DYNAMICAL EVOLUTION OF GLOBULAR CLUSTERS AFTER CORE COLLAPSE

Douglas C. Heggie
University of Edinburgh
Department of Mathematics
King's Buildings
Edinburgh EH9 3JZ, U.K.

ABSTRACT. This review describes work on the evolution of a stellar system during the phase which starts at the end of core collapse. It begins with an account of the models of Hénon, Goodman, and Inagaki and Lynden-Bell, as well as evaporative models, and modifications to these models which are needed in the core. Next, these models are related to more detailed numerical calculations of gaseous models, Fokker-Planck models, N-body calculations, etc., and some problems for further work in these directions are outlined. The review concludes with a discussion of the relation between theoretical models and observations of the surface density profiles and statistics of actual globular clusters.

1. INTRODUCTION

At the last I.A.U. conference which dealt at length with the dynamics of clusters, Hénon (1975) briefly described two new Monte Carlo models. The evolution with time of the central density in his equal-mass model (published here for the first time, with Dr. Hénon's kind permission) is shown in Fig. 1. The rise in central density during the first half of the calculation reflects the phenomenon of core collapse, described in this volume in the paper by Prof. Spitzer. This paper is concerned with the evolution of models of stellar systems during the phase following the maximum central density.

That the problem is a significant one is suggested by two arguments. First, during the collapse phase, when ρ is increasing, the time until maximum ('complete core collapse') is of the order of a hundred times the current central relaxation time (Spitzer & Shull 1975b). Since the inferred central relaxation times in a few clusters are as short as 10^7 yr (Peterson & King 1975), such clusters are predicted to reach complete collapse in the next 10^9 yr or so. Hence it is reasonable to suppose that some clusters have already done so (cf. Lightman 1982). Second, there is increasing observational evidence that the central regions of some clusters do not fit the single-component, lowered-Maxwellian models which are often used,

J. Goodman and P. Hut (eds.), Dynamics of Star Clusters, 139–160.

https://doi.org/10.1017/S0074180900147357 Published online by Cambridge University Press

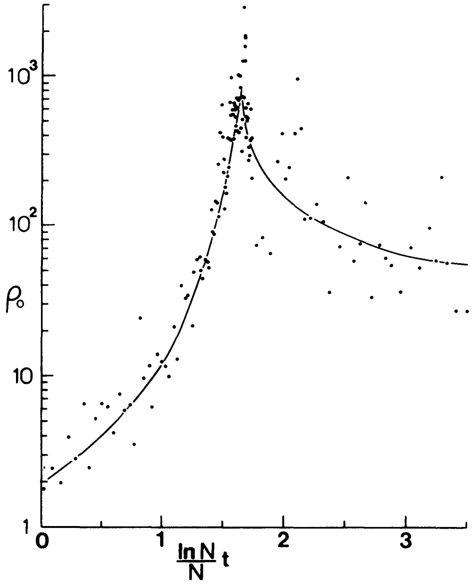

Figure 1. Evolution of central density, ρ_0, versus time, t, for Hénon's equal-mass model (Hénon 1975). Units are such that G = 1, M = 1 and E = -1/4, where G is the constant of gravitation, and M and E are total mass and energy, respectively. N is the number of stars in the system. The continuous curve was drawn by eye.

and it has been suggested that such objects may already have collapsed (Heggie 1980, Djorgovski & King 1984).

This review is mainly concerned with describing theoretical models for post-collapse evolution, and attempting to show how they fit together and how they can be related to more detailed simulations. The models are generally highly idealised and in the concluding sections we discuss avenues for future research in the direction of greater realism, and also the still tenuous link with observations.

2. MODELS

2.1 Hénon's models

The study of post-collapse evolution, as we would now call it, began with the long paper by Hénon (1961). He made several simplifications in the Fokker-Planck equation: (i) isotropic velocity distribution, (ii) self-similar evolution and (iii) a single mass component (though in other parts of his paper several mass components were considered). In searching for a solution he found it necessary to relax any assumption that the central density was finite, and finally obtained a solution behaving near the centre like the singular isothermal model, with velocity dispersion varying with time, and with a flux of energy from the central singularity. Hénon thought, with some support from

N-body simulations (von Hoerner 1960), that the energy in a real system might be supplied by the formation and evolution of binary stars.

Hénon devised two models, one tidally limited and the other (Hénon 1965) isolated. For the latter the density distribution is given in Fig. 2. The *form* of the time-evolution can be found by

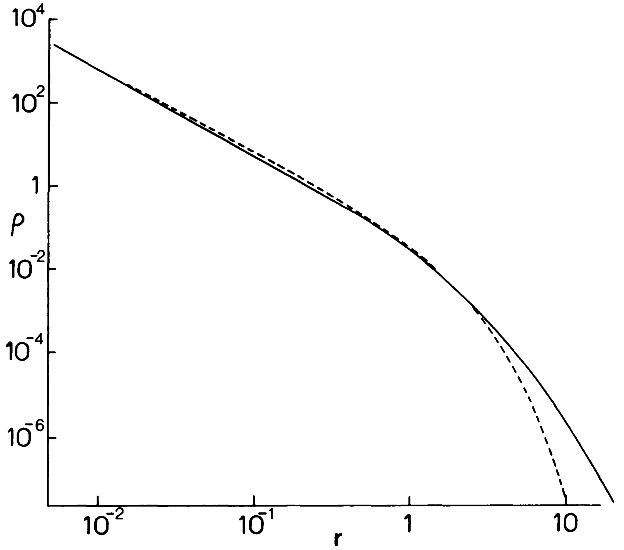

Figure 2. Density distribution (log ρ versus log r, where r is distance from the centre) for Hénon's isolated model (solid line) and Goodman's model (dashed line). Units are as in Fig. 1 for Hénon's model, (though the energy of the system changes with time), and Goodman's model has been scaled to the same total mass and the same velocity dispersion near the centre.

setting the elapsed time t equal to a typical relaxation time, and supposing that the total mass M is constant. Numerical values are given by Hénon, and yield, for example, the result

$$v_o^2(t) = v_o^2(0)\left(1 + 4.15\frac{v_o^3(0)}{GM}\frac{\ell n\ N}{N}t\right)^{-2/3}, \tag{1}$$

where v_o^2 is the central one-dimensional velocity dispersion.

