APPROXIMATE POINT SPECTRUM AND COMMUTING COMPACT PERTURBATIONS

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1. Introduction and preliminaries. Let X be an infinite-dimensional complex Banach space and denote the set of bounded (compact) linear operators on X by $B(X)$ ($K(X)$). Let $\sigma(A)$ and $\sigma_a(A)$ denote, respectively, the spectrum and approximate point spectrum of an element A of $B(X)$. Set

$$\sigma_{em}(A) = \bigcap_{K \in K(X)} \sigma(A + K),$$

$$\sigma_{ea}(A) = \bigcap_{K \in K(X)} \sigma_a(A + K),$$

$$\sigma_{eb}(A) = \bigcap_{\substack{AK=KA \\ K \in K(X)}} \sigma(A + K),$$

$$\sigma_{ab}(A) = \bigcap_{\substack{AK=KA \\ K \in K(X)}} \sigma_a(A + K).$$

$\sigma_{em}(A)$ and $\sigma_{eb}(A)$ are respectively Schechter's and Browder's essential spectrum of A (**[16]**, **[9]**). $\sigma_{ea}(A)$ is a non-empty compact subset of the set of complex numbers $\mathbb{C}$ and it is called the *essential approximate point spectrum of* A (**[13]**, **[14]**). In this note we characterize $\sigma_{ab}(A)$ and show that if f is a function analytic in a neighborhood of $\sigma(A)$, then $\sigma_{ab}(f(A)) = f(\sigma_{ab}(A))$. The relation between $\sigma_a(A)$ and $\sigma_{ab}(A)$, that is exhibited in this paper, resembles the relation between the $\sigma(A)$ and the $\sigma_{eb}(A)$, and it is reasonable to call $\sigma_{ab}(A)$ Browder's essential approximate point spectrum of A.

Throughout this paper $N(A)$ and $R(A)$ will denote respectively the null space and the range space of A. Set $\alpha(A) = \dim N(A)$ and $\beta(A) = \dim X/R(A)$. An operator $A \in B(X)$ is called *semi-Fredholm* if $R(A)$ is closed and at least one of $\alpha(A)$ and $\beta(A)$ is finite. For such an operator A we define an index $i(A)$ by $i(A) = \alpha(A) - \beta(A)$. Let $\Phi_+(X)$ denote the set of semi-Fredholm operators with $\alpha(A) < \infty$, and $\Phi_+^-(X) = \{A \in \Phi_+(X) : i(A) \leq 0\}$. Then $\sigma_{ea}(A) = \{\lambda \in \mathbb{C} : A - \lambda \notin \Phi_+^-(X)\}$ (**[14**, Theorem 3.1]). Let (G_n) be a sequence of compact subsets of $\mathbb{C}$. The limit superior, $\limsup G_n$, is the set of all λ in $\mathbb{C}$ such that every neighbourhood of λ intersects infinitely many G_n. A mapping τ defined on $B(X)$ whose values are compact subsets of $\mathbb{C}$ is said to be upper semi-continuous at A when if $A_n \to A$ then $\limsup \tau(A_n) \subset \tau(A)$ (**[11]**). The polynomial hull $\hat{E}$ of a compact subset E of the complex plane $\mathbb{C}$ is the complement of the unbounded component of $\mathbb{C} \backslash E$. Given a compact subset E of the plane, a *hole* of E is a component of $\hat{E} \backslash E$. If F is another compact set such that $\partial E \subset F \subset E$, it follows that $\partial E \subset \partial F$, $\hat{E} = \hat{F}$ and E can be obtained from F by filling in some holes of F. (Here and in what follows ∂E denotes the boundary of the set E **[15]**.) Finally $a(A)$, the *ascent* of A, is the smallest non-negative integer n such that $N(A^n) = N(A^{n+1})$. If no such n exists, then $a(A) = \infty$.

