PSEUDO-REGULARITY

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Introduction. An element x is said to be *right-quasi-regular* (r.q.r.) if there exists an element y such that $x + y + xy = 0$. This concept had its inception in the fact that (for rings with unity) if $1 + x$ has an inverse, written as $1 + y$, then $(1 + x)(1 + y) = 1, x + y + xy = 0$. Thus in rings without unity elements it seemed **(1**; **3**; **12)** profitable to consider this latter equation. Jacobson **(9)** was able to employ this concept in obtaining a structure theory for rings without chain conditions.

Our point of departure is in considering the expression $x + y + xy$ not as stemming from $(1 + x)(1 + y)$, but as a special case of the more general expression $x + x^n y + x^{n+1}y$. Our considerations seem to bear most fruit in the case $n = 1$ for commutative rings. We call an element x *right-pseudo-regular* (r.p.r.) if there exists an element y such that $x + xy + x^2y = 0$.

In §1 we show the existence of a maximal r.p.r. ideal R, called the *subradical*; and show that with some mild restrictions on the ring, it is simply the Jacobson radical J, thus obtaining a new representation of J. In general however, $R \leqslant J$. We also obtain some radical-like properties of R, as well as a definite relationship between R and J.

In §2 we use the techniques of Brown and McCoy and in the commutative case are able to show that $A - R$ is isomorphic to a subdirect sum of subdirectly irreducible rings, some of which are simple with unity (fields) and others are bound to their maximal nil ideal in the sense of Hall **(8)**.

1. An element x of a ring A shall be called *right-pseudo-regular* (r.p.r.) of degree n, if there exists an element y of A such that

$$x + x^n y + x^{n+1}y = 0.$$

It is clear that for $n = 0$ we get the familiar right-quasi-regularity. We shall be primarily interested in the case $n = 1$, and refer to it simply as r.p.r. It is also clear that if x is r.p.r. of degree n, then x is r.p.r. of degree $n - 1$

$$x + x^{n-1} . xy + x^n . xy = 0;$$

and so in particular, if x is r.p.r. it is r.q.r. (*right-quasi-regular*). The converse of this last statement is not true, since in the ring of even integers modulo 4, the element 2 is r.q.r., since

$$2 + 2 + 2.2 \equiv 0 (\text{mod } 4),$$

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but it is not r.p.r. The exact relationship between right-pseudo-regularity and right-quasi-regularity is obtained in

LEMMA 1. *An element x of a ring A is* r.p.r. *of degree n, $x + x^n y + x^{n+1} y = 0$, if and only if x is* r.q.r. *and there exists an element x' such that $x^n x' = x$.*

Proof. If x is r.p.r. of degree n, then clearly x is r.q.r. and $x^n x' = x$ with $x' = -y - xy$. Conversely if x is r.q.r., $x + z + xz = 0$, and if there exists an x' such that $x^n x' = x$, then setting $y = -x' - x'z$ we find that

$$\begin{aligned} x + x^n y + x^{n+1} y &= x + x^n(-x' - x'z) + x^{n+1}(-x' - x'z) \\ &= x - x - xz - x^2 - x^2 z \\ &= -x(z + x + xz) = 0. \end{aligned}$$

COROLLARY 1. *An element x of a ring A is* r.p.r. *if and only if x is* r.q.r. *and there exists an element x' in A such that $xx' = x$.*

COROLLARY 2. *If x is in xA for every x of A, then right-quasi-regularity and right-pseudo-regularity are equivalent concepts.*

A more unexpected result is

LEMMA 2. *Right-pseudo-regularity of degree* 2 *and right-pseudo-regularity of degree n for all $n > 1$, are equivalent concepts.*

Proof. Clearly, if x is r.p.r. of degree n, $n > 1$, it is r.p.r. of degree 2. Conversely, if x is r.p.r. of degree 2, then by Lemma 1, x is r.q.r. and there exists an x' such that $x^2 x' = x$. Notice that this is precisely strong regularity. Then

$$x^n \,.\, x'^{n-1} = x^{n-2} \,.\, x^2 x' \,.\, x'^{n-2} = x^{n-1} \,.\, x'^{n-2} = \ldots = x^2 x' = x.$$

Thus there is an element $w = x'^{n-1}$ such that $x^n w = x$. Therefore by Lemma 1, x is r.p.r. of degree n.

