INVERSE SEMIGROUPS AS EXTENSIONS OF SEMILATTICES

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1. Introduction. Let S be an inverse semigroup with semilattice of idempotents E, and let ρ be a congruence on S. Then ρ is said to be *idempotent-determined* [**2**], or I.D. for short, if $(a, b) \in \rho$ and $a \in E$ imply that $b \in E$. If, further, ρ is a group congruence, then clearly ρ is the minimum group congruence on S, and in this case S is said to be *proper* [**8**]. Let $T = S/\rho$.

Let ρ be an I.D. congruence on S; the homomorphism $\rho^\natural$ will also be called I.D.. Green [**2**] has given the structure of S in terms of T, E, and certain mappings. In the case where T is a group, that is when S is proper, two structure theorems for S have been given. One, due to the present author [**5**], bears some resemblance to Green's result. The other, due to McAlister [**3**], is extremely concrete and shows that S is isomorphic to a semigroup which involves T acting by order-automorphisms on a poset containing a copy of E as an order-ideal.

The present paper is concerned with carrying out McAlister's programme for the case where ρ is an arbitrary I.D. congruence, so that S is now an arbitrary inverse semigroup; with generalising the embedding theorems for proper inverse semigroups given in [**5, 6**]; and with expanding and slightly improving Green's result, using ideas from [**5**].

It will be found that S can be embedded in a certain way in an inverse semigroup $L = L(S)$ arising from the action of T on a poset by partial order-isomorphisms whose domains and ranges are order-ideals. This embedding is surjective exactly in the case where T is a group, that is S is proper. Furthermore, L can be embedded in an inverse semigroup $\bar{L}$ arising from a similar action of T on a semilattice. Thus S is embedded in $\bar{L}$, a fact which overlaps with some results due to Reilly [**7**].

It is also shown that S can be embedded in an inverse semigroup M, on which there is defined an I.D. congruence $\bar{\rho}$ extending ρ, such that each $\bar{\rho}$-class has a maximum element under the natural partial order, and $T = M/\bar{\rho}$. This is then used to yield a slight improvement in Green's theory.

Finally, the L-semigroups definable over an inverse semigroup S are seen to form a category with initial and terminal object.

2. The embedding of S in L. The notation and terminology of Clifford and Preston [**1**] will be used, and the basic results on inverse semigroups contained therein assumed. Any order-theoretic statement made about an inverse semigroup refers to the natural partial order. The identity congruence will be denoted by i.

The first proposition generalises a result in [**8**] and is implicit in [**2**].

PROPOSITION 2.1. *Let ρ be a congruence on S. Then ρ is* I.D. *if and only if $\rho \cap \mathscr{R} = i$.*

Proof. Suppose that ρ is I.D. and let $(a, b) \in \rho \cap \mathscr{R}$. Now $\mathscr{R}$ is a left congruence, so

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that $(a^{-1}a, a^{-1}b) \in \rho \cap \mathscr{R}$. Hence $a^{-1}b \in E$, since ρ is I.D. Now $aa^{-1} = bb^{-1}$, and so $b = aa^{-1}b \leqq a$; similarly, $b \leqq a$. Hence $\rho \cap \mathscr{R} = i$.

Conversely, suppose that $\rho \cap \mathscr{R} = i$ and let $(e, x) \in \rho$ where $e = e^2$. Then $(e, xx^{-1}) \in \rho$, so that $(xx^{-1}, x) \in \rho \cap \mathscr{R}$. Hence $x = xx^{-1} \in E$, and therefore ρ is I.D.

For the remainder of the paper, ρ denotes an I.D. congruence on S. Following Proposition 2.1, S is coordinatised by the map $a \to (aa^{-1}, a\rho)$ (see [2]).

Let $\mathscr{X}$ be a poset. A non-empty subset A of $\mathscr{X}$ is called an *order-ideal* of $\mathscr{X}$ if $b \in \mathscr{X}$ and $b \leqq a \in A$ imply that $b \in A$; and A is called a *subsemilattice of* $\mathscr{X}$ if given $a, b \in A$ their infimum in $\mathscr{X}$, denoted $a \wedge b$, exists and is in A.

Let $K_{\mathscr{X}}$ be the inverse subsemigroup of $\mathscr{I}_{\mathscr{X}}$ consisting of those $\alpha \in \mathscr{I}_{\mathscr{X}}$ whose domain $\Delta\alpha$ and range $\nabla\alpha$ are order-ideals of $\mathscr{X}^-$, and where α is an order-isomorphism from $\Delta\alpha$ onto $\nabla\alpha$. We say that an inverse semigroup T *acts suitably on* $\mathscr{X}$ if there exists a homomorphism $\phi : T \to K_{\mathscr{X}}$. In this case, for $t \in T$, we write Δt for $\Delta(t\phi)$ and ∇t for $\nabla(t\phi)$. If $a \in \Delta t$, we let T act on the left and write $t.a$ or ta for $(t\phi).a$. All other mappings will act on the right, as usual.