2.2 Goodman's model

In the past few years it has become quite common to approximate a star cluster by a self-gravitating gas (Hachisu *et al* 1978) with a coefficient of heat conduction designed to mimic the transport of energy by relaxation (Lynden-Bell & Eggleton 1980). This was used by Goodman (1984) for further development along the lines of Hénon's models. In particular he investigated the effect of the loss of mass which must accompany the emission of energy if binaries are responsible for this (Ozernoy & Dokuchaev 1982, Dokuchaev & Ozernoy 1982).

In many respects the results change only slightly. For instance, close to the centre the solution again deviates very little from the singular isothermal model. On the other hand, Fig. 2 shows that the outer parts differ substantially from the corresponding result of Hénon's model, although Goodman points out that the physical assumptions on which the two models are constructed may be expected to break down in different ways in the envelope of the cluster. For the dependence of the central velocity on time Goodman finds

$$v_o^2(t) = v_o^2(0)\left(1 + 11.9A\,\frac{v_o^3(0)}{GM(0)}\,\frac{1}{N(0)}\,t\right)^{-\frac{2}{3}(1+2\nu)} \qquad (2)$$

where A,ν are constants. In fact ν measures the rate of mass-loss, the favoured value being of the order of .0125. The constant A appears in the coefficient of thermal conductivity, and Goodman chooses $A = 0.31\ \ell n\ (0.4N)$ to ensure that the gaseous model evolves at the same rate as the isotropised Fokker-Planck model (Cohn 1980) in the homologous part of the *collapse* phase. (This is equivalent to the choice C = .104 of a corresponding constant in the formulation of Lynden-Bell & Eggleton (1980); the value C = .072 found by Inagaki & Lynden-Bell (1983), again by comparison with Cohn's result, is based on an incorrect expression for the relaxation time used by him.) Using Goodman's value for A, and comparing (1) and (2), we see that, in the limit of large N, the gaseous model evolves at a quite similar rate to the Fokker-Planck model.

2.3 The model of Inagaki & Lynden-Bell

The models of Hénon and Goodman have finite mass and energy, but exhibit two singularities. One, already referred to, is the central singularity in density and luminosity. But (1) and (2) also reveal a singularity at some time in the past, when the central velocity dispersion becomes infinite, and the scale radius of the models vanishes. What one would like, by contrast, is a model which joins on smoothly in the past to the density and velocity profile left behind by the core collapse. This is what is accomplished by the model of Inagaki and Lynden-Bell (1983). It evolves forward from an instant at which the density profile everywhere is the power-law profile $\rho = Ar^{-\alpha}$, where A is a constant and $\alpha \simeq 2.208$, left behind after core collapse in gaseous models (Lynden-Bell & Eggleton 1980).

The model still has an isothermal singularity at the centre (Fig. 3). It also has infinite mass and energy, and can therefore be applicable only to the inner parts of a system. The variation with time of the central velocity dispersion is given by

$$v_o^2(t) = v_o^2(0)\left(1 + 1.2\,\frac{G^2 m\rho_*(0)\,\ell nN}{v_*^3(0)}\,t\right)^{-2\,\frac{\alpha-2}{6-\alpha}}, \qquad (3)$$

https://doi.org/10.1017/S0074180900147357 Published online by Cambridge University Press

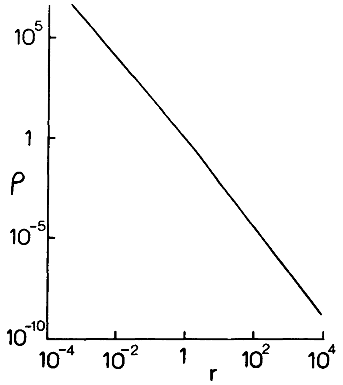

Figure 3. Density distribution in the model of Inagaki & Lynden-Bell. The variables used are the authors' homology variables ρ^*, r^*, which is equivalent (if G = 1) to scaling so that $v_o^2 = 2\pi$ and $L_o = 12\sqrt{2}\pi^3 m C \ln N \times (\alpha - 2)$, where L_o is the central flux of energy, m is stellar mass, C is discussed in § 2.2, and $\alpha \approx 2.21$. (For Hénon's model, in the scaling of Fig. 2, the corresponding values are $v_o^2 = 0.400$, $L_o = .184\, m \ln N$.) The asymptotic slope is −2 (small r), and −2.21 (large r).

where the suffix * indicates quantities evaluated at the edge of the isothermal region, defined as the radius where $v_* = v_o \exp(-0.1)$. The scale radius increases with time and the scale density decreases, in such a way that the outer part of the system, where $\rho \propto r^{-\alpha}$, does not change. Thus one can think of the isothermal region, where $\rho \propto r^{-2}$, eating outward through the halo.

2.4 The central density

Maintenance of the strict self-similarity of any of the models discussed so far is dependent on the emission of energy at a precisely determined rate (see caption to Fig. 3). Besides the lack of realism in supposing that all the energy is emitted at one point, it is inconceivable that any given mechanism would emit precisely the amount of energy required. But it is reasonable (up to a point; cf. § 4.1 below) to suppose that energy is emitted at different places at a rate depending on local values of ρ and v. Under suitable conditions it is then found that the central parts of the isothermal region may be non-singular, the central density adjusting itself so that the core (i.e. the region where the density is not far from its central value) altogether emits energy at the rate that is required for the self-similar evolution of the above models. Further details on how the central density may be determined have been given elsewhere (Heggie 1984).

Two general consequences of this argument should be mentioned. In the first place the evolution is no longer self-similar, because in general the radius, r_*, of the edge of the isothermal region will evolve at a different rate from the core radius, r_c, defined by

$$r_c^2 = \frac{9v_o^2}{4\pi G\rho_o} \tag{4}$$

(which agrees with the definition in King 1966 when the concentration parameter is large). Second, for a wide range of idealised energy production mechanisms, the rate at which the central density decreases in post-collapse evolution is much smaller than the rate of increase during collapse; while the density (ρ_*) at the edge of the isothermal region in post-collapse evolves at much the same rate as the central density (ρ_o) in the collapse phase (Inagaki & Lynden-Bell 1983); *after* collapse the central density ρ_o lags further and further behind ρ_*. Similar remarks may be made about the reexpansion of the core radius and the core mass, but the central velocity dispersion *does* evolve on much the same time-scale in both collapse and post-collapse phases.