A right ideal Q will be called r.p.r. of degree n if all of its elements are r.p.r. of degree n. To consider the existence of maximal r.p.r. of degree n right ideals we shall make use of the following

LEMMA 3. *If $x \neq 0$, is* r.p.r., *$x + xy + x^2 y = 0$, then its right-pseudo-inverse* (r.p.i.) *y is not in the Jacobson radical J.*

Proof. If y is in J, then $y + xy$ is in J and there exists an element z such that

$$y + xy + z + (y + xy)z = 0.$$

We have

$$\begin{aligned} 0 &= x + xy + x^2 y + (x + xy + x^2 y)z \\ &= x + x(y + xy + z + yz + xyz) \\ &= x + x \,.\, 0 = x. \end{aligned}$$

This contradicts the fact that $x \neq 0$ and therefore y is not in J.

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If we can show the existence of a maximal right ideal R_n, which is r.p.r. of degree n, for every n, then by Lemma 2, $R_2 = R_3 = \ldots = R_n$. This is in fact true but they are all equal to zero!

THEOREM 1. *If, for any $n > 1$, Q is a right ideal all of whose elements are* r.p.r. *of degree n, then $Q = 0$.*

Proof. Let Q be r.p.r. of degree 2. If x is in Q then $x + x^2y + x^3y = 0$. Then x is r.p.r. with r.p.i. xy. But xy is in Q; and since Q is a right ideal all of whose elements are r.q.r., Q is in J, and xy is in J. This contradicts Lemma 3 unless $x = 0, Q = 0$.

Though Theorem 1 proves that there are no right ideals all of whose elements are r.p.r. of degree n, $n > 1$, there may be many elements which are r.p.r. of degree n. Let A be a division ring. Then every element $\neq -1$ is r.q.r. Furthermore $x^2x^{-1} = x$ and thus by Lemma 1, every element $\neq -1$ is r.p.r. of degree 2 and thus by Lemma 2, every element $\neq -1$ is r.p.r. of degree n for every n.

We shall now show the existence of a maximal r.p.r. ideal, which is not always zero. The first step is to show that the sum of two r.p.r. right ideals is again an r.p.r. right ideal. To this end we prove a slightly more general result akin to Kaplansky's (**10**, Lemma 1).

LEMMA 4. *If x is* r.p.r. *and if a belongs to an* r.p.r. *right ideal Q, then $x + a$ is* r.p.r.

Proof. By Lemma 1, Corollary 1, it is sufficient to show that $x + a$ is r.q.r. and that there exists an element v such that $(x + a)v = x + a$. The fact that $x + a$ is r.q.r. follows immediately from Kaplansky's lemma, since x being r.p.r. is also r.q.r. and Q being an r.p.r. right ideal is an r.q.r. right ideal.

To find the element v, we first define $u = a - ax'$ where $xx' = x$. Since a is in Q, u is in Q, and there exists an element u' such that $uu' = u$. Define $v = x' + u' - x'u'$. Then $(x + a)v = x + a$ follows from $xv = x$ and

$$av = ax' + (a - ax')u' = ax' + uu' = ax' + u = a.$$

We now define R to be the join of all the r.p.r. right ideals of the ring A. By Lemma 4, R is an r.p.r. right ideal. It is clear that R is the set of all elements that generate r.p.r. right ideals, i.e., all x such that $xi + xa$ is r.p.r. for every integer i and every element a of A. We now show that R is a two-sided ideal.

Let x be any element in R and a be any element in A. We must show that ax is in R, i.e., that $axi + axb$ is r.p.r. for every integer i and every b of A. Since x is in R, $xi + xb$ is in R. Let $y = xi + xb$. Then it is sufficient to show that ay is r.p.r. for any y in R. Since R is a right ideal, ya is in R and therefore there exists a z such that $ya + yaz + (ya)^2z = 0$. Then

$$\begin{aligned} ay &+ (-ay - ayazy) + ay(-ay - ayazy) \\ &= ay - ay - a(yaz + ya + yayaz)y = 0 - a \,.\, 0 \,.\, y = 0. \end{aligned}$$

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Therefore ay is r.q.r. Furthermore, since y is in R, y is r.p.r., there exists a y' such that $yy' = y$. Therefore $ay \,.\, y' = ay$. Therefore, by Lemma 1, Corollary 1, ay is r.p.r. We have proved

THEOREM 2. *If A is an arbitrary ring, the join R of all the* r.p.r. *right ideals of A is an* r.p.r. *two-sided ideal.*