Recall that in $\mathscr{I}_{\mathscr{X}}$, and so in $K_{\mathscr{X}}$, $\alpha \leqq \beta$ if and only if (i) $\Delta\alpha \subseteq \Delta\beta$ and (ii) $\beta|\Delta\alpha = \alpha$.

Of necessity we follow McAlister's theory; the argument is refined or the theory generalised at those points where the fact that T is no longer necessarily a group comes into play.

Proceeding as in [3], therefore, let $\{D_i | i \in I\}$ be the set of $\mathscr{D}$-classes of S and pick an idempotent $f_i \in D_i$ for each $i \in I$. Denote by H_i the $\mathscr{H}$-class containing f_i, and let $\bar{f}_i = f_i\rho^\natural$. Further, for each $i \in I$, pick representatives r_{iu} of the $\mathscr{H}$-classes contained in the $\mathscr{R}$-class of f_i with f_i the representative of its class; denote this set of representatives by E_i.

From now on we use $r_{iu}, r_{iv}, \ldots$ to denote elements of E_i, and $h_i, h_i', \ldots$ to denote elements of H_i, for some $i \in I$.

Each element of S can be uniquely expressed in the form $r_{iu}^{-1}h_i r_{iv}$, and the idempotents of S are precisely the elements $r_{iu}^{-1}r_{iu}$; they are all distinct.

Let $k_{iu} = r_{iu}\rho^\natural$, $g_i = h_i\rho^\natural$ and $G_i = H_i\rho^\natural$. By Proposition 2.1, $G_i \approx H_i$ and for fixed i the elements k_{iu} are all distinct.

The following trivial result, and the one derived from it by applying $\rho^\natural$, will be used below without comment.

LEMMA 2.2. *Let* $a = r_{iu}^{-1}h_i r_{iv}$. *Then* $aa^{-1} = r_{iu}^{-1}r_{iu}$, $a^{-1}a = r_{iv}^{-1}r_{iv}$, *and* $f_i r_{iu} = r_{iu}$.

Finally, for each $i, j \in I$, let $B_{ij} = \{k_{ju} | r_{ju}^{-1}r_{ju} \leqq f_i\}$.

The next three technical lemmas which we quote are taken from [3], the second having been slightly adapted. Their proofs in [3] can be carried over without difficulty.

LEMMA 2.3. [3, Lemma 2.1.] $r_{iu}^{-1}r_{iu} \geqq r_{jv}^{-1}r_{jv}$ *if and only if* $G_j k_{jv} = G_j k_{jw} k_{iu}$ *for some* $k_{jw} \in B_{ij}$.

LEMMA 2.4. [3, Lemma 2.2.] *If* $k_{iu} g_j k_{jv} \in G_i$ *for some* $k_{iu} \in B_{ji}$ *and* $k_{jv} \in B_{ij}$, *then* $i = j$ *and* $k_{jv} = \bar{f}_j$.

LEMMA 2.5. [3, Lemma 2.3.] *If* $k_{jv} \in B_{ij}$ *and* $k_{nw} \in B_{jn}$, *then* $G_n k_{nw} g_j k_{jv} = G_n k_{nu}$ *for some* $k_{nu} \in B_{in}$.

As in [3], therefore, it follows from Lemma 2.3 that the semilattice E is isomorphic to the set $\mathscr{Y} = \{(i, G_ik_{iu}) | i \in I, k_{iu} \in E_i\rho^\natural\}$, where $(i, G_ik_{iu}) \geqq (j, G_jk_{jv})$ if and only if $G_jk_{jv} = G_jk_{jw}k_{iu}$ for some $k_{jw} \in B_{ij}$.

Let $\mathscr{X} = \{(i, G_ix) | i \in I, xx^{-1} = \bar{f}_i\}$ under the ordering $(i, G_ix) \geqq (j, G_jy)$ if and only if $G_jy = G_jzx$ for some $z \in B_{ij}$. T acts on $\mathscr{X}$ by partial transformations as follows:

$$\Delta t = \{(i, G_ix) \in \mathscr{X} \,|\, x^{-1}x \leqq t^{-1}t\},$$

and, for $(i, G_ix) \in \Delta t$, $t.(i, G_ix) = (i, G_ixt^{-1})$.

Lemma 2.6. *$\mathscr{X}$ is a poset, and $\mathscr{Y}$ is a subsemilattice and order-ideal of $\mathscr{X}$. T acts suitably on $\mathscr{X}$ and $\mathscr{X} = T\mathscr{Y}$. For each $t \in T$, $\mathscr{Y} \cap t(\mathscr{Y} \cap \Delta t) \neq \square$.*

Proof. Note that Lemmas 2.3, 2.4 and 2.5 hold. The argument given in [3] prior to Lemma 2.4 there, adapted slightly by noting that $\bar{f}_j \in B_{jj}$ and $g_j = g_j\bar{f}_j$, shows that $\geqq$ is a well-defined relation on $\mathscr{X}$. The relevant parts of the proof of [3, Lemma 2.4], with a similar small adaptation, show that $\geqq$ is reflexive, transitive and antisymmetric, and that $\mathscr{Y}$ is a subsemilattice and order-ideal of $\mathscr{X}$.

