Can. J. Math., Vol. XXVII, No. 3, 1975, pp. 679-690

UNEQUIVOCAL RINGS

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1. Unequivocality. For any radical property Q, a nonzero simple ring (all rings in this paper are assumed to be associative) must make up its mind so to speak and must be either Q radical or Q semi-simple. Every Q thus divides the class of all nonzero simple rings into two disjoint classes. Conversely any partition of the nonzero simple rings into two disjoint classes leads to at least two radicals [**1**, p. 16].

To extend this type of connection we consider all nonzero rings R which have the property that for any radical Q, R is either Q radical or Q semi-simple. Such rings are called unequivocal [**4**, p. 10]. The class of nonzero simple rings is properly contained in the class of all unequivocal rings.

Unequivocality can be internally characterized because it really depends on the relationship between the ring and its ideals or subideals, rather than on the class of all possible radical properties. A simple ring has no proper nonzero ideals and this is what makes it unequivocal rather than how it relates to various radicals.

THEOREM 1. *The following are all equivalent:*

(1) *R is unequivocal.*

(2) *R is S_I radical for every nonzero ideal I of R* (where S_I is the lower radical determined by I).

(3) *R is S_J radical for every nonzero subideal J of R.*

(4) *R is semi-simple re the upper radical determined by any nonzero homomorphic image H of R.*

(5) *For any nonzero subideal J of R and for any nonzero homomorphic image H of R, there exists a nonzero subideal K of H such that K is isomorphic to a homomorphic image of J. Symbolically:*

$$\begin{array}{ccc} R & \rightarrow & H \\ | & & | \\ J & \rightarrow & K. \end{array}$$

Proof. If R is unequivocal and J is any nonzero subideal of R then R is S_J radical, for if not then R is S_J semi-simple. Then every nonzero ideal of R is S_J semi-simple [**1**, p. 125, Corollary 2] and thus every nonzero subideal is S_J semi-simple. In particular J would have to be S_J semi-simple and since it is S_J radical and nonzero, we have a contradiction. Thus (1) implies (3). Clearly

Received January 2, 1974 and in revised form, April 17, 1974. This research was supported by the Canada Council.

https://doi.org/10.4153/CJM-1975-076-5 Published online by Cambridge University Press

(3) implies (2). To see that (2) implies (1), suppose there is a radical W such that $W(R) \neq 0$ and $W(R) \neq R$. Now $W(R)$ is an ideal of R and thus by (2), R is $S_{W(R)}$ radical. Since $S_{W(R)} \leqq W$, R must be a W radical ring. Thus $W(R) = R$, a contradiction. Thus R is unequivocal.

To get (1) implies (4), suppose R is unequivocal and H a nonzero homomorphic image of R. If R is radical for some radical then so is H. Thus since H is semi-simple re the upper radical determined by H, R must also be semi-simple. Conversely, suppose we have (4) and suppose there is a radical W such that $W(R) \neq 0$ and $\neq R$. Then $R/W(R)$ is W semi-simple. Let T be the upper radical determined by $R/W(R)$. Then $W \leqq T$. By (4), R is T semi-simple and therefore it is W semi-simple. Thus $W(R) = 0$, a contradiction. Thus R is unequivocal.

Finally we must get (1) if and only if (5). If R is unequivocal, J a nonzero subideal of R and H a nonzero homomorphic image of R, then R is S_J radical by (3). Then H is S_J radical and this means that H must have a nonzero subideal K which is a homomorphic image of J (see [**8**, Lemma 1]).

Conversely if we have (5) then for any nonzero subideal J of R, every nonzero homomorphic image H of R has a nonzero subideal K which is a homomorphic image of J. Therefore R is in S_J (see [**7**]). Thus (5) implies (3) and the theorem is established.

We recall that every ring R has a so-called torsion ideal

$$T = \{x \text{ in } R\text{: the characteristic of } x \text{ is nonzero}\}$$

and that R/T is torsion free, i.e., $my = 0$ implies $m = 0$ or $y = 0$ for y in R/T and m an integer. If $T = 0$ then R itself is torsion free. If $T = R$ then R is isomorphic to a direct sum of R_p's where

$$R_p = \{x \text{ in } R\text{: the characteristic of } x \text{ is } p^n,\ n \text{ a nonnegative integer}\},$$

where p is a prime. R_p is a so-called p ring.

The other additive feature of a ring is divisibility. Every ring R has a maximal divisible ideal D (for every nonzero x in D and any positive integer n, there exists a y in D such that $x = ny$) and R/D is reduced i.e. has no nonzero divisible ideals, in fact no nonzero divisible subgroups for D is, additively, the maximal divisible subgroup of R. R/D may have elements of infinite height (but not if it is torsion free).