We shall call R the right subradical of A. Most of the time $R < J$, but using Corollary 2, Lemma 1 we have

THEOREM 3. *If x is in xA for every x of A, or of J, then $J = R$.*

By considering left-pseudo-regularity we could, by exactly the same techniques, prove the existence of a maximal l.p.r. two-sided ideal L, which we call the left subradical of A. We would find that if A had a left unity, or if x was in Ax for every x of J, then $J = L$. It should be clear that though J enjoys certain left-right symmetric properties, the ideals R and L have no such well-roundedness. Of course if A has a unity element, then $J = R = L$; however in the general case, R and L are different. To see this, consider the set A of all two by two matrices of the form

$$\begin{pmatrix} a & 0 \\ b & 0 \end{pmatrix}$$

where a and b are integers mod 4. Then A contains 16 elements. The Jacobson radical J has 8 elements, namely those with $a = 0$ or 2, and $b = 0$, 1, 2 or 3. Furthermore, A has a right unity,

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix},$$

(in fact A has four different right unity elements) and therefore by Theorem 3, $J = R$. However A does not have a left unity and one can easily see that L has only the one element

$$\begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

Though R is occasionally equal to J, it is often equal to 0. By Lemma 3, it is clear that if A is a radical ring, $A = J$, then $R = 0$. And also therefore, there is no nonzero ring which is equal to its right subradical.

The right subradical R has the usual radical-like properties.

THEOREM 4. *The difference ring $A - R$ is sub-semi-simple, that is, it has zero subradical.*

Proof. Let $\bar{R}$ be the subradical of $\bar{A} = A - R$. If $\bar{x}$ is in $\bar{R}$, then there exists an element $\bar{y}$ in $\bar{A}$ such that $\bar{x} + \bar{x}\bar{y} + \bar{x}^2\bar{y} = \bar{0}$, that is, $x + xy + x^2y$ is in R. Then there exists an element z in A such that

$$x + xy + x^2y + (x + xy + x^2y)z + (x + xy + x^2y)^2z = 0.$$

Rewriting this we have

$$x + (x + x^2)(y + z + yz + yxz + xyz + yxyz + yx^2yz) = 0.$$

Therefore x is r.p.r. Furthermore, since $\bar{x}$ is in $\bar{R}$, $\bar{x}i + \bar{x}\bar{a}$ is in $\bar{R}$ for every integer i and every $\bar{a}$ in $\bar{A}$. As for x, we can show that $xi + xa$ is r.p.r. for every i and a, and therefore x is in R, $\bar{x} = \bar{0}$, $\bar{R} = \bar{0}$.

Jacobson has shown **(9)** that $J_n = J(A_n)$, where A_n is the set of all n by n matrices with elements in A, J_n is the set of all n by n matrices with elements in J, and $J(A_n)$ is the Jacobson radical of A_n. The corresponding result for subradicals is true and the proof is straightforward.

THEOREM 5. *The subradical $R(A_n)$ is equal to R_n.*

LEMMA 5. *The subradical $R = RA^n$ for every integer n.*

Proof. Since R is an ideal, $RA^n \leqslant R$. Conversely, if x is in R, there exists an element x' such that $x = xx'$, x is in RA. Therefore $R \leqslant RA$, $R \leqslant RA^n$, $R = RA^n$.

LEMMA 6. *The subradical $R \leqslant M_l$, the intersection of all the maximal left ideals.*

Proof. Jacobson has shown **(9)** that $J . A \leqslant M_l$. Since $R \leqslant J . A$, $R \leqslant M_l$.

We shall now obtain a more definite relationship between J and R.

LEMMA 7. *Let A be a non-nilpotent ring with the descending chain condition on right ideals, having all its idempotents in the centre. In particular A may be any commutative ring with* d.c.c. *Then if $A^n = A^{n+1}$, A^n has a unity element. In particular if $A = A^2$, A has a unity element.*

Proof. By d.c.c. **(2)** there exists an idempotent e such that

$$A = Ae + B$$

where B is the set of all $x - xe$ for x in A, and B is nilpotent. Furthermore $Ae . B = B . Ae = 0$ since e is assumed to be in the centre. Therefore $A^2 = (Ae)^2 + B^2$,

$$A^m = (Ae)^m + B^m.$$

Since $A > A^2 > \ldots > A^m > \ldots$ is a descending chain of right ideals, there exists an integer n such that $A^n = A^{n+1}$. It is clear that $B^n = B^{n+1}$, since if x is in B^n then $xe = 0$. And since $x = a + b$, with a in $(Ae)^{n+1}$ and b in B^{n+1}, we have $xe = 0 = ae + be = a + 0$. Therefore $a = 0$ and $x = b$ in B^{n+1}. Therefore $B^n \leqslant B^{n+1} \leqslant B^n$. Since B is a nilpotent ideal, $B^n = 0$, $A^n = (Ae)^n$. But $(Ae)^n = A^ne^n = A^ne$. Then $A^n = A^ne$, e is a unity element for A^n. It is clear that $e = e^n$ is in A^n.