Let $(i, G_ix) \in \mathscr{X}$, where $x^{-1}x \leqq t^{-1}t$; then $(xt^{-1})(xt^{-1})^{-1} = xt^{-1}tx^{-1} = xx^{-1} = \bar{f}_i$. Hence $t.(i, G_ix) \in \mathscr{X}$. Suppose further that $(j, G_jy) \in \mathscr{X}$, where $(j, G_jy) \leqq (i, G_ix)$. Then $y = g_jzx$ for some $z \in B_{ij}$, so that

$$y^{-1}y = x^{-1}z^{-1}g_j^{-1}g_jzx \leqq x^{-1}x \leqq t^{-1}t.$$

Hence $(j, G_jy) \in \Delta t$, and therefore Δt is an order-ideal of $\mathscr{X}$. Moreover $t.(i, G_ix) \geqq t.(j, G_jy)$, and $t.(i, G_ix) \in \Delta t^{-1}$ since $(xt^{-1})^{-1}xt^{-1} \leqq (t^{-1})^{-1}t^{-1}$.

Hence $\nabla t \subseteq \Delta t^{-1}$. On the other hand if $(i, G_iz) \in \Delta t^{-1}$, then $zz^{-1} = \bar{f}_i$ and $z^{-1}z \leqq tt^{-1}$. Let $w = zt$; then $ww^{-1} = ztt^{-1}z^{-1} = zz^{-1} = \bar{f}_i$, $w^{-1}w \leqq t^{-1}t$, and $z = wt^{-1}$. Hence $(i, G_iw) \in \Delta t$ and $t.(i, G_iw) = (i, G_iz), t^{-1}(i, G_iz) = (i, G_iw)$.

Thus $\nabla t = \Delta t^{-1}$ is an order-ideal, and t is a partial order-isomorphism with domain Δt and range ∇t, having inverse t^{-1}. If $s \in T$, it is easily shown that $\Delta(ts) = s^{-1}(\nabla s \cap \Delta t) = \Delta(t \circ s)$, and clearly therefore T acts suitably on $\mathscr{X}$.

Let $(i, G_ix) \in \mathscr{X}$. Then $(i, G_i\bar{f}_i) \in \mathscr{Y} \cap \Delta x^{-1}$, and $x^{-1}.(i, G_i\bar{f}_i) = (i, G_ix)$. Thus $\mathscr{X} = T\mathscr{Y}$.

Let $t \in T$, where $t = k_{iu}^{-1}g_ik_{iv}$, say. Then $(i, G_ik_{iu}) \in \mathscr{Y} \cap t(\mathscr{Y} \cap \Delta t)$, since $t.(i, G_ik_{iv}) = (i, G_ik_{iu})$.

Suppose now that we are given a poset $\mathscr{X}$ containing a subsemilattice and order-ideal $\mathscr{Y}$, and an inverse semigroup T, which together have the properties listed in the statement of Lemma 2.6.

Lemma 2.7. *For each $t \in T$, $\mathscr{Y} \cap t(\mathscr{Y} \cap t^{-1}(\mathscr{Y} \cap \Delta t^{-1})) \neq \square$.*

Proof. By hypothesis there exists $a \in \mathscr{Y} \cap \Delta t^{-1}$ such that $b = t^{-1}a \in \mathscr{Y}$. Then $b \in \nabla t^{-1} = \Delta t$, and $tb = tt^{-1}a = a$. Hence $a \in \mathscr{Y} \cap t(\mathscr{Y} \cap t^{-1}(\mathscr{Y} \cap \Delta t^{-1}))$.

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Define $L = L(T, \mathscr{X}, \mathscr{Y})$ to be

$$\{(a, t) \mid t \in T, a \in \mathscr{Y} \cap t(\mathscr{Y} \cap t^{-1}(\mathscr{Y} \cap \Delta t^{-1}))\}$$

under the multiplication

$$(a, t)(b, s) = (t(t^{-1}a \wedge b), ts).$$

By Lemma 2.7, for each $t \in T$ there exists $(a, t) \in L$.

Whenever T is a group, that is whenever S is proper and ρ is the minimum group congruence on S, then T acts on $\mathscr{X}$ by order-automorphisms, and $L = P(T, \mathscr{X}, \mathscr{Y})$ as defined in [3].

LEMMA 2.8. *Let $a \in \mathscr{X}, t \in T$. Then $(a, t) \in L$ if and only if* (i) $a \in \mathscr{Y} \cap \Delta t^{-1}$ *and* (ii) $t^{-1}a \in \mathscr{Y}$.

Proof. This follows easily from the elementary observation that $a \in \nabla t$ if and only if $a \in \Delta t^{-1}$, and then $a = tt^{-1}a$.