Since it is well known that the class of rings whose underlying additive group is torsion (resp. a p-group, resp. a divisible group) is a radical class, it is easy to show that:

THEOREM 2. *If R is unequivocal then either R is torsion free or R is a p-ring, for some prime p.*

THEOREM 3. *If R is unequivocal then either R is divisible or R is reduced. Combining these two results we have:*

THEOREM 4. *There are four kinds of unequivocal rings:*

(1) *Divisible torsion free.*
(2) *Reduced torsion free.*
(3) *Divisible p-rings.*
(4) *Reduced p-rings.*

2. Examples. Every nonzero simple ring is unequivocal and since we do not have a complete catalogue of simple rings it is unlikely that we will be able to classify all unequivocal rings. We will at least discuss a few examples.

If R is unequivocal then $R \oplus R$ and in fact any finite direct sum of copies of R, is unequivocal. In fact we can prove:

LEMMA 1. *If R is the discrete direct sum of the rings C_a, a in A, then R is unequivocal if and only if every C_a is unequivocal and $S_R = S_{C_a}$ for every a.*

Proof. Every C_a is an ideal of R and if R is unequivocal then R is in S_{C_a} or $S_R \leqq S_{C_a}$. On the other hand every C_a is a homomorphic image of R and thus is in S_R, i.e., $S_{C_a} \leqq S_R$. Therefore $S_R = S_{C_a}$ for every a in A. Furthermore if I is any nonzero ideal of C_a then it is a subideal of R. Then R is in S_I. Since C_a is a homomorphic image of R it must also be in S_I. Thus C_a is unequivocal (Theorem 1); and this is also true for every a in A.

Conversely we assume that every C_a is unequivocal, that every $S_{C_a} = S_R$ and that R is the discrete direct sum of the C_a. Let I be any nonzero ideal of R and let R/K be any nonzero homomorphic image of R. Define I_b to be the projection of I into C_b. Then each I_b is an ideal of C_b though I_b need not be contained in I. Since I is nonzero, I_b is nonzero for some b in A. Also I can be mapped homomorphically onto I_b. Since $K \neq R$ there must exist a d in A such that $C_d \nsubseteq K$. Define $K_d = C_d \cap K \neq C_d$. Then $C_d/K_d = [C_d + K]/K$ is a nonzero ideal of R/K. Since $S_{C_d} = S_{C_b}$, C_d/K_d must have a nonzero subideal which is an image of I_b (and therefore of I). Thus R/K must have a nonzero subideal which is an image of I. Therefore R is in S_I and by Theorem 1, R is unequivocal.

Remarks. If we have a class of unequivocal rings C_a with a in some index set A and if $S_{C_a} = S_{C_b}$ for every a and b in A, then R equal to the discrete direct sum of the C_a will be unequivocal, i.e. we do not have to assume that $S_R = S_{C_a}$ for every a in A because that will always be true (in the proof of the converse we only used the fact that $S_{C_a} = S_{C_b}$). Therefore finite direct sums (or discrete ones in the infinite case) of copies of, for example, the same simple ring, are unequivocal and of course not simple.

The C_a need not all be the same, e.g. if $C_1 = S \oplus S$, $C_2 = S \oplus S \oplus S$, for a simple ring S, then $R = C_1 \oplus C_2 = S \oplus S \oplus S \oplus S \oplus S$ is unequivocal.

There should be some connection between two rings B and C if $S_B = S_C$ and this connection should be even more intimate if we assume that B and C are both unequivocal. The precise nature of this connection seems to be an open question.

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LEMMA 2. *If R is a minimal ideal of some overring T then R is unequivocal.*

Proof. If R is not semi-simple for some radical S then $S(R)$ is nonzero and is an ideal of T [**1**, Theorem 47, p. 124,]. Since R is minimal we must have $S(R) = R$ and therefore R is unequivocal.

LEMMA 3 [Andrunakievic]. *If R is a minimal ideal of some overring T then R is simple or $R^2 = 0$.*

Proof. Take any nonzero x in R. Let (x) be the ideal of R generated by x. Now $R^3 \subseteq (x)$ since R must be equal to the ideal of T generated by x. If $R^2 \neq 0$ then R^2 must equal R (since R^2 is an ideal of T and R is minimal) and then $R^3 = R^2 = R = (x)$. Therefore R is simple. Otherwise $R^2 = 0$.

