{Q}(-1))$ is zero, and Pulte [17, Corollary 6.7] that

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$$\mathbb{E}_e|_{\sigma(H^1\wedge\langle\overline{\xi}_\Delta\rangle)} = (2g-2)[e] - K$$

under the identification (cf. [7, Section 4.3.3])

$$\mathrm{Ext}_{\mathrm{MHS}(\mathbb{Q})}(\sigma(H^1\wedge\langle\overline{\xi}_\Delta\rangle), \mathbb{Q}(-1)) \cong \mathrm{CH}_0(X)_{\mathrm{hom}}\otimes\mathbb{Q}. \qquad \blacksquare$$

3.4 Darmon-Rotger-Sols formula

Let $\Delta \in \mathrm{CH}_1(X\times X)$ be the diagonal of X and

$$p_i\colon X\times X \to X \quad (i=1,2)$$

be the projection to the i-th component. For $Z \in \mathrm{CH}_1(X\times X)$, put

$$Z_{12} = (p_1)_*(Z\cdot\Delta) = (p_2)_*(Z\cdot\Delta),$$
$$Z_1 = (p_1)_*(Z\cdot(X\times\{e\})), \quad Z_2 = (p_2)_*(Z\cdot(\{e\}\times X)) \in \mathrm{CH}_0(X).$$

Put

$$P_Z = Z_{12} - Z_1 - Z_2 - (\deg(Z_{12}) - \deg(Z_1) - \deg(Z_2))[e] \in \mathrm{Jac}(X).$$

Then the point P_Z is related to the extension $\mathbb{E}_e^\infty$ as follows. Let ξ_Z be the $H^1\otimes H^1$-Künneth component of the class of Z in $H^2(X\times X)$. Consider the map

$$\xi_Z^{-1}\colon \mathrm{Ext}_{\mathrm{MHS}(\mathbb{Z})}((H^1)^{\otimes 3}, \mathbb{Z}(-1)) \to \mathrm{Ext}_{\mathrm{MHS}(\mathbb{Z})}(H^1(-1), \mathbb{Z}(-1)) \cong J_0(X) = \mathrm{Jac}(X),$$

where the first arrow is the pullback along the morphism $H^1(-1)\to(H^1)^{\otimes 3}$ defined by $\omega\mapsto\omega\otimes\xi_Z$. Then we have the following.

Proposition 3.4 [5, Corollary 2.6] *For any* $Z\in\mathrm{CH}_1(X\times X)$, *we have*

$$\xi_Z^{-1}(\mathbb{E}_e^\infty) = \left(\int_\Delta \xi_Z\right)([\infty]-[e]) - P_Z$$

in $\mathrm{Jac}(X)$.

4 Proof of Theorem 1.1

There are $3N$ points on F_N

$$P_i = (0:\zeta_N^i:1), \quad Q_i = (\zeta_N^i:0:1), \quad R_i = (\xi_N\zeta_N^i:1:0), \quad (i\in\mathbb{Z}/N\mathbb{Z}),$$

where we put $\xi_N = \exp(\pi i/N)$. Fix P_0 as the base point; then the above points are torsion points in $\mathrm{Jac}(F_N)$ [8]. Therefore, for the base point $\pi_N^{a,b}(P_0)$, the images of these points under $(\pi_N^{a,b})_*$ are also torsion in $\mathrm{Jac}(C_N^{a,b})$. We shall continue to use the notation as in the previous section, specializing $X = C_N^{a,b}$, $e=\pi_N^{a,b}(P_0)$ and $\infty = \pi_N^{a,b}(Q_0)$.

Lemma 4.1 *Let* K_C *(resp.* g*) be the canonical divisor (resp. genus) of* $C_N^{a,b}$*. Then* $K_C-(2g-2)[e]$, $K_C - 2g[\infty]+2[e] \in \mathrm{Jac}(C_N^{a,b})$ *are torsion points.*