COROLLARY 2.9. *Suppose $(a, t) \in L$ and $s \geqq t$. Then $(a, s) \in L$.*

Proof. Since $s^{-1} \geqq t^{-1}$, $\Delta t^{-1} \subseteq \Delta s^{-1}$ and $s^{-1}a = t^{-1}a$. The result now follows from Lemma 2.8.

THEOREM 2.10. *L is an inverse semigroup. If $\pi_2 : L \to T$ is the second projection $(a, t) \mapsto t$, then π_2 is an* I.D. *surjective homomorphism.*

Proof. Let $(a, t), (b, s) \in L$. By Lemma 2.8, $t^{-1}a \in \mathscr{Y}$ so that $t^{-1}a \wedge b$ exists and is in $\mathscr{Y}$. Since ∇t^{-1} and Δs^{-1} are order-ideals and $t^{-1}a \in \nabla t^{-1}$, $t^{-1}a \wedge b \in \nabla t^{-1} \cap \Delta s^{-1}$. Hence $t(t^{-1}a \wedge b) \in \Delta s^{-1}t^{-1} = \Delta(ts)^{-1}$. Moreover $t(t^{-1}a \wedge b) \leqq tt^{-1}a = a \in \mathscr{Y}$, so that $t(t^{-1}a \wedge b) \in \mathscr{Y}$. Further, $s^{-1}t^{-1}t(t^{-1}a \wedge b) = s^{-1}(t^{-1}a \wedge b) \leqq s^{-1}b \in \mathscr{Y}$, by Lemma 2.8. By Lemma 2.8 again, therefore,

$$(a, t)(b, s) = (t(t^{-1}a \wedge b), ts) \in L,$$

and L is closed under multiplication.

Let $(c, r) \in L$. It is easily seen that

$$(a, t)[(b, s)(c, r)] = [(a, t)(b, s)](c, r)$$

if and only if

$$t(t^{-1}a \wedge s(s^{-1}b \wedge c)) = ts(s^{-1}(t^{-1}a \wedge b) \wedge c);$$

that is, if and only if

$$t^{-1}a \wedge s(s^{-1}b \wedge c) = s(s^{-1}(t^{-1}a \wedge b) \wedge c); \tag{1}$$

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that is, if and only if

$$s^{-1}(t^{-1}a \wedge s(s^{-1}b \wedge c)) = s^{-1}(t^{-1}a \wedge b) \wedge c. \tag{2}$$

Now the left hand side of (2) $\leqq s^{-1}(t^{-1}a \wedge ss^{-1}b) = s^{-1}(t^{-1}a \wedge b)$; and further, the left hand side of (2) $\leqq s^{-1}s(s^{-1}b \wedge c) = s^{-1}b \wedge c \leqq c$. Hence the left hand side of (2) $\leqq$ the right hand side of (2). Applying s on the left, we deduce that the left hand side of (1) $\leqq$ the right hand side of (1).

On the other hand, the right hand side of (1) $\leqq s(s^{-1}b \wedge c)$; and further, the right hand side of (1) $\leqq ss^{-1}(t^{-1}a \wedge b) = t^{-1}a \wedge b \leqq t^{-1}a$. Hence the right hand side of (1) $\leqq$ the left hand side of (1). Equality follows, so that the multiplication is associative.

It is easily seen that the set of idempotents $\mathscr{E}$ of L is given by $\mathscr{E} = \{(a, t) | t = t^2\}$ and that the elements of $\mathscr{E}$ commute. In fact, if (a, t) and $(b, s) \in \mathscr{E}$, then

$$(a, t)(b, s) = (a \wedge b, ts).$$

Some routine checking then shows that L is an inverse semigroup with $(a, t) \in L$ having inverse $(t^{-1}a, t^{-1})$. Note that $(a, t)(a, t)^{-1} = (a, tt^{-1})$, so that if $(a, t)\mathscr{R}(b, s)$ then $a = b$. Clearly π_2 is a surjective homomorphism, and if further $(a, t)\pi_2 = (b, s)\pi_2$, then $t = s$. Hence π_2 is I.D., by Proposition 2.1.

REMARK. Let (a, t), $(b, s) \in \mathscr{E}$; then their product $(a \wedge b, ts) \in \mathscr{E}$. Since $ts \leqq t$, it follows from Corollary 2.9 and Theorem 2.10 that $(a \wedge b, t) \in \mathscr{E}$.

Define the projection $\pi_1 : \mathscr{E} \to \mathscr{Y}$ by $(a, t)\pi_1 = a$. Then π_1 is a homomorphism with range $\{a \in \mathscr{Y} | a \in \Delta r \text{ for some } r \in T\}$. If $t \neq s$, then we may assume that $ts < t$, and $(a \wedge b, t)\pi_1 = (a \wedge b, ts)\pi_1$.

Hence π_1 is injective if and only if T has exactly one idempotent, that is if and only if T is a group. In this case, T acts by order-automorphisms on $\mathscr{X}$ and π_1 is surjective.

We now have the main theorem of this section, which describes how S is embedded in the corresponding L.