LEMMA 8. *If B is an ideal of A, then the radical of the ring B, $J(B)$, is equal to $J \cap B$, where J is the Jacobson radical of A.*

This result is due to Perlis **(13)**.

LEMMA 9. *If B is an ideal of A, then $R(B) \leqslant R \cap B$, where R is the subradical of A.*

The proof uses Lemma 8 and is straightforward. Note that it is impossible to prove that $R(B) = R \cap B$, since if we take $B = J$, $R(B) = 0$, whereas $R \cap B = R \cap J = R$.

THEOREM 6. *If A is a ring with* d.c.c. *on right ideals, having all its idempotents in the centre, then $R = JA^{n-1}$, where n is the smallest integer such that $A^n = A^{n+1}$. When $n = 1$, $R = J$.*

Proof. If A is nilpotent, $A^n = 0$, $R = JA^{n-1} = 0$ by the remark just before Theorem 4. If A is not nilpotent, by d.c.c. there exists a least integer n such that $A^n = A^{n+1} \neq 0$. Then by Lemma 7, A^n has a unity element and by Theorem 3, $R(A^n) = J(A^n)$. By Lemma 9, $R \geqslant R(A^n)$. By Lemma 8, $J(A^n) = J \cap A^n$. Therefore

$$R \geqslant R(A^n) = J(A^n) = J \cap A^n \geqslant JA^{n-1}.$$

Conversely, by Lemma 5, $R = RA^{n-1}$. Thus $R \leqslant JA^{n-1}$. Therefore $R = JA^{n-1}$.

By similar techniques we can show that the left subradical L is contained in M_r, the intersection of all the maximal right ideals, and that $L = A^{n-1} J$.

Discussion of Theorem 6. Theorem 6 is not true without d.c.c., as the following example, mentioned to the author in a discussion with Professor Zassenhaus, proves. Let x_α be a basis for a commutative algebra, where the α's are real, $0 \lneq \alpha \lneq 1$. Define $x_\alpha x_\beta = x_{\alpha+\beta}$ if $\alpha + \beta < 1$, and equal to 0 if $\alpha + \beta \geqslant 1$. Then it is clear that every element is nilpotent. Thus $A = J$, and $R = 0$. However $x_\alpha = x_{\frac{1}{2}\alpha} x_{\frac{1}{2}\alpha}$, and therefore $A = A^2$. To be sure, A is nil, but not nilpotent.

Whether the theorem is true if A has d.c.c., but not the restriction that the idempotents lie in the centre, seems to be an open question. Since every ring with d.c.c. can be expressed **(7)** as $A = M + M^*$ where M is the maximal regular ideal, and M^* is bound to its radical in the sense of Hall **(8)**, and $MM^* = M^*M = 0$, the condition $A = A^2$ implies $M^* = M^{*2}$. Thus the first step seems to be to decide whether there exists a ring, say B, with d.c.c., bound to its radical, without a right or left unity element and such that $B = B^2$.

Another attack on this question can be made using a technique due to Baer **(4)**. Since we can write $A = Ae + B$, where B is the set of all $x - xe$ with x in A, and with B in J, we embed Ae in a maximal left ideal F (we assume if necessary a.c.c.). Then $A = (F, J)$. Since J is contained in every maximal *modular*[1] left ideal, if F were modular, $A = F$, a contradiction. Thus either Ae is already A, in which case A has a right unity, $R = J$, and we are well away to proving Theorem 6; or F is not modular. The condition $A = A^2$, together

[1]A left ideal L is called *modular* if there exists an element e in the ring A, such that $xe - x$ is in L for every x of A.

with d.c.c., imply that every maximal ideal is modular and that for every maximal left ideal, and in particular for F, $A - F$ is an irreducible A-module. Then $A - F$ is A-isomorphic to $A - Q$ where Q *is* a maximal modular left ideal of A. It is not clear though that F must be modular.