2.5 Evaporative models

The evaporative model of core evolution, which assumes that the core can be described by two time-dependent parameters such as mass and radius (see, e.g., Lightman & Shapiro 1978), has proved a useful tool for the study of core collapse. The core evolution is implicitly assumed to be entirely homologous, the structure varying from one time to another by variations in the two parameters. Before it was realised that post-collapse evolution was essentially non-homologous (unless discussion is restricted to regions well *outside* the core), a number of simple generalisations of the model had been proposed which dealt with post-collapse evolution. Some discussions (Hills 1975, Alexander & Budding 1979, Dokuchaev & Ozernoi 1982) were concerned with binaries, and those by Shapiro (1977) and Duncan & Shapiro (1982) with a cluster containing a central massive black hole. The latter problem will be discussed below (§ 3.2), and the further remarks here will be confined to the discussions involving binaries.

In Hills' work it was assumed that the fraction of binaries, f_b, was constant and uniform, and the rate at which they emitted energy varied essentially as

$$\dot{E} \sim \frac{f_b E}{T_R \ln N} \tag{5}$$

where E is the energy of the cluster and T_R is a relaxation time. Now in the evaporative model, and in models where the assumption of self-similarity after collapse is *not* made (Heggie 1984), core collapse ends when $\dot{E}$ becomes comparable with the rate at which energy is exchanged by two-body relaxation, which varies as E/T_R. Thus in Hills' model the arrest of core collapse depends crucially on the variation of $\ln N$ in (5). In fact it seems likely that variations in f_b would be far more important.

Perhaps the strongest indication that the simplest evaporative

https://doi.org/10.1017/S0074180900147357 Published online by Cambridge University Press

models are inappropriate when significant heating by binaries is included is the equation

$$\dot{N} = -\frac{\alpha N}{T_R}$$

for N, interpreted as the number of stars in the core, where α is a constant. This forces the number of stars in the core to decrease, contrary to the findings of more detailed models and simulations.

It is convenient to mention here some models by Retterer (1984), though they are much more elaborate than evaporative ones, especially in taking account of the detailed distribution of the binding energies of the binaries. Even the cluster is described by an additional parameter (which can be thought of as the central potential). Unfortunately, the method fails when the central potential becomes too large, and only small systems could be followed past core collapse.

2.6 Synthesis of the models

Something has been said already about the regimes, in space and time, to which the validity of the various models may be restricted. Now we shall attempt to show how in principle they may be put together to provide a relatively complete semi-quantitative description of post-collapse evolution, in much the same way that the evaporative model covers the collapse phase.

We suppose that, at the end of core collapse, the outer parts of the cluster (well beyond the *initial* core radius) have not changed very much. At the very centre is an isothermal region with a small but finite core radius, and between here and the outer regions is the power law profile $\rho \propto r^{-\alpha}$. (Numerical experiments indicate that the constant of proportionality can be fixed to about a factor of two by supposing that the profile joins smoothly onto the initial profile at the point with the same logarithmic slope.)

During the first phase of the post-collapse evolution, the edge of the isothermal region gradually expands through the power-law profile, and the central density and core radius adjust themselves as described in § 2.4. The central velocity dispersion varies as in eq. (3), and, after a time comparable to the duration of the collapse phase, reduces to its initial value; however the central density is now much greater than it was originally. Between the current core radius and the original core radius is a large region with approximately the r^{-2} isothermal profile, and well beyond the original core radius the density profile has still not greatly altered from its original appearance. (Numerical simulations, described in § 3.1 below, indicate that densities may be higher by a factor of up to 10).

Next, at the beginning of the second phase of post-collapse evolution, the structure of the system approaches that of the Hénon model described in § 2.1 (or Goodman's, § 2.2), though the central density adjusts according to the discussion of § 2.4. Thus, outside the current core radius is a large region with the isothermal r^{-2} density profile, which at larger radii gives way to one of the limiting forms

https://doi.org/10.1017/S0074180900147357 Published online by Cambridge University Press

$\ln\rho \sim -\frac{9}{4}\ln(r/\alpha_1) - (r/\alpha_1)^{1/2}$ (for Hénon's isolated model) or

$\ln\rho \sim -2\ln(r/\alpha_2) - r/\alpha_2$ (for Goodman's model), where α_1, α_2 are radius scales (Goodman 1983). The central velocity dispersion evolves as in eq. (1) or (2). If, as we have supposed, the system is isolated, then it expands indefinitely at an ever diminishing rate.

3. SIMULATIONS

So far we have reviewed a number of models for the post-collapse evolution of star clusters. The solution of the corresponding equations generally requires some numerical work, but at least the models have a kind of universal significance independent of special initial conditions. We now turn to simulations, which often permit the inclusion of more realistic detail than the idealized models, and also require the specification of initial conditions and, generally, much more numerical effort. We discuss these simulations roughly in order of increasing elaboration.

3.1 Gaseous Models

These are the most idealized types of formulation, and also the easiest. However, there are serious disagreements between those of Bettwieser *et al* (Bettwieser & Sugimoto 1984, Bettwieser & Fritze 1984) and the present author. The former authors find that under quite a wide range of circumstances the post-collapse evolution is oscillatory, rather resembling the possibility of repeated core collapse previously considered by Fall & Malkan (1978). The present author, on the other hand, observes only a steady expansion in the post-collapse phase, even though models have been computed with the same equations, initial conditions, boundary conditions and energy generation as those of Bettwieser *et al*. There are differences in the scaling, the variables, and the numerical methods, and it is probably in one of these that the disagreement originates. Since the oscillating models are described by Bettwieser and by Sugimoto elsewhere in this volume, the following remarks will be confined to a recent calculation by the author. They must be regarded as tentative as long as the source of the disagreement remains undiscovered.