2. Characterization of $\sigma_{ab}(A)$.

THEOREM 2.1. *$\lambda \notin \sigma_{ab}(A)$ if and only if $A-\lambda \in \Phi_+^-(X)$ and $a(A-\lambda)<\infty$.*

Proof. If $\lambda \notin \sigma_{ab}(A)$, there is a $K \in K(X)$ such that $AK=KA$ and $\lambda \notin \sigma_a(A+K)$. In particular, $A+K-\lambda \in \Phi_+^-(X)$ and $a(A+K-\lambda)=0$. Adding the operator $-K$ to $A+K-\lambda$, we see that $A-\lambda \in \Phi_+^-(X)$ ([**7**, Theorem 5.26 of Chapter IV]) and $a(A-\lambda)<\infty$ ([**3**, Theorem 2]). To prove the converse suppose that $A-\lambda_0 \in \Phi_+^-(X)$ and that $a(A-\lambda_0)<\infty$. If $\lambda_0 \notin \sigma(A)$, then $\lambda_0 \notin \sigma_{ab}(A)$ and the proof is complete. Suppose that $\lambda_0 \in \sigma(A)$. Then λ_0 is an isolated point of $\sigma_a(A)$ ([**10**, Lemma 2.5]). Now $0<\alpha(A-\lambda_0)<\infty$ implies that Kato's number $v(A-\lambda_0:I)$ is finite ([**6**, Theorem 3]). Following Zemánek's method of removing jumping points ([**18**, Theorem 7.1]), while applying Kato's reduction theorem ([**6**, Theorem 4]), we conclude that the space X decomposes into a direct sum of two closed subspaces X_0 and X_1. These subspaces are $(A-\lambda_0)$-invariant, hence A-invariant, and have the following properties: (we quote only those relevant to our problem). The space X_1 is finite dimensional (and $A-\lambda_0$ is nilpotent on it). If A_0 is the restriction of A to X_0 considered as an operator from X_0 into itself then $\alpha(A_0-\lambda)$ is constant on a neighbourhood of λ_0, and hence it is 0. Let F be the finite rank operator defined by $F=I$ on X_1, $F=0$ on X_0. Hence, $AF=FA$, $\alpha(A+F-\lambda_0)=0$ and $A+F-\lambda_0 \in \Phi_+(X)$ ([**7**, Theorem 5.26 of Chapter IV]). Thus, $\lambda_0 \notin \sigma_a(A+F)$ and the proof is complete.

COROLLARY 2.2. *$\lambda \in \sigma_a(A)\backslash\sigma_{ab}(A)$ if and only if λ is an isolated point of $\sigma_a(A)$, an eigenvalue of A of finite multiplicity, $a(A-\lambda)<\infty$ and $R(A-\lambda)$ is closed.*

COROLLARY 2.3. *Let $\lambda \in \sigma_a(A)$ be an isolated point of $\sigma_a(A)$ and let $a(A-\lambda)=\infty$. Then $\lambda \in \sigma_{ea}(A)$.*

Proof. Let λ be an isolated point of $\sigma_a(A)$, $a(A-\lambda)=\infty$ and $\lambda \notin \sigma_{ea}(A)$. Then $0<\alpha(A-\lambda)<\infty$, $R(A-\lambda)$ is closed and Kato's number $v(A-\lambda:I)$ is finite ([**6**, Theorem 3]). Let us apply the operator F from the proof of Theorem 2.1. Then, by [**3**, Theorem 2], $a(A-\lambda)<\infty$, which provides a contradiction. This completes the proof.

COROLLARY 2.4. *$\sigma_{ab}(A)=\sigma_{ea}(A)\cup\{$limit points of $\sigma_a(A)\}$.*

COROLLARY 2.5. *Let $A \in B(X)$. Then*

(i) $\sigma_{ea}(A) \subset \sigma_{ab}(A) \subset \sigma_{eb}(A)$,

(ii) $\partial\sigma_{eb}(A) \subset \partial\sigma_{ab}(A) \subset \partial\sigma_{ea}(A)$,

(iii) $\hat{\sigma}_{ea}(A)=\hat{\sigma}_{ab}(A)=\hat{\sigma}_{eb}(A)$,

(iv) *$\sigma_{ab}(A)$ ($\sigma_{eb}(A)$) can be obtained from $\sigma_{ea}(A)$ ($\sigma_{ab}(A)$) by filling in some holes of $\sigma_{ea}(A)$ ($\sigma_{ab}(A)$).*