With regard to the following properties:

(a) A has a right unity element;

(b) For every x in A, x is in xA;

(c) $A = A^2$;

it is interesting to observe that (a) implies (b), and (b) implies (c). Lemma 7 proves that with d.c.c. and idempotents in the centre, (c) implies (a) and thus that the three conditions are equivalent. The above-mentioned example due to Zassenhaus shows that (c) does not imply (b) without d.c.c. To see that (b) does not imply (a) without d.c.c., consider the set of all infinite diagonal matrices, elements in a field, each matrix having only a finite number of nonzero entries.

2. Following Brown and McCoy (**5**; **6**), we associate with every element a in A, the ideal

$$R'(a) = \{ax - a^2x + \sum y_i a z_i - \sum y_i a^2 z_i\}.$$

We call an element a, R'-regular if a is in $R'(a)$. We call an ideal I, R'-regular if every element of I is R'-regular. In this way we obtain the set R'' of all elements that generate R'-regular ideals. The set R'' is simply a special case of Brown and McCoy's F-radical. If a is an element of the subradical R, then $-a$ is also in R,

$$-a - ab + a^2b = 0, \quad a = a(-b) - a^2(-b),$$

a is R'-regular. Thus it is clear that $R \leqslant R''$. In the commutative case $R = R''$, though in general they are different. From (**5**) we have the following important results about R''.

THEOREM 7. *The set R'' is an ideal of A.*

THEOREM 8. $R''(A - R'') = 0$.

THEOREM 9. *The ring $A - R''$ is isomorphic to a subdirect sum of subdirectly irreducible rings each having their $R'' = 0$.*

THEOREM 10. *A subdirectly irreducible ring A, has its $R'' = 0$ if and only if there exists an element $e \neq 0$ in the minimal ideal K of A such that $R'(e) = 0$.*

THEOREM 11. *A has its $R'' = 0$ if and only if it is isomorphic to a subdirect sum of subdirectly irreducible rings each having their $R'' = 0$.*

Let A be a subdirectly irreducible ring with $R'' = 0$. Then by Theorem 10, there exists an element $e \neq 0$ in the minimal ideal K of A such that $R'(e) = 0$. Thus

$$\{ex - e^2x + \sum y_i e z_i - \sum y_i e^2 z_i\} = 0.$$

Therefore $ex = e^2x$ for every x in A. Thus $e^2 = e^3$, $e^3 = e^4$, $e^2 = e^4$. We will use

LEMMA 10. *If there exists a nonzero idempotent e', both in the centre and in the minimal ideal K of a subdirectly irreducible ring A, then A is simple with e' as unity.*

Proof. Consider the Peirce decomposition of A for e'. $A = A_1 + A_2$, where $A_1 = Ae'$, and A_2 is the set of all $x - xe'$ for x in A and is the set of all x such that $xe' = 0$. Since A_2 is an ideal which cannot contain e', and since e' is in every nonzero ideal, $A_2 = 0$. Therefore $A = A_1 = Ae'$. Since e' is in K, $A = K$, A is simple ($A^2 = Ae' . Ae'$ contains e' and is therefore not zero), and has e' as unity element.

Therefore if $e^2 \neq 0$ and e is in the centre, in the subdirectly irreducible ring A with $R'' = 0$, K contains a nonzero idempotent in the centre, and A is a simple ring with unity. Otherwise $e^2 = 0$, and then $ex = 0$ for every x in A. Thus $eA = 0$, $JA = 0$. We have proved

THEOREM 12. *If a ring A has all its idempotents in the centre, then it has $R'' = 0$ if and only if A is isomorphic to a subdirect sum of subdirectly irreducible rings some of which are simple with unity, others (call them B_i) have the property that $K_i B_i = 0$, where K_i is the minimal ideal of B_i.*

With some mild conditions on A we can remove the nonsimple components.

THEOREM 13. *If a ring A has all its idempotents in the centre and either of the following properties:* (a) *if $xA \leqslant I$, then x is in I, for every ideal I*; (b) *every ideal M such that $A - M$ is subdirectly irreducible, is modular; then A has $R'' = 0$ if and only if A is isomorphic to a subdirect sum of simple rings with unity.*

Proof. Consider the components B_i of Theorem 12, that are not simple, i.e., the ones that satisfy $K_i B_i = 0$. For every i, there exists an ideal M_i in A such that $A - M_i \cong B_i$. If A satisfies (a), then $e_i A \leqslant M_i$ implies e_i is in M_i. Thus if $e_i B_i = 0$ in B_i, $e_i A \leqslant M_i$, e_i is in M_i, $e_i = 0$ in B_i. This contradicts the fact that $e_i \neq 0$ in B_i. Therefore there are no nonsimple components. Finally, if A satisfies (b), then $A - M_i$ has a unity element. Therefore $e_i B_i$ cannot be zero.