The initial conditions are a Plummer model, and units are those of Fig. 1. The system is surrounded by an impermeable enclosure of radius 20 times the half-mass radius. Energy is generated at a rate

$$\varepsilon \propto \rho^2 v^{-1} \qquad (6)$$

per unit mass, a choice which was made for the purposes of comparison with the results of Bettwieser & Sugimoto. Details of calculations with a different choice of initial conditions and other assumptions on the energy generation are given in Heggie 1984. Our purpose here is to illustrate some of the statements in § 2.6.

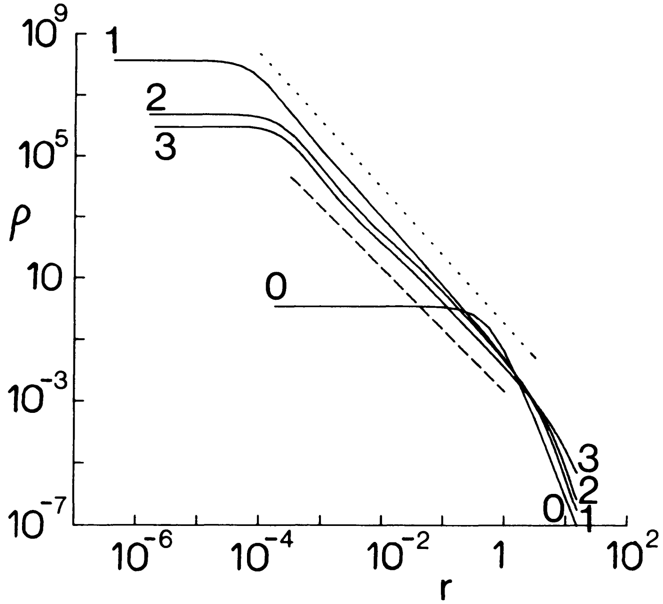

Figure 4. Density profiles log ρ against log r at different times in the evolution of a gaseous model. Initial and boundary conditions and units are stated in the text. The times of the profiles 0, 1, 2, 3 are given by $C \frac{\ln N}{N} t = 0$, .167, .305, 1.111 . The dotted and dashed straight lines have logarithmic slope -2.21, -2 respectively.

Some density profiles are shown in Fig. 4, where profile 0 is the initial Plummer model. Close to the time of maximum central density (profile 1) the core contains some 10^{-4} of the total mass (a value governed by the choice of the constant of proportionality in (6)). It is surrounded by an extensive halo with logarithmic slope approximately -2.21, and beyond that is a steeper envelope still reflecting initial conditions. (One difference between gaseous models and Monte-Carlo simulations is that in the former there appears to be no sign of the formation of an envelope with logarithmic slope -3.5, contrary to what was found by Spitzer & Shull 1975a.) After a comparable time interval (profile 2) the central velocity dispersion has reduced to slightly below its initial value, but the central density is still very high, and the logarithmic slope of the bulk of the profile has changed from -2.21 to approximately -2. The slight differences between this profile and profile 3, taken much later, show how slow the evolution of the system becomes.

The quantitative rate of evolution of the system can be judged to some extent from the dependence of the central velocity dispersion on time (Fig. 5). After the maximum value of v_o is attained the dependence soon settles down to a form like (3), except that the coefficient must be reduced from 1.2 to about 0.6, and for α we have taken the value 2.22, instead of the value 2.208 of Lynden-Bell & Eggleton.

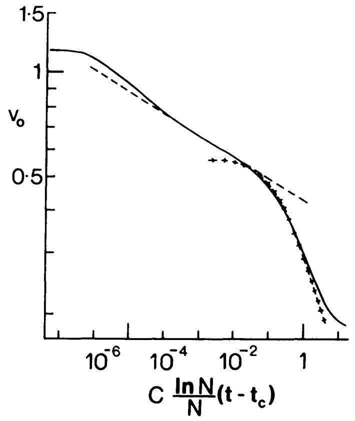

Figure 5. Dependence of root mean square central velocity v_o against time, given as $C\,\frac{\ell n\ N}{N}\,(t - t_c)$, where t_c is the time of maximum value of v_o. The scaling is as in Fig. 1, and the two theoretical curves are explained in the text.

(The larger value of α is consistent with the evolution of the model during core collapse, and the difference may be due to the relatively coarse time-step used in this calculation, but the reason for the discrepancy in the coefficient is unknown. Eq. (3), modified in this way, is plotted as the straight dashed line in Fig. 5. Note, however, that the origin of time is different in Fig. 5 and eq. (3).) The simulations begin to deviate again from this theoretical relation when v_o reduces to approximately the value it had at the beginning of core collapse, and thereafter for a period it follows quite closely eq. (2), which is the curved theoretical relation in Fig. 5. (Here we have set $\nu = 0$, since there is no mass-loss in the simulation, and also the origin of t has been chosen to optimise the fit; this was done by plotting v^{-3} against t. Such optimization is justifiable, since there is no reason to suppose that the time at which Goodman's solution becomes singular will equal the time of maximum v_o in the simulation.) Thereafter, the rate of evolution slows down as the entire system within the enclosing wall becomes more and more nearly isothermal. By this time it is evident that the boundary conditions are playing an important role in the structure of the system.

3.2 Fokker-Planck Models

It was Hénon (1975) who conjectured that the details of the mechanism of energy-generation did not matter, within wide limits, as far as the evolution of the rest of the system is concerned, though it is now realized that this can only apply to regions of the system outside the core. Hénon's conjecture was also confirmed by Stodółkiewicz (1982) in several Monte-Carlo solutions of the Fokker-Planck equation, modified in several different (but artificial) ways so as to avoid the central singularity which would otherwise develop. While gross features of the density profile and its evolution (at and beyond the

https://doi.org/10.1017/S0074180900147357 Published online by Cambridge University Press

innermost 10% of the mass) are virtually unaffected by the nature of the singularity, the escape of stars may be influenced by it.

Hénon (1975) also gave some results of two Monte-Carlo simulations with an artificial source of energy, and two density profiles, respectively early and late in the post-collapse phase, are shown in Fig. 6. These make the qualitative point that post-collapse evolution in this model is relatively very slow.