(v) *If $\sigma_{ea}(A)$ is connected, $\sigma_{ab}(A)$ is connected, and if $\sigma_{ab}(A)$ is connected, $\sigma_{eb}(A)$ is connected.*

Proof. It is sufficient to prove (ii). Since $\partial\sigma_{eb}(A) \subset \partial\sigma_{em}(A)$ ([**15**, Theorem 1(b)]) and $\partial\sigma_{em}(A) \subset \partial\sigma_{ea}(A)$ ([**13**, Theorem 1]), then $\partial\sigma_{eb}(A) \subset \partial\sigma_{ab}(A)$. Suppose $\lambda_0 \in \partial\sigma_{ab}(A)$ and $\lambda_0 \notin \sigma_{ea}(A)$. Hence, $0<\alpha(A-\lambda_0)<\infty$ and $R(A-\lambda_0)$ is closed. Then there exists an $\varepsilon > 0$ such that $0 < |\lambda_0 - \lambda| < \varepsilon$ implies that $R(A-\lambda)$ is closed and $\alpha(A-\lambda)$ is constant ([**7**, p. 243]). Since $\lambda_0 \in \partial\sigma_{ab}(A)$, we have, by Theorem 2.1, that the constant is 0. Thus, λ_0 is an isolated point of $\sigma_a(A)$, and again by Theorem 2.1 we have that $a(A-\lambda_0) = \infty$. Hence, $\lambda_0 \in \sigma_{ea}(A)$ (Corollary 2.3). This is a contradiction, and the proof is complete.

The following example was used by Salinas ([**15**]) in another context. We use it to show that in general $\sigma_{ea}(A) \neq \sigma_{ab}(A)$.

EXAMPLE 2.6. Let H be a separable Hilbert space, and let V be a unilateral shift of multiplicity one on H; also let $N \in B(H)$ be any quasinilpotent operator. Set $A = V \oplus V^* \oplus N$. If we denote by D the closed unit disc in $\mathbb{C}$ we have $\sigma_{ab}(A) = D$, while $\sigma_{ea}(A) = \partial D \cup \{0\}$.

Proof. Salinas showed that $\sigma_{em}(A) = \partial D \cup \{0\}$ and $\sigma_{eb}(A) = D$. Hence, by ([**13**, Theorem 2.1]) we have that $\sigma_{ea}(A) = \partial D \cup \{0\}$. Suppose $0 < |\lambda| < 1$ and $\lambda \notin \sigma_{ab}(A)$. Then $a(A-\lambda) < \infty$ (Theorem 2.1). Now $\lambda \notin \sigma_{eb}(A)$ ([**16**], [**9**, Theorem 1(4)]). This is a contradiction, and the proof is complete.

From the proof of Theorem 2.1 and Corollary 2.3 we have the following $\Phi_+^-(X)$-version of a T. J. Laffey and T. T. West theorem ([**8**, Proposition 2]).

COROLLARY 2.7. *Let $A \in \Phi_+^-(X)$. Then the following statements are equivalent:*

(i) *$A = V + F$, where $\alpha(V) = 0$, F is finite rank and $VF = FV$;*
(ii) *there exists a finite rank projection P commuting with A such that $\alpha(A \mid N(P)) = 0$;*
(iii) *there exists $\varepsilon > 0$ such that $\alpha(A+\lambda) = 0$ for $0 < |\lambda| < \varepsilon$;*
(iv) *$a(A) < \infty$.*