We now turn our attention to the commutative case. Here $R = R''$.

McCoy **(11)** made a study of all commutative subdirectly irreducible rings and considered them in two large classes: those having at least one element not a divisor of zero, and others all of whose elements are divisors of zero. We obtain more information about them in

THEOREM 14. *If A is a commutative subdirectly irreducible ring with subradical R zero, and with at least one element not a divisor of zero, then A is a field.*

If A is a commutative subdirectly irreducible ring all of whose elements are divisors of zero, then its subradical is zero and it is bound to its maximal nilideal N, and therefore also bound to its Jacobson radical J. Furthermore, if A has either d.c.c. *or* a.c.c., *A is nilpotent.*

Proof. By Theorem 12, a commutative subdirectly irreducible ring A with subradical R zero is either a simple ring with unity or $eA = 0, e \neq 0$. Therefore if A is fortunate enough to possess an element which is not a divisor of zero, $eA \neq 0$, A is simple with unity, A is a field.

From **(11)** we learn that if A is a commutative subdirectly irreducible ring all of whose elements are divisors of zero, then the minimal ideal $K = (e, 2e, \ldots pe = 0)$ and $eA = 0$. Therefore $KA = 0$ and by Theorem 12, $R = 0$. Furthermore, if $A \neq N$, the maximal nilideal, then let x be any element not in N. Since in the commutative case N is the set of *all* nilpotent elements, x is not nilpotent, the ideal $xA \neq 0$, and the ideal $x^2A \neq 0$. Since e is in every nonzero ideal, $e = x^2y$. Then

$$(xy)^2 = x^2y \,.\, y = ey = 0,$$

since $eA = 0$. Therefore xy is nilpotent, xy is in N. Thus $x \,.\, xy = e \neq 0$, $xN \neq 0$. Thus if $xN = 0$, x must be in N, i.e. A is bound to N. Then of course A is bound to J, since if $xJ = 0$, $xN = 0$, x is in $N \leqslant J$.

Assume now that A has d.c.c. If A is not nilpotent, and therefore not nil, there exists an element x which is not nilpotent. Consider the chain $xA > x^2A > \ldots$, and get an integer n such that $x^nA = x^{n+1}A$. Thus there is an element y such that $x^{n+1} = x^{n+1}y$. Since $x^{n+1}A$ is a nonzero ideal, e is in it, $e = x^{n+1}z$. Then

$$ey = 0 = x^{n+1}yz = x^{n+1}z = e \neq 0.$$

This contradiction shows that x must be nilpotent, A is nil and therefore nilpotent.

Assume now that A has a.c.c. If A is not nil, then again let x be any non-nilpotent element. Then all the ideals x^iA are nonzero and therefore each of them contains e. Therefore

$$e = xy_1 = x^2y_2 = \ldots = x^ny_n = \ldots.$$

Define V_i to be the set of all z that annihilate x^i. It is clear that the V_i are ideals and that $V_1 < V_2 < \ldots < V_i < \ldots$. Since

$$ex = 0 = x^iy_ix = x^{i+1}y_i,$$

y_i is in V_{i+1}, but is not in V_i. Therefore this is a properly ascending chain which does not stop after a finite number of steps. (Since x is not nilpotent, no V_i is equal to the whole ring.) This contradicts a.c.c. and therefore A is nil. By a result of Zassenhaus as yet unpublished, which states that in the presence of a.c.c. the maximal nil ideal is nilpotent, A is nilpotent.

Combining Theorems 12 and 14 we have our main result:

THEOREM 15. *If A is a commutative ring whose subradical R is zero, then A is isomorphic to a subdirect sum of subdirectly irreducible rings $A_1, A_2, \ldots$, $B_1, B_2, \ldots$, where the A_i are fields and the B_j are bound to their maximal nil-ideals. If in addition, for every ideal M such that $A - M$ is subdirectly irreducible, $A - M$ satisfies either* d.c.c. *or* a.c.c., *and in particular if A satisfies* d.c.c. *or* a.c.c., *then the B_j are nilpotent.*

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