LEMMA 4. *If S is any simple ring then S^0, the zero ring on the additive group of S, is unequivocal.*

Proof. If $S^2 = 0$ then $S^0 = S$, which is simple. If $S^2 \neq 0$ then we consider the ring $T = \begin{bmatrix} S & S \\ 0 & S \end{bmatrix}$ of upper triangular matrices with entries in S. Then $K = \begin{bmatrix} 0 & S \\ 0 & 0 \end{bmatrix}$ is an ideal of T and is isomorphic to S^0. To show that K is a minimal ideal of T we note that if $k = \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$ is any nonzero element of K then $x \neq 0$ and so $SxS = S$. Then for any y in S we have $y = \sum_1^n u_i x v_i$ and

$$\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix} = \sum_1^n \begin{bmatrix} u_i & 0 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix} \begin{bmatrix} 0 & 0 \\ 0 & v_i \end{bmatrix},$$

whence $\begin{bmatrix} 0 & y \\ 0 & 0 \end{bmatrix}$ belongs to the ideal in T generated by $\begin{bmatrix} 0 & x \\ 0 & 0 \end{bmatrix}$. Thus K is a minimal ideal of T, K is unequivocal and S^0 is unequivocal.

In addition to all simple rings, all discrete direct sums of any number of copies of the same simple ring, all zero rings which are minimal ideals and, in particular, all zero rings on the additive group of a simple ring (e.g. (rationals)^0), all hearts of subdirectly irreducible rings and discrete direct sums of any number of copies of the same unequivocal ring, there are several other unequivocal rings that should be singled out.

A. *The zero ring on an infinite cyclic additive group*, $Z_\infty{}^0$. This ring is isomorphic to each of its nonzero ideals I and therefore it is S_I radical and thus it is unequivocal. Note that $Z_\infty{}^0$ is never a minimal ideal for if it is generated by x and if it is an ideal of T, then for every t in T, $tx = ix$ for some integer i. Then if we let I be the ideal of T generated by nx with $n > 1$, we have $t.nx = n.ix = i.nx$ in I and thus I is an ideal of T and $I \subsetneq Z_\infty{}^0$. This ring is reduced and torsion free.

B. *The zero ring on the additive group p^∞* [**1**, pp. 14, 15], $Z_p{}^0(\infty)$. This ring

https://doi.org/10.4153/CJM-1975-076-5 Published online by Cambridge University Press

is isomorphic to each of its nonzero homomorphic images and is therefore S_I radical for every nonzero ideal I. Thus it is unequivocal. It is a divisible p-ring.

Remarks. $Z_\infty{}^0$ is unequivocal but $Z_\infty{}^0/6Z_\infty{}^0$ is not (because it is a torsion ring but not a p-ring). Similarly $Z_p{}^0(\infty) \oplus Z_p{}^0(\infty)$ is unequivocal but its ideal $Z_p{}^0(\infty) \oplus (Z/pZ)^0$ is not unequivocal because it is not divisible and not reduced (Z here means the integers). Thus the class of unequivocal rings is neither homomorphically closed nor hereditary.

Z, the ring of integers, is not unequivocal. Let $I = 2Z$ and let $H = Z/I$. Then I cannot be homomorphically mapped onto a nonzero subideal of Z/I, i.e. onto Z/I itself, because the only possibility is $I/2I = 2Z/4Z$ and this is a zero ring and not a field. In fact mZ is not unequivocal, for every positive integer m.

C. *A, the ring of all finite sums $\sum a_\alpha x_\alpha$, a_α rational, where $x_\alpha x_\beta = x_{\alpha+\beta}$ if $\alpha + \beta < 1$ and $x_\alpha x_\beta = 0$ if $\alpha + \beta \geqq 1$, where the α's are real between* 0 *and* 1.