3. Spectral mapping theorem for $\sigma_{ab}(A)$.

THEOREM 3.1. *If A is any operator and p is any polynomial, then*

$$\sigma_{ab}(p(A)) = p(\sigma_{ab}(A)).$$

Proof. Let $\lambda \notin p(\sigma_{ab}(A))$ and $p(t) - \lambda = c(t-\lambda_1)\dots(t-\lambda_n)$, $c \neq 0$. Thus, $p(A) - \lambda = c(A-\lambda_1)\dots(A-\lambda_n)$, where $A - \lambda_i \in \Phi_+^-(X)$ and $a(A-\lambda_i) < \infty$ for $i = 1, \dots, n$ (Theorem 2.1). Then $p(A) - \lambda \in \Phi_+^-(X)$ ([**17**, Theorem 6.6, Theorem 3.5, Theorem 2.3 of Chapter V]). Let us show that $a(p(A)-\lambda) < \infty$. By ([**5**, Proposition 38.7]) it is sufficient to prove that $p(A) - \lambda$ is injective on the subspace $U = \bigcap_{n=1}^{\infty} (p(A)-\lambda)^n(X)$. We shall use the method of mathematical induction. This is true for $n = 1$. Suppose that this is true for all polynomials of degree $n-1$. Set $p(A) - \lambda = q(A)(A-\lambda_1)$ where $q(t)$ is polynomial of degree $n-1$. Let $x \in U$ and $(p(A)-\lambda)x = 0$. Then $(A-\lambda_1)(q(A)x) = 0$ and $q(A)x \in \bigcap_{n=1}^{\infty} (A-\lambda_1)^n(X)$. Hence, by ([**5**, Proposition 38.7]) $q(A)x = 0$. Since $x \in \bigcap_{n=1}^{\infty} (q(A))^n(X)$ and $q(t)$ is a polynomial of degree $n-1$, we have that $x = 0$. Thus, we see by Theorem 2.1 that $\lambda \notin \sigma_{ab}(p(A))$. This shows that $\sigma_{ab}(p(A)) \subset p(\sigma_{ab}(A))$. We now turn to the proof of the opposite inclusion. Suppose that $\lambda \in p(\sigma_{ab}(A))$ and $\lambda \notin \sigma_{ab}(p(A))$.

Then $p(A)-\lambda \in \Phi_+^-(X)$ and $a\ (p(A)-\lambda)<\infty$ (Theorem 2.1). By ([**4**, p. 20]) $A-\lambda_i \in \Phi_+(X)$ for $i=1,\ldots,n$. Let $\lambda_j \in \sigma_{ab}(A)$ and $\lambda=p(\lambda_j)$. Now, λ is an isolated point of $\sigma_a(p(A))$ (Corollary 2.2) and by ([**4**]) λ_j is an isolated point of $\sigma_a(A)$. Thus, $A-\lambda_j \in \Phi_+^-(X)$ ([**7**, Theorem 5.22 of Chapter IV]), and by Corollary 2.3 we have $a(A-\lambda_j)<\infty$. Again, by Theorem 2.1, we have that $\lambda_j \notin \sigma_{ab}(A)$, which provides a contradiction. This completes the proof.

To show that if f is an analytic function defined on a neighbourhood of $\sigma(A)$, then $f(\sigma_{ab}(A))=\sigma_{ab}(f(A))$ we shall apply K. Oberai's method ([**12**]). First we shall prove two following statements which are of particular interest.

THEOREM 3.2. *Let $A \in B(X)$. Then the mapping $A \to \sigma_{ab}(A)$ is upper semi-continuous.*

Proof. Let $A_n \to A$. We have to show that $\limsup \sigma_{ab}(A_n) \subset \sigma_{ab}(A)$. It is enough to show that if $0 \notin \sigma_{ab}(A)$, then $0 \notin \limsup \sigma_{ab}(A_n)$. Let $0 \notin \sigma_{ab}(A)$. Then 0 is an isolated point of $\sigma_a(A)$ and $A \in \Phi_+^-(X)$ (Theorem 2.1). By ([**7**, Theorem 5.22 of Chapter IV]) there exists an $\varepsilon>0$ and an integer n_1 such that $A_n-\lambda \in \Phi_+^-(X)$ for $|\lambda|<\varepsilon$ and for $n \geq n_1$. We may assume that $\alpha(A-\lambda)=0$ for $0<|\lambda|<\varepsilon$. Let $n \geq n_1$. Hence, by ([**7**, p. 243]) $\alpha(A_n-\lambda)$ are constant for all $|\lambda|<\varepsilon$ except for an isolated set. Let $0<|\lambda_0|<\varepsilon$. Set $S_0=\{\lambda \in \mathbb{C}:|\lambda|=|\lambda_0|\}$ and $m(A)=\inf\{\|Ax\|:\|x\|=1\}$. Now, $m(A-\lambda)>0$ for $\lambda \in S_0$. Since $m(A-\lambda)$ is a continuous function of λ ([**2**, p. 19]) and S_0 is compact, there exists a $\mu_0 \in S_0$ such that $m(A-\mu_0)=\inf\{m(A-\lambda):\lambda \in S_0\}$. Let n_2 be an integer such that for $n \geq n_2$ we have $\|A_n-A\|<m(A-\mu_0)$. Hence, for $n \geq n_0=\max(n_1,n_2)$ and $\lambda \in S_0$ we have $m(A-\lambda)-\|A-A_n\| \leq m(A_n-\lambda)$ ([**2**, Lemma 2.2]). Thus, $m(A_n-\lambda)>0$ and $\alpha(A_n-\lambda)=0$ for all $|\lambda|<\varepsilon$ except for an isolated set and for $n \geq n_0$. Therefore, for $n \geq n_0$ we see that $\sigma_{ab}(A_n) \cap \{\lambda \in \mathbb{C}:|\lambda|<\varepsilon\}$ is empty (Corollary 2.3 and Theorem 2.1). Thus we have $0 \notin \limsup \sigma_{ab}(A_n)$, and the proof is complete.

