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# OSCILLATION OF FIRST-ORDER DELAY DIFFERENTIAL EQUATIONS

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**Abstract**

This paper is concerned with the oscillation of first-order delay differential equations

$$x'(t) + p(t)x(\tau(t)) = 0,$$

where $p(t)$ and $\tau(t)$ are piecewise continuous and nonnegative functions and $\tau(t)$ is nondecreasing. A new oscillation criterion is obtained.

## 1. Introduction

We are concerned with delay differential equations of the form

$$x'(t) + p(t)x(\tau(t)) = 0, \quad t \geq t_0, \tag{1.1}$$

where $p(t) \geq 0$ is a piecewise continuous function and $\tau(t)$ is a nondecreasing piecewise continuous function, $\tau(t) < t$ for $t \geq t_0$ and $\lim_{t\to\infty} \tau(t) = \infty$.

As is customary, a solution of (1.1) is said to be oscillatory if it has arbitrarily large zeros. Hereafter for convenience we shall assume that inequalities and equations about values of functions are satisfied eventually for all large $t$.

Two well-known oscillation criteria for (1.1) are, respectively,

$$\alpha := \liminf_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds > \frac{1}{e} \tag{1.2}$$

and

$$\beta := \limsup_{t\to\infty} \int_{\tau(t)}^{t} p(s)\,ds > 1 \tag{1.3}$$

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(see [9, 11]). Concerning the constant $1/e$ in (1.2), it is pointed out in [9] that if the inequality

$$\int_{\tau(t)}^{t} p(s)\,ds \le \frac{1}{e}$$

holds eventually, then (1.1) has a nonoscillatory solution.

It is obvious that there is a gap between the conditions (1.2) and (1.3) when the limit $\lim_{t\to\infty}\int_{\tau(t)}^{t} p(s)\,ds$ does not exist. How to fill the gap is an interesting problem which has been recently investigated by several authors. See [1–4, 6–8, 10, 12–18]. Of them, the best results are, respectively, the condition

$$\beta > \frac{\ln\lambda_1 + 1}{\lambda_1} - \frac{1-\alpha-\sqrt{1-2\alpha-\alpha^2}}{2} \tag{1.4}$$

derived in [6], where $\lambda_1$ is the smaller root of the equation

$$\lambda = e^{\alpha\lambda}, \tag{1.5}$$

and the condition

$$\int_0^{\infty} p(t)\ln\left(e\int_t^{t+\tau} p(s)\,ds\right)ds = \infty \tag{1.6}$$

obtained in [12] in the case $\tau(t) = t-\tau$, $\tau > 0$.

The purpose of this paper is to develop a new oscillation criterion of the form

$$\limsup_{t\to\infty}\left\{\min_{\tau(t)\le s\le t}\int_{\tau(s)}^{s} p(\xi)\,d\xi\right\} > \frac{1+\ln\lambda_1}{\lambda_1} - \frac{1}{\lambda_2} \tag{1.7}$$

for (1.1), which improves (1.3) and is related to but independent of (1.4) and (1.6), where $\lambda_1$ is the smaller and $\lambda_2$ the greater root of (1.5).

When $\alpha = 1/e$, it is obvious that $\lambda_1 = \lambda_2 = e$. In this case, (1.7) reduces to

$$\limsup_{t\to\infty}\left\{\min_{\tau(t)\le s\le t}\int_{\tau(s)}^{s} p(\xi)\,d\xi\right\} > \frac{1}{e}. \tag{1.8}$$

The constant $1/e$ in the right-hand side of (1.8) is "best possible" and cannot be further improved. Furthermore, the left-hand side of (1.8) cannot be weakened to $\beta = \limsup_{t\to\infty}\int_{\tau(t)}^{t} p(s)ds$ in general (see [17]).

For a discussion on the significance of oscillation properties in applications, see the monograph [5, p. 288].

## 2. Oscillation criteria for (1.1)

Throughout this section, let $\alpha$ be defined by (1.2) and let $\lambda_1$ be the smaller and $\lambda_2$ the greater root of (1.5).

LEMMA 2.1 ([8]). *Assume that* (1.1) *has an eventually positive solution* $x(t)$. *Set*

$$w(t) = \frac{x(\tau(t))}{x(t)}.$$

*Then*

$$\lambda_1 \le \liminf_{t\to\infty} w(t) \le \lambda_2. \tag{2.1}$$

LEMMA 2.2. *Assume that* (1.1) *has an eventually positive solution* $x(t)$. *Set*

$$B(t) = \max\left\{\frac{x(s)}{x(\tau(s))} : \tau(t) \le s \le t\right\}.$$

*Then*

$$\liminf_{t\to\infty} B(t) \ge \frac{1}{\lambda_2}. \tag{2.2}$$

PROOF. Assume, for the sake of contradiction, that (2.2) is not true. Then there exists an increasing sequence $\{t_n\}$ with $t_n \to \infty$ as $n \to \infty$ such that

$$\lim_{n\to\infty} B(t_n) = \liminf_{t\to\infty} B(t) = \mu < \frac{1}{\lambda_2}.$$