As before, let S be an inverse semigroup with semilattice of idempotents E, and let ρ be an I.D. congruence on S. Suppose that $\mathscr{X}$, $\mathscr{Y}$ and T are as defined prior to Lemma 2.6. Let $L = L(S)$, where $L(S) = L(T, \mathscr{X}, \mathscr{Y})$, and define the map $\psi : S \to L$ as follows:

$$(r_{iu}^{-1} h_i r_{iv})\psi = ((i, G_i k_{iu}), k_{iu}^{-1} g_i k_{iv}).$$

THEOREM 2.11. *ψ is an injective homomorphism such that $\psi\pi_2 = \rho^{\natural}$. For each $a \in \mathscr{Y}$ there exists $(a, t) \in L$ with the following property:*

$$(a, s) \in L \text{ and } s \leqq t \text{ imply that } s = t: \tag{3}$$

and $S\psi$ is the set of all such (a, t). Moreover, given $a \in \mathscr{Y}$, then $(a, s) \in L$ if and only if there exists $(a, t) \in S\psi$ with $t \leqq s$.

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Proof. The last paragraph of the proof of Lemma 2.6 shows that ψ indeed maps into L. The argument given prior to [3, Lemma 2.5] shows that ψ is injective, and clearly $\psi\pi_2 = \rho^\natural$.

Let

$$p = r_{iu}^{-1} h_i r_{iv}, q = r_{jx}^{-1} h_j r_{jy}$$

be elements of S. Following the first part of the proof of [3, Lemma 2.5],

$$r_{iv}^{-1} r_{iv} r_{jx}^{-1} r_{jx} = r_{nw}^{-1} r_{nw}, \quad \text{for some} \quad r_{nw} \in E_n,$$

where

$$r_{nw} r_{iv}^{-1} h_i^{-1} r_{iu} = h_n r_{nz} \quad \text{for some} \quad r_{nz} \in E_n, h_n \in H_n.$$

Further,

$$pq = r_{nz}^{-1} h'_n r_{nc} \quad \text{for some} \quad r_{nc} \in E_n, h'_n \in H_n.$$

Thus $(n, G_n k_{nz})$ is the first coordinate of $(pq)\psi$, and

$$(n, G_n k_{nz}) = (n, G_n k_{nw} k_{iv}^{-1} g_i^{-1} k_{iu}).$$

Now

$$k_{nw}^{-1} k_{nw} \leqq k_{iv}^{-1} k_{iv} = (k_{iu}^{-1} g_i k_{iv})^{-1}(k_{iu}^{-1} g_i k_{iv}),$$

so that

$$(n, G_n k_{nz}) = k_{iu}^{-1} g_i k_{iv}.(n, G_n k_{nw}) = p\rho^\natural.\{(i, G_i k_{iv}) \wedge (j, G_j k_{jx})\}.$$

As seen in the last paragraph of the proof of Lemma 2.6, $(i, G_i k_{iv}) = (p\rho^\natural)^{-1}(i, G_i k_{iu})$, and it follows that $(pq)\psi$ and $p\psi.q\psi$ have the same first coordinate. Their second coordinates are also equal, so that ψ is a homomorphism.

Given $a = (i, G_i k_{iu}) \in \mathscr{Y}$ therefore, take $g_i \in G_i$ and $k_{iv} \in E_i\rho^\natural$. Letting $t = k_{iu}^{-1} g_i k_{iv}$, it follows that $(a, t) \in S\psi$. On the other hand, by Lemma 2.8, $((i, G_i k_{iu}), t) \in L$ if and only if $k_{iu}^{-1} k_{iu} \leqq tt^{-1}$ and $t^{-1}(i, G_i k_{iu}) = (i, G_i k_{iw})$ for some $k_{iw} \in E_i\rho^\natural$. The latter conditions hold if and only if $k_{iu}^{-1} k_{iu} \leqq tt^{-1}$ and $k_{iu} t = g'_i k_{iw}$, for some $g'_i \in G_i$.

Let $m = k_{iu}^{-1} g'_i k_{iw}$; then $((i, G_i k_{iu}), m) \in S\psi$. If $m \leqq t$, then $m = mm^{-1}t = k_{iu}^{-1} k_{iu} t$, so that

$$g'_i k_{iw} = k_{iu} m = k_{iu} t; \quad \text{also} \quad k_{iu}^{-1} k_{iu} = mm^{-1} \leqq tt^{-1}.$$

Conversely, if $k_{iu} t = g'_i k_{iw}$ then $mm^{-1} t = k_{iu}^{-1} k_{iu} t = k_{iu}^{-1} g'_i k_{iw} = m$, so that $m \leqq t$.

Suppose now that $((i, G_i k_{iu}), m') \in S\psi$, where $m' \leqq m$. Then $m' = m'm'^{-1}m = k_{iu}^{-1} k_{iu} m = m$. Since to any $(a, t) \in L$ there corresponds $(a, s) \in S\psi$ with $s \leqq t$, this suffices to complete the proof.