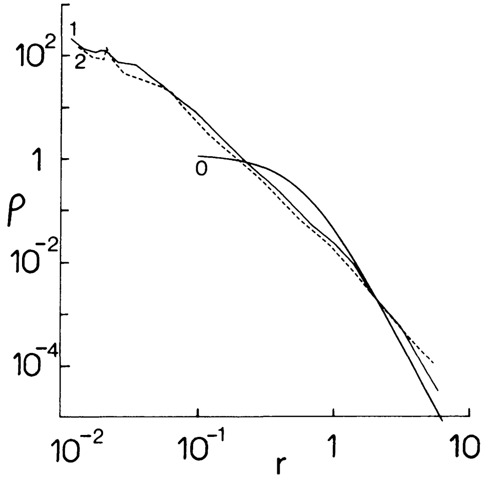

Figure 6. Density profiles, log ρ against log r, in Hénon's equal-mass model. The times of the Monte-Carlo profiles (1 and 2) are given by $\frac{\ell n\ N}{N}\ t = 1.73$ and 2.66, respectively, and the units used are those of Fig. 1. (Unpublished data presented here by kind permission of Dr. Hénon.) Large parts of the earlier profile can be fitted by a power law of the form $\rho \propto r^{-2.44}$. Also shown (0) is the analytic profile corresponding to the initial conditions.

We now come to solutions of the Fokker-Planck equations in which the artificial energy sources discussed so far are replaced by more realistic processes, in particular the formation and evolution of binary stars (Stodółkiewicz 1983; Cohn, this volume), or the presence of a central black hole (Marchant & Shapiro 1980, Duncan & Shapiro 1982). This work of Stodółkiewicz contains many important clues to the complicated behaviour of binaries, such as: the spatial distribution and numbers of binaries, the importance of binary-binary reactions when the total number is not small, the very restricted size of the region within which energetic interactions take place, the importance of the loss of mass by escaping binaries (as in Goodman's model) and so on. Cohn's calculations are more idealized, but show very beautifully the homological form of post-collapse expansion of the Hénon model.

The emission of energy by a central black hole is, in some respects,

https://doi.org/10.1017/S0074180900147357 Published online by Cambridge University Press

a much simpler process, in that the energy flux can be given fairly accurately as a relatively simple function of parameters in the core. This permits some fairly detailed comparison (Heggie 1984) between the simple models outlined in §§ 2.3-4 and the Monte-Carlo simulations, thanks also to the significant amount of quantitative information published by Duncan & Shapiro.

These Monte-Carlo results have been interpreted also in terms of a simple extension of the evaporative model for core evolution, in which the core is characterized by its radius and central density. But now there is an additional equation for the mass of the black hole, and additional terms for the fluxes of mass and energy which it produces (Shapiro 1977) and for mass encompassed by the core as it reexpands (Duncan & Shapiro 1982; see also McMillan, Lightman & Cohn 1981). With a suitable (and reasonable) choice of a free parameter which this last effect requires, good agreement with the growth of the black hole is obtained. Consistency with the evolution of the core parameters is less satisfactory, probably because of the homological assumption implicit in the model. The considerations of §2.4 account for these aspects of the simulations more successfully.

Density profiles are given by Marchant & Shapiro (1980), but the post-collapse phase was followed for only about 10% of the total collapse time. During this early post-collapse phase, the cusp around the black hole, with logarithmic slope -1.75, eats out through the steeper profile left behind by core collapse. This would be difficult to distinguish from the isothermal (slope -2) profile of the models discussed in § 2.

3.3 Hybrid Simulations

Very simple considerations (e.g. Lightman & Shapiro 1978) suggest that a single binary can emit energy at a rate comparable to the energy flux in core collapse, if the core contains around 100 stars. Furthermore, numerical simulations confirm that such considerations do correctly predict the conditions under which collapse ends and re-expansion begins (Duncan & Shapiro 1982, Heggie 1984). Therefore, the conditions for validity of the Fokker-Planck equation (especially, the assumption that N is large) are violated at the beginning of the post-collapse phase. Furthermore, since reexpansion of the core is much slower than its collapse (if the mechanisms of energy generation that have been studied are typical in this regard), this difficulty persists for some time. For this and other reasons (e.g. correct treatment of the behaviour of binary stars) it is desirable to simulate the central parts of the system by direct N-body integration, though one of the 'simpler' techniques based on the Fokker-Planck equation may be used further out. Precisely such a hybrid scheme has recently been described (McMillan & Lightman 1984a,b), and it represents an exciting and notable advance in the techniques of simulation.

Though still not fully implemented, a restricted version of the scheme has been applied to a very short interval of time around maximum core collapse. Several interesting aspects of the evolution of binary stars were observed, indluding oscillation of the core. In one cycle of the oscillation, core heating by a binary first causes expansion of the core, until a sufficiently energetic interaction ejects it from the core, whereupon a modest contraction occurs until a new energetic binary forms. (Such oscillations are causally unrelated to those observed by Bettwieser *et al.*) In an interesting discussion of the behaviour of binaries in post-collapse evolution, the authors predict that such oscillations will die out, even though the fraction of time during which an active binary is present diminishes steadily.

3.4 N-body simulations

Whether the results of N-body simulations, with at most about 1000 stars, are applicable to globular star clusters is not the point at issue here (but see the paper by Goodman in this volume). Rather, now that there is a much better theoretical understanding of post-collapse evolution, albeit in an idealized form, it would be worth reexamining the evidence of these simulations in order to learn how comprehensive these theories are, since N-body simulations, even more than hybrid ones, remove many of the simplifying assumptions on which the theory is based. (By post-collapse evolution, in this context, we mean the phase following the appearance of very energetic binaries in a dense core, as this is usually taken as the manifestation of core collapse.)

Hénon himself (Hénon 1965) compared his homological model with the early 25-body models of von Hoerner (1963). The fit is extraordinarily good, over 7 orders of magnitude in density. The fact that such a result could be obtained, in a simulation so far removed from the assumption of large N on which Hénon's model was based, suggests that much could be learned by reexamining the much larger systems that have been studied since von Hoerner's time.