For a given $\lambda \in (\mu, 1/\lambda_2)$, there exists an integer $N > 0$ such that

$$B(t_n) < \lambda, \quad n \ge N. \tag{2.3}$$

Since $-\lambda \ln \lambda < \ln \lambda_2/\lambda_2 = \alpha$, it follows from the definition of $\alpha$ that there exists an integer $N_1 > N$ such that

$$\int_{\tau(t)}^{t} p(s)ds > -\lambda \ln \lambda, \quad t \ge t_{N_1}. \tag{2.4}$$

Next we prove that

$$\frac{x(t)}{x(\tau(t))} < \lambda, \quad t \ge t_{N_1}. \tag{2.5}$$

In fact, if (2.5) is not true, then by (2.3) there exist an integer $n_1 \ge N_1$ and $T$ with $t_{n_1} \le T < t_{n_1+1}$ such that

$$\frac{x(t)}{x(\tau(t))} < \lambda \text{ for } t \in [\tau(t_{n_1}), T) \quad \text{and} \quad \frac{x(T)}{x(\tau(T))} = \lambda.$$

By (1.1), we have

$$\int_{\tau(T)}^{T} p(s)\,ds = -\int_{\tau(T)}^{T} \frac{x'(s)}{x(\tau(s))}\,ds \le \ln\frac{x(\tau(T))}{x(T)} \cdot B(T) \le -\lambda\ln\lambda,$$

which contradicts (2.4) and so (2.5) holds. By (2.5), we have

$$\liminf_{t\to\infty}\frac{x(\tau(t))}{x(t)} = \liminf_{t\to\infty} w(t) \ge \frac{1}{\lambda} > \lambda_2,$$

which contradicts (2.1) again and so the proof is complete.

THEOREM 2.1. *Assume that* $0 < \alpha \le 1/e$ *and*

$$\limsup_{t\to\infty}\left\{\min_{\tau(t)\le s\le t}\int_{\tau(s)}^{s} p(\xi)\,d\xi\right\} > \frac{1+\ln\lambda_1}{\lambda_1} - \frac{1}{\lambda_2}. \tag{2.6}$$

*Then all solutions of* (1.1) *oscillate.*

PROOF. Assume, for the sake of contradiction, that (1.1) has an eventually positive solution $x(t)$. For any given $\theta \in (0, 1)$, by Lemma 2.1 and (1.2),

$$\int_{\tau(t)}^{t} p(s)\,ds \ge \theta\alpha \quad \text{and} \quad \frac{x(\tau(t))}{x(t)} \ge \theta\lambda_1$$

for all sufficiently large $t$, and consequently for $\tau(t) \le s \le t$

$$\begin{aligned}
\frac{x(\tau(s))}{x(\tau(t))} &= \exp\left(\int_{\tau(s)}^{\tau(t)} p(\xi)\frac{x(\tau(\xi))}{x(\xi)}\,d\xi\right) \\
&\ge \exp\left(\theta\lambda_1\int_{\tau(s)}^{\tau(t)} p(\xi)\,d\xi\right) \\
&= e^{(\theta-1)\lambda_1}\exp\left[(1-\theta)\lambda_1 + \theta\lambda_1\int_{\tau(s)}^{\tau(t)} p(\xi)\,d\xi\right] \\
&\ge e^{(\theta-1)\lambda_1}\exp\left(\lambda_1\int_{\tau(s)}^{\tau(t)} p(\xi)\,d\xi\right),
\end{aligned} \tag{2.7}$$

since $\int_{\tau(s)}^{s} p(\xi)\,d\xi \le 1$. Integrating (1.1) from $\tau(t)$ to $t$ and using (2.7), we obtain

$$\begin{aligned}
x(\tau(t)) - x(t) &= \int_{\tau(t)}^{t} p(s)x(\tau(s))\,ds \\
&\ge e^{(\theta-1)\lambda_1}x(\tau(t))\int_{\tau(t)}^{t} p(s)\exp\left(\lambda_1\int_{\tau(s)}^{\tau(t)} p(\xi)\,d\xi\right)ds,
\end{aligned}$$

and so

$$1 \geq \frac{x(t)}{x(\tau(t))} + e^{(\theta-1)\lambda_1} \int_{\tau(t)}^{t} p(s) \exp\left(\lambda_1 \int_{\tau(s)}^{\tau(t)} p(\xi)\, d\xi\right) ds. \tag{2.8}$$