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COROLLARY 2.12. *ψ is surjective if and only if S is proper and ρ is the minimum group congruence on S.*

Proof. Suppose that ψ is surjective, and let $t, s \in T$ with $t \leqq s$. As noted prior to Lemma 2.8, $(a, t) \in L$ for some $a \in \mathscr{Y}$. By Corollary 2.9, $(a, s) \in L$. Since $L = S\psi$ by hypothesis, it follows from Theorem 2.11 that $t = s$. Hence each element of T is maximal in T, so that T is a group. As noted in §1, this implies that S is proper and that ρ is necessarily the minimum group congruence on S.

The converse has been proved in [3].

Not every semigroup $L(T, \mathscr{X}, \mathscr{Y})$ is of the form $L(S)$ for some inverse semigroup S. However one can give necessary and sufficient conditions that this should be so.

Finally we note that [3, Theorem 2.7] has an obvious generalisation which we will not state here.

3. Two other embedding theorems.

We now generalise two embedding theorems for proper inverse semigroups (see [5, 6]).

Let $\mathscr{X}$, $\mathscr{Y}$ and T have the properties listed in the statement of Lemma 2.6, and let $L = L(T, \mathscr{X}, \mathscr{Y})$.

Define $\bar{\mathscr{X}}$ to be the set of (non-empty) order-ideals A of $\mathscr{X}$ such that $A \subseteq t(\mathscr{Y} \cap \Delta t)$ for some $t \in T$. For each $a \in \mathscr{X}$, let $\bar{a} = \{b \in \mathscr{X} | b \leqq a\}$.

LEMMA 3.1. *$(\bar{\mathscr{X}}, \cap)$ is a semilattice on which T acts suitably. The map $j: a \mapsto \bar{a}$ is an order-isomorphic embedding of $\mathscr{X}$ in $\bar{\mathscr{X}}$ which preserves the action of T, and $\bar{\mathscr{X}}$ is a conditional $\vee$-completion for $\mathscr{X}j$.*

Proof. Let $s, t \in T$. It is clear that $\mathscr{Y} \cap \Delta t$ is a non-empty order-ideal and subsemilattice of $\mathscr{X}$.

Since ∇t is an order-ideal of $\mathscr{X}$ and since t is an order-isomorphism on Δt, it follows that $t(\mathscr{Y} \cap \Delta t)$ is a subsemilattice and order-ideal of $\mathscr{X}$. Moreover, $\mathscr{Y} \cap t(\mathscr{Y} \cap \Delta t) \neq \Box$.

Let $a \in A \in \bar{\mathscr{X}}$ where $A \subseteq t(\mathscr{Y} \cap \Delta t)$, and $b \in B \in \bar{\mathscr{X}}$, where $B \subseteq s(\mathscr{Y} \cap \Delta s)$. Let $c \in \mathscr{Y} \cap t(\mathscr{Y} \cap \Delta t)$, $d \in \mathscr{Y} \cap s(\mathscr{Y} \cap \Delta s)$. Then $a \wedge c$ exists in $\mathscr{X}$ and lies in $\mathscr{Y} \cap t(\mathscr{Y} \cap \Delta t)$; similarly $b \wedge d$ exists in $\mathscr{X}$ and lies in $\mathscr{Y} \cap s(\mathscr{Y} \cap \Delta s)$. Hence $e = (a \wedge c) \wedge (b \wedge d)$ exists in $\mathscr{X}$ and lies in $\mathscr{Y}$. Since e is a common lower bound of a and b, $e \in A \cap B$. Thus $A \cap B \neq \Box$, and it easily follows that $A \cap B \in \bar{\mathscr{X}}$.

For each $t \in T$, let $\bar{\Delta} t = \{A \in \bar{\mathscr{X}} | A \subseteq \Delta t\}$; for example, $t^{-1}(\mathscr{Y} \cap \Delta t^{-1}) \in \bar{\Delta} t$. Clearly $\bar{\Delta} t$ is an order-ideal of $\bar{\mathscr{X}}$. For each $A \in \bar{\Delta} t$, define tA to be the set $\{ta | a \in A\}$. Then tA is an order-ideal of $\mathscr{X}$ and if $r \in T$ is such that $A \subseteq r(\mathscr{Y} \cap \Delta r)$, then $tA \subseteq tr(\mathscr{Y} \cap \Delta(tr))$. Hence $tA \in \bar{\mathscr{X}}$, and $tA \subseteq \bar{\Delta} t^{-1}$. On the other hand, if $B \in \bar{\Delta} t^{-1}$, then $t^{-1}B \in \bar{\Delta} t$ and $tt^{-1}B = B$. Clearly, therefore, t is an order-isomorphism with domain $\bar{\Delta} t$ and range $\bar{\nabla} t = \bar{\Delta} t^{-1}$. Given $A \in \bar{\mathscr{X}}$ and $s \in T$, $A \in \bar{\Delta}(ts)$ if and only if $A \subseteq \Delta(ts)$; that is, if and only if $A \subseteq \Delta s$ and $sA \subseteq \Delta s^{-1} \cap \Delta t$. It easily follows that T acts suitably on $\bar{\mathscr{X}}$.