4. UNSOLVED PROBLEMS

The purpose of this section is to indicate a selection of problems and avenues for further research. Some of these are specific questions which could be answered using existing techniques after a few days' work; others require a prolonged program of research of which even the outline may be quite unclear.

4.1 Gaseous Models

The use of gaseous models in this problem has proved so effective that further research along this highly idealized direction is worthwhile. Among the problems for solution may be mentioned the following:

(i) The value of C: while the formal dependence of the thermal conductivity on density and velocity dispersion is not in doubt, the constant of proportionality should be determined; perhaps this could be done by comparing the rate of evolution of a slightly perturbed isothermal system in the gaseous and Fokker-Planck models.

https://doi.org/10.1017/S0074180900147357 Published online by Cambridge University Press

(ii) Gravothermal oscillations: the reason for the discrepancy between the author's results and those of Bettwieser *et al* should be settled; a third independent integration would be a useful contribution.

(iii) Stochastic energy sources: so far the energy-sources used have been time-independent functions of density, velocity dispersion and location. A time-dependent source should be tried, perhaps along lines suggested by the discussion of McMillan & Lightman (1984b).

(iv) Tidal effects: study of tidally truncated systems would be useful to check whether the Inagaki-Lynden-Bell solution (modified centrally) gives way to the gaseous analogue of the original Hénon model (Hénon 1961), just as isolated systems follow the Goodman model (Fig. 5). The self-similar gaseous analogue of the tidally truncated Hénon model should be found.

(v) Anisotropy: though Larson's model for the evolution of stellar systems (Larson 1970 a,b) does not deal with energy transport in the same way as the gaseous model, it is formally very similar, and looks almost as simple. It was also designed to deal with an anisotropic distribution of velocities, and post-collapse evolution should be studied with such a method. To begin with, searches for self-similar solutions analogous to those of Inagaki & Lynden-Bell and of Goodman would be desirable. (It is perhaps worth stating that no anisotropic self-similar solution for core collapse has yet been produced either.)

(vi) Rotation: to drop the assumption of spherical symmetry and discuss axisymmetric rotating clusters with two spatial dimensions would greatly complicate the gaseous model. But clusters do rotate, and it might at least be worth treating rotation as a perturbation by including small P_2 (cos θ) terms in the equation, provided (as in (i)) a rational method could be found for determining the coefficient of angular-momentum transport.

(vii) Mass spectrum: work has begun (Bettwieser & Inagaki 1984) on the inclusion of stars of different mass, but comparison with Fokker-Planck simulations indicates that further work is needed. The most delicate problem is the rational estimation of coefficients for heat conduction and for energy exchange between different populations.

(viii) Dissipative effects: it would be desirable, and easily possible in principle, to incorporate these into the gaseous model, by adding a dissipative term in the same manner as the energy generation term (Bettwieser & Fritze 1984). Given a satisfactory solution to (vii) above, it would also be possible to treat the formation and segregation of two-body binaries.

4.2 Fokker-Planck models

While simulations based on the Fokker-Planck equation are much more time-consuming than those using the gaseous model, there is no doubt that it is a superior model of the essential relaxation processes which drive much of the evolution. The following problems appear to be desirable or feasible, and sometimes both:

(i) Self-similar models: in unpublished work the author obtained a self-similar isotropic model for core collapse which agrees quantitatively with the results of Cohn (1980), and Hénon's models presumably relate to late post-collapse evolution. It would be desirable to find the Fokker-Planck analogue of the Inagaki-Lynden-Bell solution.

(ii) Simulations with simple energy-sources: these have been helpful in developing the theory of gaseous models, and it would be desirable to study and interpret a comparable suite of Fokker-Planck simulations, along the lines of those described by Cohn in this volume. (The difference between this and the artificial energy sources used by Hénon and Stodółkiewicz is that the energy generation would depend in a known way on the local density and distribution of velocities.)

(iii) Realistic effects: Fokker-Planck models have generally tended to be less idealized than gaseous models, and some models of Stodółkiewicz which go well beyond core collapse (see, for example, his paper in this volume) also include such effects as anisotropy, tidal truncation, stellar mass spectrum, mass-loss in stellar evolution, tidal shocking, and formation and evolution of binaries. The ultimate aim of such realism is, of course, comparison with observation. However, unless the history of all existing observable globular clusters has been identical, many different simulations would be needed. An alternative approach would be to introduce the various realistic effects one at a time, rather as is suggested above for gaseous models, so that a theoretical understanding of their effects could be developed.

4.3 Other remarks on simulations

(i) N-body integrations: it is worth repeating the suggestion made at the end of § 3 that renewed attempts should be made at the theoretical interpretation of these simulations, based on the picture that is now emerging from more highly idealized calculations. These and the hybrid simulations are invaluable for elucidating the detailed behaviour of binaries.

(ii) Presentation of results: no author is to be blamed because he did not publish those details of his models which, ten years later, turn out to be of interest. But it is worth stating that there is current interest in density- and velocity-profiles, and on the time-dependence of central values of velocity dispersion and density. Some of these are primarily of theoretical interest, but the importance of velocity and density profiles surely extends also into the domain of comparison with observations. Another topic to which thought should be given is scaling of results, since a lack of uniformity here makes comparison of published results by different authors time-consuming and sometimes impossible. The scaling often used in N-body simulations (see Fig. 1) has much to recommend it for most purposes. One further point on the presentation of results is the use of tables; those given by Hénon (1961, 1965) make his models much more useful than models given only in a graphical representation.

5. COMPARISON WITH OBSERVATIONS

5.1 Lowered Maxwellian models

It is customary to compare observations of the stellar distribution in globular clusters with lowered Maxwellian models (King 1966), which are tidally truncated, single-component isotropic models with a nearly isothermal core. Such models provide a satisfactory fit to the surface brightness profiles of most clusters, and it is usually only when the attempt to fit such a model fails that the possibility that the cluster has passed core collapse is entertained. What evidence do we have that a cluster with a collapsing core should resemble a lowered Maxwellian model, or that a cluster in post-collapse evolution should not?

