Take a radius of a circle AB. Bisect it in C, and on AC describe the equilateral triangle ACD. Mark off the chord AL equal to AC. With centre L and radius LD describe an arc cutting the circumference in O. Then CO is approximately the side of the hendecagon. The angle subtended by CO at the centre is 32° 44′ 29″ instead of 32° 43′ 38″. It is to be noted that the angle which AO subtends at B is 21° 12′ 42″. The angle which the side of the regular 17-sided figure would subtend at B is 21° 10′ 35″.

The calculation of this result suggested the following approximation. The length of the side of the 11-gon is equal to one-fifth the diagonal of the circumscribing square.

This gives an angle 32° 51′ 35″.

On the history and degree of certain geometrical approximations.

Part II.*

By A. J. Pressland, M.A.

§ 1. Since the former paper on this subject was read, Prof. Cantor has published the second volume of his history of Mathematics. This has necessitated various additions to the paper, which can perhaps be best given as an appendix.

On page 413 Prof. Cantor says that the construction of Dürer's pentagon is found in a book called *Geometria deutsch*, which was lately discovered in the town library at Nürnberg, and gives 1487 as the upper limit to its date. The construction is said to be "mit unverrücktem Zirckel," the same expression that Schwenter applies to Dürer's solution.

Lionardo da Vinci (1452-1519) gave several methods of accurate and of approximate construction. Thus in fig. 18. the arc $ba = 1/6$, the arc $bc = 1/3$, the arc $cf = 1/8$, and the arc $af = 1/24$ of the circumference (p. 271).

Two constructions for the pentagon are also given by Lionardo. In fig. 19 the arcs are all of the same radius and the arc *am* is approximately 1/5 of the circumference, the value on calculation being found to be 72° 25′ (p. 272).

* This was read at the Sixth Meeting, 8th April, 1892.

The second approximate solution consists in drawing an equilateral triangle ABC, and the perpendicular AD on BC. The side AC is assumed the chord of the regular pentagon in a circle radius DA. The angle so subtended is 70° 31′ 42″.

The following construction for the nonagon is due to Lionardo, (fig. 20). Let bna be an equilateral triangle; make $ng = ah$. Describe a circle with g as centre, radius ga. As the angle $bga = bka = 40° \ 12' \ 28''$, the line ba is nearly the chord of the regular nonagon in this circle.

The expressions for the side of the 11-gon and 13-gon given by Schwenter are to be found in Dürer.

§ 2. The construction given by Abû'l Wafâ for the regular heptagon is known as the Indian rule. It has been applied in different forms, as above to the 11-gon and 13-gon, to the construction of regular figures.

It admits, however, of various modifications.

(i) Considering it under the following aspect, "the n^{th} part of the regular s-gon is the side of the regular p-gon in the same circle," the following results can be obtained :—

$s_{11} = 9/32$	of the diameter	gives an angle	3′ 26″ too small.
$s_{13} = 1/4$	,, ,,	gives an angle	1° 15′ too large.

These are due to Dürer.

$s_{17} = 11/60$	of the diameter	gives an angle	20° 7′ 40″·6.
		instead of	21° 10′ 35″
$s_{18} = s_3/5$		gives an angle	19° 56′ 54″
		instead of	20°
$s_{19} = 1/6$	of the diameter	gives an angle	19° 11′ 17″·4
		instead of	18° 56′ 41″.

These are due to Schwenter.

Beyond these the following may be worth mention :—

* $s_{29} = s_3/8$	giving an angle	12° 25′ 45″·2
	instead of	12° 24′ 49″·6.
$s_{31} = s_4/7$	giving an angle	11° 34′ 43″
	instead of	11° 36′ 46″.
$s_5 = 6s_{32}$	giving an angle	72° 2′ 42″
	instead of	72° 0′ 0″.

This might be written $s_5 = 3s_{16}$ but the approximation is not so near.

$$s_{25} = s_6/4 \quad \text{giving an angle } 14^\circ\ 21'\ 42'' \quad \text{instead of } 14^\circ\ 24'\ 0.$$

(ii.) Considering the rule generalised thus, "The side of the n-gon minus the side of the s-gon is the side of the r-gon," the following results are found.

$$s_3 - s_7 = s_7$$

$$*\ s_{10} - s_{17} = s_{25} \quad \text{giving } 14^\circ\ 23'\ 32''\cdot 4 \quad \text{instead of } 14^\circ\ 24'\ 0.$$

$$s_7 - s_{16} = s_{13} \quad \text{giving } 27^\circ\ 38'\ 6'' \quad \text{instead of } 27^\circ\ 41'\ 32''.$$

$$*\ s_{24} + s_{34} = s_{14} \quad \text{giving } 25^\circ\ 42'\ 54'' \quad \text{instead of } 25^\circ\ 42'\ 51''\cdot 4.$$