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Let $a \in \mathscr{X} = T\mathscr{Y}$. Then $a = sb$ for some $s \in T$ and $b \in \mathscr{Y} \cap \Delta s$. Hence $\bar{a}$ is an order-ideal of $\mathscr{X}$ such that $\bar{a} \subseteq s(\mathscr{Y} \cap \Delta s)$. Let $t \in T$. Then $a \in \Delta t$ if and only if $\bar{a} \in \bar{\Delta} t$, and in this case $\overline{ta} = t.\bar{a}$.

The rest of the result follows from the proof of [**6**, Lemma 1.2].

Following Lemma 3.1, let $A \in \bar{\mathscr{X}}$, where $A \subseteq t(\mathscr{Y} \cap \Delta t)$ say. Then $A \in \bar{\nabla} t$, so that $A = tB$ for some $B \in \bar{\Delta} t$. Hence $\bar{\mathscr{X}} = T\bar{\mathscr{X}}$, and it follows from Lemmas 2.6 and 3.1 that we can form $\bar{L} = L(T, \bar{\mathscr{X}}, \bar{\mathscr{X}})$.

THEOREM 3.2. *The map* $k : (a, t) \mapsto (\bar{a}, t)$ *is an injective homomorphism from* L *into* $\bar{L}$.

Proof. Let $(a, t) \in L$. By Lemma 2.8, $a \in \mathscr{Y} \cap \Delta t^{-1}$ and $t^{-1}a \in \mathscr{Y}$. Hence $\bar{a} \in \bar{\mathscr{X}} \cap \bar{\Delta} t^{-1}$ and $t^{-1}.\bar{a} \in \bar{\mathscr{X}}$. Thus k maps into $\bar{L}$ and clearly it is injective.

Let $(b, s) \in L$. Then

$$\overline{t(t^{-1}a \wedge b)} = t\cdot\overline{(t^{-1}a \wedge b)} = t(\overline{t^{-1}a} \cap \bar{b}) = t(t^{-1}\cdot\bar{a} \cap \bar{b}),$$

and it follows that k is a homomorphism.

By Theorems 2.11 and 3.2, any inverse semigroup S can be embedded in an $L(T, \mathscr{X}, \mathscr{Y})$ and so in $L(T, \bar{\mathscr{X}}, \bar{\mathscr{X}})$. For results of a similar nature, see [**7**, Propositions 1.4 and 3.6].

Now let S be an inverse semigroup with semilattice of idempotents E, let ρ be an I.D. congruence on S, and let $T = S/\rho$. Then $\alpha = \rho^\natural$ is an isotone homomorphism onto T. In this case, following [**4**, Theorem 3], $j : s \mapsto Es$ is an embedding of S into the semigroup

$$M = \{EX | \text{there exists } s' \in S \text{ such that } \square \neq X \subseteq s'\rho\},$$

where the operation on M is set multiplication, and $\beta : EX \mapsto s'\rho$ is a homomorphism from M onto T with $\alpha = j\beta$. Moreover, M is a partially ordered semigroup under inclusion and if $\bar{\rho} = \beta \circ \beta^{-1}$, then each $\bar{\rho}$-class has a maximum element.

Recall that $EW = WE$ for any non-empty subset W of S.

THEOREM 3.3. *M is an inverse semigroup, and inclusion is the natural partial order on M. Moreover, $\bar{\rho}$ is* I.D..

Proof. If $Y \subseteq S$, let $Y^{-1} = \{y^{-1} | y \in Y\}$. In [**9**], Schein showed that

$$C = \{EX | \square \neq X \subseteq S;\ XX^{-1}, X^{-1}X \subseteq E\}$$

is an inverse semigroup under set multiplication, with $Y \in C$ having inverse Y^{-1} (see the Note following Theorem 1 in [**5**]).

If $Y \in M$ then $Y^{-1} \in M$, and it follows that M is an inverse subsemigroup of C.

Suppose $F = EX \in M$, where $\square \neq X \subseteq s\rho$ for some $s \in S$. If $F = F^2$, then $EX = EX^2$. Applying $\rho^\natural$, we deduce that $E\rho^\natural.s\rho^\natural = E\rho^\natural.s^2\rho^\natural$. Hence $s\rho^\natural = s^2\rho^\natural$, so that $s\rho \subseteq E$ since ρ is I.D.. Therefore $F \subseteq E$.

Let $Y, Z \in M$, where $Y = YY^{-1}Z$. By the preceding paragraph $YY^{-1} \subseteq E$, and $EZ = Z$. Hence $Y \subseteq Z$.

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Conversely, if $Y \subseteq Z$, then $Y = YY^{-1}Y \subseteq YY^{-1}Z$, while $Y^{-1}Z \subseteq Z^{-1}Z \subseteq E$; hence $YY^{-1}Z \subseteq Y$. It follows that inclusion is the natural partial order.