Proof. We know that $A^2 = A$. Let I be any nonzero ideal in A. Then some nonzero element is in I, say $\sum_{i=1}^n a_{\alpha_i} x_{\alpha_i}$, with $\alpha_1 < \alpha_2 < \ldots < \alpha_n$. If we multiply by $cx_{1-\alpha_2}$ we get $ca_{\alpha_1}\, x_{1-(\alpha_2-\alpha_1)}$ in I. Since $a_{\alpha_1} \neq 0$ and c is any rational we have $rx_{1-(\alpha_2-\alpha_1)}$ in I for any rational r. If we multiply $\sum_1^n a_{\alpha_i} x_{\alpha_i}$ by $cx_{1-\alpha_2-(\alpha_2-\alpha_1)}$ we get $ca_{\alpha_1} x_{1-2(\alpha_2-\alpha_1)} + \sum_2^n ca_{\alpha_i} x_{1-(\alpha_2-\alpha_1)+(\alpha_i-\alpha_2)}$ in I. Since $\alpha_i - \alpha_2 \geqq 0$ for every $i \geqq 2$, we have $1 - (\alpha_2 - \alpha_1) + (\alpha_i - \alpha_2) \geqq 1 - (\alpha_2 - \alpha_1)$. Then all the terms in the $\sum_2^n$ can be removed by appropriate multiples of $x_{1-(\alpha_2-\alpha_1)}$, and we can conclude that $rx_{1-2(\alpha_2-\alpha_1)}$ is in I for any rational r. We continue in this way, obtaining $rx_{1-t(\alpha_2-\alpha_1)}$ in I, until we reach the integer t that satisfies

$$\frac{1-\alpha_2}{\alpha_2-\alpha_1} \leqq t < \frac{1-\alpha_1}{\alpha_2-\alpha_1}.$$

Since

$$\frac{1-\alpha_1}{\alpha_2-\alpha_2} - \frac{1-\alpha_2}{\alpha_2-\alpha_1} = 1,$$

there is precisely one such integer. Now $\alpha_1 < 1 - t(\alpha_2 - \alpha_1) \leqq \alpha_2$ and thus all the terms in $\sum_2^n a_{\alpha_i} x_{\alpha_i}$ can be removed by appropriate multiples of $x_{1-t(\alpha_2-\alpha_1)}$. Then $a_{\alpha_1} x_{\alpha_1}$ is in I. If any nonzero multiple of x_α is in I then every multiple of x_β is in I for any $\beta > \alpha$. Therefore every nonzero proper ideal I of A is of the form:

$I = \{tx_\delta + \sum a_\beta x_\beta$ where $\beta > \delta > 0$ and where t ranges over some additive subgroup T of the rationals $Q.\}$

Note that T may be 0 or all of Q. Then every nonzero proper homomorphic image of A is of the form:

$A/K = \{cx_\alpha + \sum a_\beta x_\beta$ where $0 < \beta < \alpha$ and where c ranges over some additive quotient group Q/S of the rationals$\}$

where $x_u x_v = 0$ for $u + v > \alpha$; $x_u x_v = x_{u+v}$ for $u + v < \alpha$; and when $u + v = \alpha$ then $ax_u bx_v = abx_\alpha$ where ab is the representative of the coset $ab + S$ in Q/S.

We want to show that any nonzero ideal I can be homomorphically mapped onto a nonzero subideal of A/K for any nonzero A/K. If $I \not\subseteq K$ then

$$\frac{I+K}{K} \cong \frac{I}{I \cap K}$$

works. Thus we need only consider the cases when $I \subseteq K$.

Take any τ such that $\delta < \tau < 2\delta$ and let

$$B = \{gx_\tau + \sum a_\beta x_\beta,\ \beta > \tau, \text{ and where } g \text{ ranges over } S\}.$$

Then B is an ideal of I and

$$I' = I/B = \{tx_\delta + \sum a_\beta x_\beta + hx_\tau \text{ where } \delta < \beta < \tau, \text{ where } t \text{ ranges over } T \text{ and } h \text{ over } Q/S\}.$$

Now I' is a zero ring since $\delta + \delta > \tau$.

Next we take any σ such that $\alpha/2 < \sigma < \alpha$ and let

$$J' = \{cx_\sigma + dx_\alpha + \sum a_\beta x_\beta \text{ where } \sigma < \beta < \alpha, \text{ where } c \text{ ranges over } T \text{ and } d \text{ over } Q/S\}.$$

Then J' is also a zero ring since $\sigma + \sigma > \alpha$. Now I' is isomorphic to J'. They are both zero rings and additively there is a one-to-one correspondence between (δ, τ) and (σ, α). Thus we can set up a one-to-one correspondence between I' and J' which preserves addition and of course preserves multiplication.

Thus in all cases I can be mapped to a subideal of A/K and thus every nonzero homomorphic image of A has a nonzero subideal which is a homomorphic image of I. Therefore A is in S_I for every nonzero ideal I of A, and by Theorem 1, A is unequivocal. This ring A is divisible and torsion free.

Note. A similar proof establishes that if the a_α are in Z/pZ for a prime p, the ring is also unequivocal, and this ring is a reduced p ring.