The lowered Maxwellian model is an approximation to the solution of the equation

$$\frac{\partial f}{\partial t} = \left(\frac{\partial f}{\partial t}\right)_c ,$$

(King 1965), where $f(\varepsilon)$ is the stellar distribution function expressed as a function of energy ε per unit mass, and the right-hand side is the Fokker-Planck collision term, in which the diffusion terms are evaluated for a Maxwellian, and this equation is satisfied at the centre of the cluster. By contrast, the Fokker-Planck equation for a cluster with an isotropic distribution of velocities is

$$\overline{\frac{\partial f}{\partial t}} + \overline{\frac{\partial \phi}{\partial t}}\,\frac{\partial f}{\partial \varepsilon} = \left(\overline{\frac{\partial f}{\partial t}}\right)_c \tag{7}$$

where ϕ is potential per unit mass, and the bar denotes a phase-space average for a star of energy ε (see, for example, Hénon 1961). Thus the lowered Maxwellian is an approximation to an equation which differs from the Fokker-Planck equation, and so its inherent dynamical justification is doubtful. Therefore we should avoid drawing any dynamical inferences from the success, or lack of success, of attempts to fit lowered Maxwellians to observational data. Rather, if the attempt is to be made at all to do this with single-component, isotropic models (and more will be said about this below), then the models used should be those obtained by integration of equations such as (7), and not an approximation to a solution of an equation which is not used to study dynamical evolution.

It is likely that deviations of solutions of (7) from lowered Maxwellian models would be most pronounced in regions far from both the core and the tidal boundary. The core of the lowered Maxwellian model is nearly isothermal, and after a few central relaxation times it would be a satisfactory fit to the core of a solution of (7). On

https://doi.org/10.1017/S0074180900147357 Published online by Cambridge University Press

the other hand simple isothermal models might be equally satisfactory for this purpose.

5.2 Interpretation of surface brightness profiles

Even if we abandon lowered Maxwellian models and use solutions of (7), it is probable that the results of comparisons between observations and such simple models would be misleading, because the models differ from more realistic simulations in ways which may be essential. From the point of view of comparison with observations, it can be argued that the idealisation which is most suspect is the lack of a spectrum of masses, and the reason for this is observational. In their discussion of M15, Illingworth & King (1977) proposed that the distribution of visible stars was partly controlled by a centrally concentrated population of heavier, invisible neutron stars. There is some attraction in interpreting this population as a collapsed core (Heggie 1980), as only the heavier stars will undergo the process of core collapse. If so, an understanding of the observable properties of post-collapse clusters depends essentially on the presence of different masses. Indeed it has been shown by Larson (1984) that approximate two-component models provide a relatively natural explanation of the observed velocity dispersion in the cores of several globular clusters, as well as being consistent with the central parts of the surface brightness profiles.

Besides M15, a second cluster in which there is much interest is NGC 6624 (Djorgovski & King 1984). A large part of the surface brightness profile, starting just outside the seeing disc, has a logarithmic slope of −1, which would be given by projection of an isothermal. Thus the central parts of this cluster could be fitted by one of the single-component post-collapse models of § 2. If this interpretation is correct, it should be noted that the interpretations of M15 and NGC 6624 as post-collapse objects are essentially different. On the other hand, Grindlay's surface brightness profile of NGC 6624 (see his paper in this volume) resembles that of M15, and an isothermal profile is, he asserts, a poor fit.

5.3 Statistical comparisons

Another approach to the problem of seeking observational evidence on post-collapse evolution was pioneered by Lightman, Press & Odenwald (1978), who studied how a population of clusters (characterised by two core parameters) would change as the core evolved. While there is no theoretical support for the kind of evolution they considered (i.e. prompt dissipation of a cluster very soon after core collapse), their approach is well worth following up. An indication of what changes are needed if current ideas of post-collapse evolution are adopted is furnished by Cohn & Hut (1984). There are two reasons why their results are illustrative rather than definitive, however. First, they suppose that the *central* relaxation time is a constant multiple of the time until, or since, maximum collapse, whereas in post-collapse evolution it is the relaxation time at the *edge* of the isothermal region which behaves in this way. Secondly, if at least two mass-components are needed for an understanding of surface-brightness profiles, and it is only the less massive, brighter, uncollapsed component that we observe, then an understanding of the evolution of such systems will surely be needed also for the adequate interpretation of the statistics of the cores.

6. CONCLUDING REMARKS

The process of idealization has been quite successful in leading to an understanding of some essential ideas in the theory of post-collapse evolution. Indeed we can have some confidence that the idealized problem of the evolution of an isolated system containing a large number of point masses can even be described quite simply in a semi-quantitative manner. But it is sobering to realize that the essence of these ideas (and much of the detail) was already contained in Hénon's paper of 1961. He made the assumptions of isotropy and self-similarity (which are both wrong), obtained a solution with singularities in both space and time (which might have impelled one to reject it), and devised a *post hoc* explanation in terms of binary stars. And yet the ideas have stood the test of time, and it may fairly be claimed on Hénon's behalf that post-collapse evolution was essentially understood several years before the collapse phase was!

7. REFERENCES

Alexander, M.E. & Budding, E.: 1979, *Astron. Astrophys.* 73, 227.
Bettwieser, E. & Fritze, U.: 1984, submitted to *P.A.S.J.*
Bettwieser, E. & Inagaki, S.: 1984, 'A Gas Model for Multi-Component Star Clusters: 1. Method and Comparison with Fokker-Planck,' in preparation.
Bettwieser, E. & Sugimoto, D.: 1984, *M.N.R.A.S.*, in press.
Cohn, H.: 1980, *Ap. J.* 242, 765.
Cohn, H. & Hut, P.: 1984, *Ap. J. Lett.* 277, L45.
Djorgovski, S. & King, I.R.: 1984, *Ap. J. Lett.* 277, L49.
Dokuchaev, V.I. & Ozernoy, L.M.: 1982, *Astron. Astrophys.* 111, 16.
Duncan, M.J. & Shapiro, S.L.: 1982, *Ap. J.* 253, 921.
Fall, S.M. & Malkan, M.A.: 1978, *M.N.R.A.S.* 185, 899.
Goodman, J.: 1984, *Ap. J.* 280, 298.
Hachisu, I., Nakada, Y., Nomoto, L. & Sugimoto, D.: 1978, *Prog. Theor. Phys.* 60, 393.
Heggie, D.C.: 1980, in J.E. Hesser (ed.), *Star Clusters*, Reidel, Dordrecht, p. 401.
Heggie, D.C.: 1984, *M.N.R.A.S.* 206, 179.
Hénon, M.: 1961, *Ann. d'Astrophys.* 24, 369.
Hénon, M.: 1965, *Ann. d'Astrophys.* 28, 62.
Hénon, M.: 1975, in A. Hayli (ed.), *Dynamics of Stellar Systems*, Reidel, Dordrecht, p. 133.
Hills, J.G.: 1975, *Astron. J.* 80, 1075.