THEOREM 3.3. *Let $A \in B(X)$ and let f be an analytic function defined on a neighbourhood of $\sigma(A)$. Then*

$$\sigma_{ea}(f(A)) \subset f(\sigma_{ea}(A)).$$

Proof. $\Phi_+(X)$ is an F-semigroup with index i ([**4**, p. 20]). Also

$$\sigma_{ea}(A)=\sigma_{e\alpha}(A) \cup \left(\bigcup_{n\geq 1} F_n\right), \tag{3.1}$$

where $\sigma_{e\alpha}(A)=\{\lambda \in \mathbb{C}:A-\lambda \notin \Phi_+(X)\}$, $F_n=\{\lambda \in \sigma(A):A-\lambda \in \Phi_+(X),\ i(A-\lambda)=n\}$. Now suppose that $\mu \notin f(\sigma_{ea}(A))$. Then $\mu-f(\lambda)$ has no zeros on $\sigma_{ea}(A)$ and in particular has no zeros on $\sigma_{e\alpha}(A)$. Applying ([**4**, Theorem 1]) we conclude that $\mu-f(A) \in \Phi_+(X)$ and

$$i(\mu-f(A))=\sum_n n\alpha_n,$$

where α_n is the number of isolated zeros of $\mu-f(\lambda)$ on F_n counted according to their multiplicities. From (3.1) it follows that $\alpha_n=0$ for $n \geq 1$. Thus $i(\mu-f(A)) \leq 0$, which implies that $\mu \notin \sigma_{ea}(f(A))$. This completes the proof.

THEOREM 3.4. *Let $A \in B(X)$ and let f be an analytic function defined on a neighbourhood of $\sigma(A)$. Then*

$$f(\sigma_{ab}(A)) = \sigma_{ab}(f(A)).$$

Proof. Let $(p_n(t))$ be a sequence of polynomials converging uniformly to $f(t)$ on a neighbourhood of $\sigma(A)$. We have

$$\begin{aligned} f(\sigma_{ab}(A)) &= \lim p_n(\sigma_{ab}(A)) && (\text{by ([}\mathbf{12}\text{, p. 370])}) \\ &= \lim \sigma_{ab}(p_n(A)) && (\text{by Theorem 3.1}) \\ &\subset \sigma_{ab}(f(A)) && (\text{by Theorem 3.2}). \end{aligned}$$

To prove the converse suppose that $\lambda \in \sigma_{ab}(f(A))$.

Case I. $\lambda \in \sigma_{ea}(f(A))$. By Theorem 3.3 $\lambda \in f(\sigma_{ea}(A))$. Thus, we have $\lambda \in f(\sigma_{ab}(A))$.