D. The set R of all $2m/(2n + 1)$, where m and n are integers.

Proof. This well known example is nil semisimple but Jacobson radical. First we establish that the only nonzero ideals of R are of the form 2^nR for some $n \geqq 0$. Let I be a nonzero ideal of R and let $2a$ be minimal among all positive integers that are numerators of elements of I. Then $2a/(2n + 1)$ is in I and thus $2a$ is in I. Then $(2a) \subseteq I$. Also $2a/(2b + 1) = 2a - 2a(2b/(2b + 1))$ is in I, for every b. Therefore $aR \subseteq I$. Furthermore, if a is not a power of 2, then some odd prime p divides into a. Then $2a(2/p)$ is in I and is an even integer $< 2a$, contradicting the minimality of $2a$. Thus $a = 2^n$ for some $n > 0$. To see that $I \subseteq aR = 2^nR$, let $2m/(2n + 1)$ be any element in I, and write $2m = qa + b$, with $0 \leqq b < a$. Then

$$\frac{2m}{2n+1} - \frac{qa}{2n+1} = \frac{b}{2n+1}$$

is in I and since $2b < 2a$, we must have $b = 0$. Therefore $I = 2^nR$.

Now if $I = 2^mR$ is any nonzero ideal of R and if $R/2^nR$ is any nonzero homomorphic image of R, we can map I onto a nonzero subideal of $R/2^nR$ if

$2^mR \nleq 2^nR$. Thus we consider only those cases when $2^mR \leqq 2^nR$, i.e., $m \geqq n$. Then we map 2^mR to $2^mR/2.2^mR$ and this is isomorphic to $2^{n-1}R/2^nR$, an ideal of $R/2^nR$, because both are zero rings with only two elements. If we take any element $2^m(2a/(2b+1))$ in 2^mR then if a is even, it is in 2.2^mR and thus 0 in the factor ring; and if a is odd then $2^m(2a/(2b+1))$ is equal to $2^m(2/(2b+1))$ in the factor ring and this is equal to $2^m.2$ in the factor ring because

$$2^m \cdot \frac{2}{2b+1} - 2^m \cdot 2 = 2^m \cdot 2\left(\frac{-2b}{2b+1}\right) = 0$$

in the factor ring. Thus $2^mR/2.2^mR = \{0, 2^m.2\}$. Similarly $2^{n-1}R/2^nR = \{0, 2^{n-1}.2\}$. Therefore R is in S_I for every ideal I and R is unequivocal. This ring is reduced and torsion free.

Before we tabulate our examples we shall consider the four classes of Theorem 4. If R is a divisible p-ring then it must be a zero ring. For if x and y are in R then $p^nx = 0$ for some n. Also $y = p^nz$ for some z in R since R is divisible. Then $xy = xp^nz = 0$. Furthermore [**3**, Theorem 19.1] any divisible p-group is a discrete direct sum of perhaps an infinite number of copies of p^∞. Therefore R must be a discrete direct sum of copies of $Z_p{}^0(\infty)$. Since $Z_p{}^0(\infty)$ is unequivocal R is unequivocal by Lemma 1. We thus have:

THEOREM 5. *Every divisible p ring is unequivocal and is a discrete direct sum of copies of $Z_p{}^0(\infty)$.*

For divisible torsion free rings, we know that their additive groups are discrete direct sums of copies of the rational numbers under addition [**3**, Theorem 19.1].

Reduced torsion free rings are, additively, discrete direct sums of subgroups of the additive rationals. We also know there are no simple rings in this class for there must exist a prime p such that pR is a proper nonzero ideal of R.

Our table may then look like this [this is not a complete classification]:

	Unequivocal	Not Unequivocal
Divisible p Rings	$\sum_{\oplus} Z_p{}^0(\infty)$	none
Divisible Torsion Free	All simple rings, char. o. ($R^2 = R$) All zero rings which are div. torsion free i.e. $\sum_{\oplus}(\text{Rationals})^0$ Example C.	Rationals $\oplus$ Example C ($R^2 = R$) Rationals $\oplus$ (Rationals)^0 ($R^2 \neq R$)
Reduced Torsion Free	$\sum_{\oplus} Z_\infty{}^0$ Example D	Z mZ
Reduced p Rings	All simple rings R, $R^2 = 0$ All simple rings R, $R^2 = R$, char. p. Example C with coefficients in Z/pZ. $\sum_{\oplus}(Z/pZ)^0$	$Z/pZ \oplus$ Ex. C on Z/pZ ($R^2 = R$) $Z/pZ \oplus (Z/pZ)^0$ ($R^2 \neq R$)

3. Partitions of the unequivocal rings. Every radical partitions the unequivocal rings into two disjoint classes, an upper and a lower class. An arbitrary partition of the unequivocals into two disjoint classes, an upper and a lower, may not lead to a radical. If S is simple then if S and $S \oplus S$ are in different parts of the partition, no radical corresponds to it. An *allowable* partition [**4**] is a partition into two disjoint classes, one designated as upper the other as lower, such that every ring in the upper class is semi-simple re the lower radical determined by the lower class.