Let $t$ be large enough so that $\int_{\tau(t)}^{t} p(s)\, ds \geq \theta\alpha$. Then there exists $t^* \in [\tau(t), t]$ such that $\int_{\tau(t)}^{t^*} p(s) = \theta\alpha$. Thus

$$\begin{aligned}
&\int_{\tau(t)}^{t} p(s) \exp\left(\lambda_1 \int_{\tau(s)}^{\tau(t)} p(\xi)\, d\xi\right) ds \\
&\quad \geq \int_{\tau(t)}^{t} p(s)\, ds + \int_{\tau(t)}^{t^*} p(s) \left[\exp\left(\lambda_1 \int_{\tau(s)}^{\tau(t)} p(\xi)\, d\xi\right) - 1\right] ds \\
&\quad = \int_{\tau(t)}^{t} p(s)\, ds + \int_{\tau(t)}^{t^*} p(s) \left\{\exp\left[\lambda_1 \left(\int_{\tau(s)}^{s} p(\xi)\, d\xi - \int_{\tau(t)}^{s} p(\xi)\, d\xi\right)\right] - 1\right\} ds \\
&\quad \geq \int_{\tau(t)}^{t} p(s) ds + e^{\theta\alpha\lambda_1} \int_{\tau(t)}^{t^*} p(s) \exp\left(-\lambda_1 \int_{\tau(t)}^{s} p(\xi)\, d\xi\right) ds - \theta\alpha \\
&\quad = \int_{\tau(t)}^{t} p(s)\, ds + \frac{e^{\theta\alpha\lambda_1} - (1 + \theta\alpha\lambda_1)}{\lambda_1}.
\end{aligned}$$

Substituting this into (2.8), we have

$$1 \geq \frac{x(t)}{x(\tau(t))} + e^{(\theta-1)\lambda_1} \left[\int_{\tau(t)}^{t} p(s)\, ds + \frac{e^{\theta\alpha\lambda_1} - (1 + \theta\alpha\lambda_1)}{\lambda_1}\right].$$

It follows that

$$e^{(1-\theta)\lambda_1} - \frac{e^{\theta\alpha\lambda_1} - (1 + \theta\alpha\lambda_1)}{\lambda_1} \geq e^{(1-\theta)\lambda_1} B(t) + \min_{\tau(t) \leq s \leq t} \int_{\tau(s)}^{s} p(\xi)\, d\xi.$$

Taking the limit superior as $t \to \infty$ and using Lemma 2.2, we obtain

$$\begin{aligned}
&e^{(1-\theta)\lambda_1} - \frac{e^{\theta\alpha\lambda_1} - (1 + \theta\alpha\lambda_1)}{\lambda_1} \\
&\quad \geq \limsup_{t\to\infty} \left(e^{(1-\theta)\lambda_1} B(t) + \min_{\tau(t) \leq s \leq t} \int_{\tau(s)}^{s} p(\xi)\, d\xi\right) \\
&\quad \geq \frac{1}{\lambda_2} e^{(1-\theta)\lambda_1} + \limsup_{t\to\infty} \left\{\min_{\tau(t) \leq s \leq t} \int_{\tau(s)}^{s} p(\xi)\, d\xi\right\}.
\end{aligned}$$

Since $0 < \theta < 1$ is arbitrarily close to 1, we let $\theta \to 1$. Then

$$\limsup_{t\to\infty} \left\{\min_{\tau(t) \leq s \leq t} \int_{\tau(s)}^{s} p(\xi) d\xi\right\} \leq 1 - \frac{e^{\alpha\lambda_1} - (1 + \alpha\lambda_1)}{\lambda_1} - \frac{1}{\lambda_2} = \frac{1 + \ln \lambda_1}{\lambda_1} - \frac{1}{\lambda_2},$$

which contradicts (2.6) and so the proof is complete.

EXAMPLE. Consider the delay differential equation

$$x'(t) + p(t)x(t-1) = 0, \quad t \geq 1, \tag{2.9}$$

where $\tau(t) = t - 1$ and

$$p(t) = \begin{cases} a + 1/e, & n^2 \leq t < n^2 + 2, \\ 1/e - 1/t^2, & n^2 + 2 \leq t < (n+1)^2, \end{cases} \quad n = 1, 2, \ldots.$$

Observe that

$$\int_{t-1}^{t} p(s)\,ds = \begin{cases} \dfrac{1}{e} + a(t - n^2) + \dfrac{1}{n^2} - \dfrac{1}{t-1}, & n^2 \leq t < n^2 + 1, \\ \dfrac{1}{e} + a, & n^2 + 1 \leq t < n^2 + 2, \\ \dfrac{1}{e} + a(n^2 + 3 - t) + \dfrac{1}{t} - \dfrac{1}{n^2 + 2}, & n^2 + 2 \leq t < n^2 + 3, \\ \dfrac{1}{e} - \dfrac{1}{t(t-1)}, & n^2 + 3 \leq t < (n+1)^2. \end{cases}$$

Clearly, if $a > 0$, then

$$\alpha = \liminf_{t\to\infty} \int_{t-1}^{t} p(s)\,ds = \frac{1}{e}, \quad \beta = \limsup_{t\to\infty} \int_{t-1}^{t} p(s)\,ds = \frac{1}{e} + a$$

and

$$\limsup_{t\to\infty} \left\{ \min_{t-1\leq s\leq t} \int_{s-1}^{s} p(\xi)\,d\xi \right\} = \frac{1}{e} + a > \frac{1}{e}.$$

Thus, according to Theorem 2.1, all solutions of (2.9) oscillate. However, none of the results mentioned in the introduction can be applied to this equation when $a < 0.2313$.

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