Case II. $\lambda \notin \sigma_{ea}(f(A))$. In this case λ is a limit point of $\sigma_a(f(A))$ (Corollary 2.4), and there exists a sequence $(\lambda_n) \subset \sigma_a(f(A))$ such that $\lambda_n \to \lambda$. Now, there exists a sequence $(\mu_n) \in \sigma_a(A)$ such that $f(\mu_n) = \lambda_n \to \lambda$ ([**4**]). Then (μ_n) contains a convergent subsequence and we may assume that $\lim \mu_n = \mu \in \sigma_a(A)$. Then $\lambda = f(\mu) \in f(\sigma_{ab}(A))$. This completes the proof of the theorem.

4. A perturbation theorem.

THEOREM 4.1. *Let $A \in B(X)$ and let $N \in B(X)$ be a quasinilpotent operator commuting with A. Then $\sigma_{ab}(A+N) = \sigma_{ab}(A)$.*

Proof. It is enough to show that if $0 \notin \sigma_{ab}(A)$, then $0 \notin \sigma_{ab}(A+N)$. Let $0 \notin \sigma_{ab}(A)$. If $0 \notin \sigma_a(A)$, then $0 \notin \sigma_a(A+N)$ ([**1**, p. 320]). Hence, we have that $0 \notin \sigma_{ab}(A+N)$. If $0 \in \sigma_a(A)$, then 0 is an isolated point of $\sigma_a(A)$ (Corollary 2.2), and therefore 0 is an isolated point $\sigma_a(A+N)$ ([**1**, p. 320]). Since $0 \notin \sigma_{ab}(A)$, we see that $0 \notin \sigma_{ea}(A+N)$ ([**19**, Theorem 7]). This implies that $0 \notin \sigma_{ab}(A+N)$ (Corollary 2.3), and the proof is complete.

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REFERENCES

1. M. D. Choi and C. Davis, The spectral mapping theorem for joint approximate point spectrum, *Bull. Amer. Math. Soc.* **80** (1974), 317–321.

2. H. A. Gindler and A. E. Taylor, The minimum modulus of a linear operator and its use in spectral theory, *Studia Math.* **22** (1962), 15–41.

3. S. Grabiner, Ascent, descent, and compact perturbations, *Proc. Amer. Math. Soc.,* **71** (1978), 79–80.

4. B. Gramsh and D. Lay, Spectral mapping theorems for essential spectra, *Math. Ann.,* **192** (1971), 17–32.

5. H. G. Heuser, *Functional analysis* (Wiley-Interscience, 1982).

6. T. Kato, Perturbation theory for nullity, deficiency and other quantities of linear operators, *J. Analyse Math.,* **6** (1958), 261–322.

7. T. Kato, *Perturbation theory for linear operators* (Springer-Verlag, 1976).

8. T. J. Laffey and T. T. West, Fredholm commutators, *Proc. Roy. Irish Acad. Sect. A,* **82** (1982), 129–140.

9. D. Lay, Characterizations of the essential spectrum of F. E. Browder, *Bull. Amer. Math. Soc.,* **74** (1968), 246–248.

10. D. Lay, Spectral analysis using ascent, descent, nullity and defect, *Math. Ann.,* **184** (1970), 197–214.

11. K. K. Oberai, On the Weyl spectrum, *Illinois J. Math.,* **18** (1974), 208–212.

12. K. K. Oberai, Spectral mapping theorem for essential spectra, *Rev. Roumaine Math. Pures Appl.,* **25** (1980), 365–373.

13. V. Rakočević, On one subset of M. Schechter's essential spectrum, *Mat. Vesnik,* **5** (1981), 389–391.

14. V. Rakočević, On the essential approximate point spectrum II, *Mat. Vesnik,* **36** (1984), 89–97.

15. N. Salinas, Operators with essentially disconnected spectrum, *Acta Sci. Math. (Szeged),* **33** (1972), 193–205.

16. M. Schechter, On the essential spectrum of an arbitrary operator I, *J. Math. Anal. Appl.,* **13** (1966), 205–215.

17. M. Schechter, *Principles of functional analysis* (Academic Press, Student edition, 1973).

18. J. Zemánek, Geometric characteristic of semi-Fredholm operators and their asymptotic behaviour, Institute of Mathematics, Polish Academy of Sciences, preprint 290 (1983).

19. J. Buoni, R. Harte and T. Wickstead, Upper and lower Fredholm spectra, *Proc. Amer. Math. Soc.,* **66** (1977), 309–314.

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