For any allowable partition we let S be the lower radical determined by the lower class and T the upper radical determined by the upper class. Then $S \leqq T$ and both correspond to the partition. When is $S < T$? We give a partial answer to this question.

We consider the extreme cases and let S_U be the lower radical determined by U, the class of all unequivocal rings. We know [**2**, Theorem 2] that if U_1 is the class of all homomorphic images of unequivocal rings then S_U is the class of all rings R such that every nonzero homomorphic image of R contains a nonzero subideal in U_1. We wish to find some S_U semi-simple rings.

LEMMA 5. *R is S_U semi-simple if and only if no unequivocal ring can be mapped homomorphically onto a nonzero subideal of R.*

Proof. If R is S_U semi-simple then it cannot have a nonzero subideal in U_1. Conversely if R has no nonzero subideal in U_1 then $S_U(R)$ is 0, else $S_U(R)$ has a nonzero subideal in U_1 and this would also be a subideal of R.

THEOREM 6. *Z, the ring of integers, is S_U semi-simple and thus is semi-simple re the lower radical determined by any class of unequivocal rings.*

Proof. We will prove that no unequivocal ring can be homomorphically mapped onto a nonzero subideal of Z. Assume then that R is unequivocal and that R maps onto a nonzero subideal of Z. But every subideal of Z is in fact an ideal and of the form mZ, for some integer $m > 0$. Since mZ is torsion free, R itself must be torsion free. If we take any prime p which is relatively prime to m, then $mZ/pmZ \cong Z/pZ$, a finite field. Then R can be homomorphically mapped onto Z/pZ.

Let Q be the upper radical determined by this field Z/pZ. Then R must be Q semisimple (Theorem 1). Then R is a subdirect sum of copies of Z/pZ (see [**1**, Theorem 46, p. 12]). Now each Z/pZ has characteristic p and therefore R has characteristic p. But R is supposed to be torsion free and therefore no such R can exist.

LEMMA 6. *The Baer lower radical is < the S_U radical, and thus if R is S_U semi-simple then R is semi-prime.*

Proof. Since Z_∞^0 is unequivocal, it is S_U radical. Since it is known that the Baer Lower radical is the smallest radical containing Z_∞^0, it must be $\leqq$ the S_U

radical. Since any field is unequivocal and thus S_U radical, and is not Baer Lower radical, the lemma is proved.

Next we let T_U be the upper radical determined by all unequivocal rings. Let F be the class of all subideals of unequivocal rings. Then we know [**2**, Theorem 6] that the class of all rings R such that every nonzero subideal of R can be mapped homomorphically onto a nonzero ring in F is the class of all T_U semi-simple rings. A ring is T_U radical if and only if it cannot be mapped onto a nonzero T_U semi-simple ring and this happens if and only if it cannot be mapped onto a nonzero ring in F. We thus have:

LEMMA 7. *R is T_U radical if and only if R cannot be homomorphically mapped onto a nonzero subideal of an unequivocal ring.*

LEMMA 8. *The T_U radical is $<$ the Brown-McCoy radical.*

Proof. If a ring R is not Brown-McCoy radical it can be mapped onto a subdirect sum of simple rings with unity. Thus R can be mapped onto a single simple ring. Then R is not T_U radical. Thus $T_U \leqq$ Brown-McCoy. Example D is Brown-McCoy radical but unequivocal and therefore T_U semi-simple. Thus $T_U <$ Brown-McCoy.

We wish to find some T_U radical rings.

LEMMA 9. *If there exists a ring R such that*

$$R = \bigcup_{n=1}^{\infty} S_n \supset \ldots \supset S_n \supset \ldots \supset S_1 \supset S_0 = 0$$

where the S_n are the only ideals of R, where each S_{n+1}/S_n is simple (i.e. the S_n are the only subideals of R) *and where for every $n \geqq 0$ there exists an $m > n$ such that S_{n+1}/S_n and S_{m+1}/S_m are not isomorphic, then R is a T_U radical ring.*

Proof. Suppose that R is not T_U radical. Then some nonzero homomorphic image R/S_n is a subideal of some unequivocal ring Q. Then S_{n+1}/S_n is also a subideal of Q. Let V be the lower radical determined by this one simple ring S_{n+1}/S_n. Since Q is unequivocal it must be V radical. It is known that such a radical V must be hereditary [**5**, Theorem 2] and therefore R/S_n must be V radical. Thus every homomorphic image of R/S_n and in particular R/S_m must be V radical. Therefore R/S_m must have a nonzero subideal which is a homomorphic image of S_{n+1}/S_n. The only possibility is for S_{n+1}/S_n to be isomorphic to S_{m+1}/S_m and this is not possible. Therefore R is a T_U radical ring.