If $s \in S$, then $E.s\rho$ is the maximum element in its $\bar{\rho}$-class, and $(E.s\rho)\bar{\rho} = \{EW \mid \square \neq W \subseteq s\rho\}$. Suppose some EW in $(E.s\rho)\bar{\rho}$ is idempotent. Then, as shown above, $W \subseteq E$ and so $s\rho \subseteq E$, since ρ is I.D.. Hence $\bar{\rho}$ is I.D..

Green [2] has shown that there exists a maximum I.D. congruence on an inverse semigroup. Clearly ρ is the maximum I.D. congruence on S if and only if $\bar{\rho}$ is the maximum I.D. congruence on M. As in [5], Sj is the set of V-irreducible elements of M.

Let $a \in S$. In M, $aj = Ea \subseteq E.a\rho$. Since inclusion is the natural partial order on M, by Theorem 3.3, $Ea = Ea.a^{-1}E.a\rho = Eaa^{-1}.a\rho$.

The coordinatisation $a \mapsto (aa^{-1}, a\rho)$ of S can be replaced by $a \mapsto (Eaa^{-1}, a\rho)$, and as seen above the latter has an interpretation in terms of set multiplication.

Let $M(E)$ be the set of non-empty order-ideals of E under set multiplication. As seen in [5], $M(E)$ is a semilattice in which E is embedded by the map $e \mapsto Ee$.

Let $H = E.a\rho$. It follows from Theorem 3.3 that $\phi_{a\rho} : F \mapsto HFH^{-1}$ is an endomorphism of $M(E)$. If $b \in S$, then $Eab(ab)^{-1} = Eaa^{-1} \wedge (Ebb^{-1})\phi_{a\rho}$.

Hence in Green's theory [2, third section] we can replace the endomorphism $\phi(e, t)$ of E by the endomorphism ϕ_t of $M(E)$, where the latter depends on only one parameter. However, it is extremely doubtful if a corresponding reaxiomatisation would present any real gain.

The above considerations generalise part of the theory of [5].

4. The category of *L*-semigroups over an inverse semigroup. In this final section we show that the L-semigroups definable over an inverse semigroup S form a category with initial and terminal object. Since the details are entirely straightforward, they are omitted.

Suppose ρ_1 and ρ_2 are I.D. congruences on S such that $\rho_1 \subseteq \rho_2$. For $i = 1, 2$, given the I.D. congruence ρ_i let $\mathscr{X}_i$, $\mathscr{Y}_i$ and T_i be as defined prior to Lemma 2.6; put $L_i = L(T_i, \mathscr{X}_i, \mathscr{Y}_i)$ and let $\psi_i : S \to L_i$ be the corresponding embedding.

There is induced a unique homomorphism $\eta : T_1 \to T_2$ such that $\rho_1^\natural \eta = \rho_2^\natural$. In turn η defines a map $\mu : \mathscr{X}_1 \to \mathscr{X}_2$ as follows: for $(i, G_i x) \in \mathscr{X}_1$, $(i, G_i x)\mu = (i, (G_i x)\eta)$. Then μ has the following properties:

(i) μ is isotone, $\mathscr{Y}_1\mu \subseteq \mathscr{Y}_2$ and $\mu|\mathscr{Y}_1$ is a semilattice homomorphism; and

(ii) for each $t \in T_1$ and $a \in \Delta t$, $(\Delta t)\mu \subseteq \Delta(t\eta)$ and $(ta)\mu = t\eta . a\mu$.

The maps η and μ define a map $\alpha : L_1 \to L_2$ by

$$, t)\alpha = (a\mu, t\eta),$$

and α is a homomorphism such that $\psi_1\alpha = \psi_2$.

The semigroups L_1 together with the homomorphisms α form the objects and morphisms, respectively, of a category, which we call the category of L-semigroups over S. It has an initial object L_0 corresponding to the minimum I.D. congruence i, and a terminal object L_∞ corresponding to the maximum I.D. congruence τ (see [2]).

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We note that in L_0, $T = S$ and we can take $\mathscr{X} = \mathscr{Y} = E$, where E is the semilattice of idempotents of S. For $t \in S$, $\Delta t = \{e \in E | e \leqq t^{-1}t\}$ and for $e \in \Delta t$, $t.e = tet^{-1}$. Then $L_0 = \{(e, t) | e \leqq tt^{-1}\}$, where in L_0, $(e, t)(f, s) = (etft^{-1}, ts)$.

On the other hand, suppose, for $i = 1, 2$, we are given a poset $\mathscr{X}_i$ having a subsemilattice and order-ideal $\mathscr{Y}_i$ and an inverse semigroup T_i having the properties listed in the statement of Lemma 2.6 and let $L_i = L(T_i, \mathscr{X}_i, \mathscr{Y}_i)$. Let $\eta : T_1 \to T_2$ be a homomorphism and $\mu : \mathscr{X}_1 \to \mathscr{X}_2$ a map satisfying the properties (i) and (ii) above. Then the map $\alpha : L_1 \to L_2$ defined by $(a, t)\alpha = (a\mu, t\eta)$ is a homomorphism. For an analogous characterisation of homomorphisms between P-semigroups, see [3].

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