Formed on the analogy of

$$s_5^2 = s_6^2 + s_{10}^2 \quad \text{(Euclid XIII. 10)}$$

and

$$s_3^2 = s_6^2 + s_4^2$$

are

$$*\ s_{11}^2 = s_8^2 - s_{12}^2 \quad \text{giving } 32^\circ\ 44'\ 44'' \quad \text{instead of } 32^\circ\ 43'\ 38''.$$

$$s_{13}^2 = s_{12}^2 - s_{32}^2 \quad \text{giving } 27^\circ\ 50'\ 16'' \quad \text{instead of } 27^\circ\ 41'\ 32''.$$

$$s_{13}^2 = s_{10}^2 - s_{16}^2 \quad \text{giving } 27^\circ\ 44'\ 6'' \quad \text{instead of } 27^\circ\ 41'\ 32''.$$

And the rougher approximation

$$s_{11}^2 - s_{13}^2 = s_{21}^2.$$

(iii.) Considering the rule generalised thus, "If AB be the side of the n-gon in a circle radius CA, and CA be bisected in D, then BD is the side of the regular s-gon," the following approximations may be obtained :—

s_6 gives s_7

s_8 gives for $2s_{17}$ the angle $21^\circ\ 13'\ 46''$ instead of $21^\circ\ 10'\ 35''$.

s_{10} gives for $2s_{19}$ the angle $19^\circ\ 6'\ 46''$ instead of $18^\circ\ 56'\ 51''$.

s_{12} gives for	s_{10}	the angle	36° 6′ 8″
		instead of	36° 0′ 0″.
s_{15} gives for	$2s_5$	the angle	70° 54′ 26″
		instead of	72° 0′ 0″.
*s_{17} gives for	s_{11}	the angle	32° 43′ 44″·8
		instead of	32° 43′ 38″·2,

an approximation first suggested by Le Clerc's construction for s_{11}.

s_{39} gives for	s_{12}	the angle	30° 12′ 48″
		instead of	30° 0′ 0″

suggesting an approximation to s_{39}.

It is to be noted that this variation of the Indian rule is used in the Almagest construction (accurate) of the regular pentagon by Ptolemy.

(iv.) The following construction, though not an adaptation of the Indian rule, was suggested by it (fig. 21).

Let BE be the side of the n-gon in a circle radius OB. Bisect OB at D. Join ED and draw OF perpendicular to it. The locus of F is a circle with OD diameter. The angle DOF may give approximations to the side of the regular polygons.

Thus

with s_6	the angle DOF = 0°.			
with s_7	the angle DOF = 8° 58′,	giving	an approximation	to s_{40}
with s_8	the angle = 16° 19′ 30″	,,	,,	to s_{11}
*with s_9	the angle = 22° 29′	,,	,,	to s_{16}
with s_{10}	the angle = 27° 44′	,,	,,	to s_{13}.

By reversing the construction of the fourth of these an extremely near approximation to s_9 is obtained, the angle being 39° 59′ 15″·5 instead of 40°.

§ 3. From the following construction certain approximations result.

Take a radius AB and draw the tangent at B. Take a point C on the circle and draw CE perpendicular to the tangent at B. Then sinCAB = tanEAB. Calling CAB θ, and EAB ϕ, this gives:

$$\theta = 40° ; \ \phi = 32° \ 43' \ 56''·7.$$

*From which s_9 gives s_{11}.

With the previous approximation to s_9 the angle obtained is 32° 43′ 32″·5. The true angle is 32° 43′ 38″·1.

The equation $\frac{1}{2}\sin\theta = \tan\psi$ gives :

$\theta =$ 40° ; $\psi = 18°$ 50′ 58″·5, thus approximating to s_{19}.
With $\theta =$ 20° ; $\phi = 18°$ 52′ 54″·1, again approximating to s_{19},

where the angle is 18° 56′ 41″.

With $\theta = \frac{360°}{19}$; $\phi = 17°$ 59′ 19″. From which s_{19} gives s_{20}.

As s_{20} can be found exactly, this construction can be reversed.

With $\theta =$ 18° ; $\phi = 17°$ 10′ 19″·3. From which s_{20} gives s_{21} the correct angle for s_{21} being 17° 8′ 34″.

§ 4. In fig. 22, AB, CD are two diameters of a circle perpendicular to each other ; AE, BF, tangents at A, B, are equal to four times the radius and the radius respectively. Join EF cutting the circle at M, N ; and join AM, AN, cutting CD at m, n. Through m, n draw parallels to AB, namely GH, IK. The pentagon CIHGK is regular. M. Henri Barral, in *Nouvelles Annales*, XI. 388-390 (1852).

The construction above is given by Herr Staudt without proof in Crelle XXIV. (1842).

Terquem in a note says, "The construction of Herr Staudt is remarkable because it indicates an analogous construction for the division of the circumference into 17 equal parts." See also *Nouvelles Annales*, XVI. 310 (1857).

Among the calculations made for this paper the following occurred :—

$$61{\cdot}5 - 10\sqrt{5} = 39{\cdot}139320225,$$

a near approximation to the length 39·13929 ... inches of the seconds pendulum in London.

On Electrolysis.

By Professor Morrison.