Remarks. Each S_n is T_U semisimple because each of its ideals, including itself, can be mapped onto a simple (i.e. unequivocal) ring.

It is easy to show that the conditions of Lemma 9 force $R^2 = R$. Example B satisfies most of the conditions of Lemma 9 but not the last one, i.e. all the S_{n+1}/S_n are isomorphic. Furthermore Example B is neither idempotent nor T_U radical.

Now we are ready to find a T_U radical ring.

Example E† Let V be a vector space over a division ring D, of dimension $\aleph_{\omega_0}$ i.e., an infinite cardinal whose subscript is the first limit ordinal. Then

$$\aleph_0 < \aleph_1 < \ldots < \aleph_n < \ldots < \aleph_{\omega_0}.$$

Let L be the ring of all linear transformations on V. Then it is known (see [**6**, pp. 789, 790; **9**, p. 360]) that the only ideals of L are

$$0 < S_0 < S_1 < \ldots < S_n < \ldots < S_{\omega_0} < L,$$

where $S_n = \{x \text{ in } L\colon O(x) < \aleph_n\}$ where $O(x)$ is the dimension of Vx. Also L/S_{ω_0} is a simple ring with unity element; S_0 is a simple ring without unity element but with minimal one sided ideals; S_{n+1}/S_n are simple rings without unity element and without minimal one sided ideals. The only ideals of S_m are the S_n with $n \leqq m$.

The ring L is not T_U radical because L/S_{ω_0} is simple. We note that L is not unequivocal because L/S_{ω_0} is not radical re the lower radical determined by S_0, since S_0 has no unity element and thus cannot be mapped into (i.e. onto) L/S_{ω_0}.

To obtain a T_U radical ring we consider S_{ω_0}. Now the only ideals of S_{ω_0} are: $0 < S_0 < S_1 < \ldots < S_n < \ldots < S_{\omega_0} = \cup S_n$. To apply Lemma 9 we must only prove that for every n there exists an $m > n$ such that S_{n+1}/S_n is not isomorphic to S_{m+1}/S_m. We do know that S_0 is not isomorphic to any of the S_{m+1}/S_m and thus we consider only the S_{n+1}/S_n.

Suppose then that S_{n+1}/S_n is isomorphic to S_{m+1}/S_m for $m > n$. We select a basis of V in sets of $\aleph_m$ elements i.e.

$$(b_{11}, b_{12}, \ldots, b_{1\Omega}), (b_{21}, b_{22}, \ldots, b_{2\Omega}), \ldots, (b_{\Omega 1}, b_{\Omega 2}, \ldots, b_{\Omega\Omega}), \ldots,$$

where $\{1, \ldots, \Omega\}$ has cardinal number $\aleph_m$. We define e_β to be 1 on the basal elements $b_{11}, \ldots, b_{\beta\Omega}$, for all $\beta \leqq \Omega$, and 0 on the remaining basal elements. This gives us a set of idempotents $\{e_\beta\}$, each one in S_{m+1}, such that

$$e_\gamma e_\delta = e_\delta e_\gamma = e_\gamma, \quad \text{for every } \gamma \text{ and } \delta, \text{ with } 1 \leqq \gamma < \delta \leqq \Omega.$$

None of the e_β are in S_m and each e_β gives us a distinct coset $e_\beta + S_m$ in S_{m+1}/S_m.

To each $e_\beta + S_m$ there must correspond a coset $x_\beta + S_n$ in S_{n+1}/S_n, and we must have:

$$(x_\beta + S_n)(x_\beta + S_n) = x_\beta + S_n$$

as well as:

$$(x_\gamma + S_n)(x_\delta + S_n) = (x_\delta + S_n)(x_\gamma + S_n) = x_\gamma + S_n$$

for all $1 \leqq \gamma < \delta \leqq \Omega$.