Proof Since

$$K_C - 2g[\infty] + 2[e] = K_C - (2g-2)[e] - 2g([\infty] - [e])$$

and $[\infty] - [e]$ is a torsion point, it suffices to show that $K_C - (2g-2)[e]$ is a torsion point. Let K_F be the canonical divisor of F_N and $R_{\pi_N^{a,b}}$ be the ramification divisor of $\pi_N^{a,b}$; that is,

$$K_F = (N-1)\sum_{i=0}^{N-1} Q_i - 2\sum_{i=0}^{N-1} R_i,$$
$$R_{\pi_N^{a,b}} = (\gcd(N,a)-1)\sum_{i=0}^{N-1} P_i + (\gcd(N,b)-1)\sum_{i=0}^{N-1} Q_i + (\gcd(N,a+b)-1)\sum_{i=0}^{N-1} R_i.$$

Then we have

$$K_F = (\pi_N^{a,b})^*(K_C) + R_{\pi_N^{a,b}}$$

up to principal divisor (cf. [11, Proposition 2.3, Chap.IV]). Therefore, we have

$$N(K_C - (2g-2)[e]) = (\pi_N^{a,b})_*\left(K_F - R_{\pi_N^{a,b}} - (2g-2)N[P_0]\right)$$

in $\mathrm{Jac}(C_N^{a,b})$. Since P_i, Q_i, and R_i are torsion in $\mathrm{Jac}(F_N)$, $K_F - R_{\pi_N^{a,b}} - (2g-2)N[P_0]$ is torsion, which finishes the proof. ■

Proof of Theorem 1.1 First, by Proposition 3.1, it suffices to show the case when $k = 1$. Secondly, consider the map

$$f\colon F_N \to F_p; \quad (x_0 : y_0 : z_0) \mapsto (x_0^{N/p} : y_0^{N/p} : z_0^{N/p}).$$

Let $\langle a\rangle \in \{0,\ldots,p-1\}$ be the representative of a. Then f descends to a map $\overline{f}\colon C_N^{a,b} \to C_p^{\langle a\rangle,\langle b\rangle}$. Since

$$f_*(\Phi_1(C_{N,e}^{a,b} - (C_{N,e}^{a,b})^-)) = \deg\overline{f}\cdot\Phi_1\left(C_{p,\overline{f}(e)}^{a,b} - (C_{p,\overline{f}(e)}^{a,b})^-\right),$$

we are reduced to the case when $N = p$.

By Lemma 4.1 and Proposition 3.3, it suffices to show that, for the specific choices of e and ∞ as above, the element $\mathbb{E}_e^\infty \in \mathrm{Ext}_{\mathrm{MHS}(\mathbb{Q})}(H^1\otimes H^1\otimes H^1, \mathbb{Q}(-1))$ is nonzero. By Lemma 2.1, the automorphism β of F_p descends to an automorphism $\widetilde{\beta}$ of $C_p^{a,b}$; let Z be the graph of $\widetilde{\beta}$. Since $[\infty] - [e]$ is torsion, it suffices to show that P_Z is non-torsion by Proposition 3.4. Since $\widetilde{\beta}\circ\pi_p^{a,b} = \pi_p^{a,b}\circ\beta$ and $\widetilde{\beta}$ has two fixed points by Lemma 2.3, we have

$$\begin{aligned}P_Z &= ([\pi_p^{a,b}(S)] + [\pi_p^{a,b}(\overline{S})] - 2[e]) - ([\widetilde{\beta}(e)] + [\widetilde{\beta}^{-1}(e)] - 2[e])\\ &= (\pi_p^{a,b})_*(([S] + [\overline{S}] - 2[P_0]) - ([\beta(P_0)] + [\beta^{-1}(P_0)] - 2[P_0])).\end{aligned}$$

The point $[\beta(P_0)] + [\beta^{-1}(P_0)] - 2[P_0]$ is a torsion point on $\mathrm{Jac}(F_p)$; hence, $(\pi_p^{a,b})_*([\beta(P_0)] + [\beta^{-1}(P_0)] - 2[P_0])$ is a torsion point on $\mathrm{Jac}(C_p^{a,b})$. However, since

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$a - b, a + 2b, 2a + b \not\equiv 0 \pmod{p}$ by the assumption $a^2 + ab + b^2 \equiv 0 \pmod{p}$, the point

$$(\pi_p^{a,b})_*([S] + [\overline{S}] - 2[P_0]) \in \mathrm{Jac}(C_p^{a,b})$$

is non-torsion by Theorem 2.4. Therefore, the point P_Z is non-torsion, which finishes the proof. ■

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