The cosets $x_\beta + S_n$ must all be distinct. If $\gamma < \delta$ then no matter which representatives we take from the cosets $x_\gamma + S_n$ and $x_\delta + S_n$, say x_γ and x_δ,

†The author would like to thank Professor O. Kegel for his help in working out this example.

we know that $x_\gamma x_\delta + S_n = x_\gamma + S_n = x_\gamma x_\gamma + S_n$. Then $x_\gamma x_\delta - x_\gamma x_\gamma$ is in S_n, or the dimension of $V(x_\gamma x_\delta - x_\gamma x_\delta) = Vx_\gamma(x_\delta - x_\gamma)$ is $< \aleph_n$. On the other hand since $x_\delta x_\delta + S_n = x_\delta + S_n \neq x_\gamma + S_n = x_\delta x_\gamma + S_n$, we know that $x_\delta x_\delta - x_\delta x_\gamma$ is not in S_n (it is of course in S_{n+1}). Thus the dimension of $V(x_\delta x_\delta - x_\delta x_\gamma) = Vx_\delta(x_\delta - x_\gamma)$ is $\aleph_n$. This allows us to conclude that the image spaces Vx_γ and Vx_δ must be distinct, for if they were equal, then we would have:

$$Vx_\gamma(x_\delta - x_\gamma) = Vx_\delta(x_\delta - x_\gamma)$$

but these spaces have different dimensions.

Again if $\gamma < \delta$ we have $x_\gamma x_\delta + S_n = x_\gamma + S_n$ or $x_\gamma x_\delta = x_\gamma + t$, with t in S_n. Then $V(x_\gamma + t) = Vx_\gamma x_\delta \leqq Vx_\delta$. Now $(x_\gamma + t) + S_n = x_\gamma + S_n$ and thus by selecting a different representative of the coset $x_\gamma + S_n$, we can guarantee that $V(x_\gamma + t) \leqq Vx_\delta$.

To finish our example, we consider x_Ω and its image space Vx_Ω. Since x_Ω is in S_{n+1}, the dimension of Vx_Ω is $\aleph_n$. For every $\beta \leqq \Omega$ we select representatives say x_β, of the cosets $x_\beta + S_n$, so that $Vx_\beta \leqq Vx_\Omega$. Since the cosets are all distinct, we know that all the image spaces Vx_β are distinct. There are $\aleph_m$ such distinct subspaces of Vx (a space of dimension $\aleph_n$). If the division ring D of V is the field of two elements, or any division ring of cardinality $\leqq \aleph_n$, then the number of distinct subspaces of Vx_Ω is $\leqq \aleph_{n+1}$.† Thus if $m > n + 1$, there are simply too many distinct subspaces to squeeze into Vx_Ω. Therefore no isomorphism can exist between S_{m+1}/S_m and S_{n+1}/S_n when $m > n+1$ and the cardinality of D is $\leqq \aleph_n$.

Thus by Lemma 9 we have:

THEOREM 7. *The ring S_{ω_0} of example E, with D of cardinality $\leqq \aleph_0$, is a T_U radical ring, and thus is radical re the upper radical determined by any class of unequivocal rings.*

Remark. If the division ring D in Example E is a field of characteristic 0 the S_{ω_0} is divisible torsion free. If D is a field of characteristic p then S_{ω_0} is a reduced p-ring.

We define a nonzero ring R to be *ambiguous* if no unequivocal ring can be mapped onto a nonzero subideal of R and if R cannot be mapped onto a nonzero subideal of an unequivocal ring. It is an open question as to whether ambiguous rings exist. If one does then we will know that $S < T$ for any allowable partition of the unequivocal rings. Note that an ambiguous ring must be Baer-Lower semi-simple and Brown-McCoy radical.

Since we are usually interested in radicals $\leqq$ the Jacobson radical, the following partial solution to the problem of when $S < T$ may be of interest.

†We are happy to assume the generalized continuum hypothesis. Prof. Osofsky claims to be able to prove this result without it and in fact to show that the result holds for any D.

THEOREM 8. *For any allowable partition of the unequivocal rings having all Jacobson semi-simple unequivocal rings in the upper class, we do have $S < T$.*

Proof. We know that S_{ω_0} of example E is T_U radical and therefore T radical. To see that it is S semisimple we must show that no unequivocal ring Q in the lower class can be mapped onto a nonzero subideal of S_{ω_0}, i.e. onto some S_n. Now S_{ω_0} and therefore S_n is Jacobson semisimple. Thus Q would have to be Jacobson semisimple–but these unequivocals are all in the upper class. Thus no such Q can exist and S_{ω_0} is S semisimple and $S < T$.

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