https://doi.org/10.1017/S0074180900147357 Published online by Cambridge University Press

Illingworth, G. & King, I.R.: 1977, *Ap. J. Lett.* 218, L109-12.
Inagaki, S. & Lynden-Bell, D.: 1983, *M.N.R.A.S.* 205, 913.
King, I.R.: 1965, *Astron. J.* 70, 376.
King, I.R.: 1966, *Astron. J.* 71, 64.
Larson, R.B.: 1970a, *M.N.R.A.S.* 147, 323.
Larson, R.B.: 1970b, *M.N.R.A.S.* 150, 93.
Larson, R.B.: 1984, 'Black-Hole Remnants in Globular Clusters', preprint.
Lightman, A.P.: 1982, *Ap. J. Lett.* 263, L19.
Lightman, A.P., Press, W.H. & Odenwald, S.F.: 1978, *Ap. J.* 219, 629.
Lightman, A.P. & Shapiro, S.L.: 1978, *Rev. Mod. Phys.* 50, 437.
Lynden-Bell, D. & Eggleton, P.P.: 1980, *M.N.R.A.S.* 191, 483.
McMillan, S.L.W. & Lightman, A.P.: 1984a, *Ap. J.*, in press.
McMillan, S.L.W. & Lightman, A.P.: 1984b, *Ap. J.*, in press.
McMillan, S.L.W., Lightman, A.P. & Cohn, H.: 1981, *Ap. J.* 251, 436.
Marchant, A.B. & Shapiro, S.L.: 1980, *Ap. J.* 239, 685.
Ozernoy, L.M. & Dokuchaev, V.I.: 1982, *Astron. Astrophys.* 111, 1.
Peaterson, C.J. & King, I.R.: 1975, *Astron. J.* 80, 427.
Retterer, J.M.: 1984, 'A Dynamical Model for Small Stellar Systems,' preprint.
Shapiro, S.L.: 1977, *Ap. J.* 217, 281.
Spitzer, L., Jr. & Shull, J.M.: 1975a, *Ap. J.* 200, 339.
Spitzer, L., Jr. & Shull, J.M.: 1975b, *Ap. J.* 201, 773
Stodółkiewicz, J.S.: 1982, *Acta Astron.* 32, 63.
Stodółkiewicz, J.S.: 1983, in J. Ruprecht & J. Palouš (eds.), *Star Clusters and Associations and their Relation to the Evolution of the Galaxy,* Astron. Inst. Czech. Acad. Sci., Prague, p.103.
von Hoerner, S.: 1960, *Z. Astrophys.* 50, 184.
von Hoerner, S.: 1963, *Z. Astrophys.* 57, 47.

https://doi.org/10.1017/S0074180900147357 Published online by Cambridge University Press

DISCUSSION

OSTRIKER: What determines the core radius in your simplified post collapse models?

HEGGIE: I assume that the rate of energy generation (per unit mass) depends in a known way on the local density and velocity dispersion, v. If the dependence on density is sufficiently steep, energy generation is nearly confined to the core, and the total rate, E, can be expressed in terms of v and the core radius, r_c. In the models of Hénon, Goodman, or Inagaki & Lynden-Bell, at any time v is determined and also the luminosity L of the central source. Hence r_c is determined by supposing the $L = \dot{E}$.

WHITE: You mentioned that Hénon's model had a precollapse density exponent of about -2.4. Could you clarify whether this is a result of the fact that his calculations allowed anisotropic velocity distributions and thus might be more "realistic" than purely isoltropic models which give -2.21 or -2.23?

HEGGIE: Actually I found -2.44 for Hénon's profile. Larson, with his fluid dynamical model, found -2.4 in one case. However Haldan Cohn tells me he found -2.23 (if I recall correctly), and this was also obtained by Duncan & Shapiro. All these models allow for anisotropy, and so there is some disagreement to sort out here.

COHN: Do I understand correctly that there is a difference in the form of the energy input rate used in your calculations and that adopted by Drs. Sugimoto and Bettwieser?

HEGGIE: There is no difference in *form*. There is a difference in the constant of proportionality, which I chose in such a way as to produce a density increase during collapse similar to that in one of their calculations which produced oscillations. The fact that I had to choose a different constant to achieve the same density increase is due in part to an error in their specification of the problem they solved. Dr. Bettwieser tells me that their cluster had a total mass of $10^5 M_\odot$, not $10^6 M_\odot$ as stated in their paper.

BETTWIESER: A comment to the question of Dr. Cohn. Recently we made simulations with very different energy sources, extended as well as point sources. We find core oscillations provided energy input is sufficiently small. Also I want again to stress the point, that in Dr. Heggie's calculations a large energy flux directed inwards is not allowed for, since his dependent variables are singular at L = -1. But the latter is necessary in order to have gravothermal expansion and not only a simple thermal expansion.

HEGGIE: I use the variable ln(1 + L), where L is the luminosity, and this does not permit $L \leq -1$. But I always observe L to be positive. If a gravothermal expansion were to begin, I would first see L become slightly negative, before the singularity at L = -1 is reached. However, L does not become negative.

SUGIMOTO: One comment and one question: The comment is concerned with Inagaki-Lynden-Bell's self-similar solution. As you criticized it correctly, the time dependence of the energy generation is determined by the overall expansion. However, the energy generation is controlled by a completely different mechanism, i.e., by binary hardening. Thus

https://doi.org/10.1017/S0074180900147357 Published online by Cambridge University Press

it will deviate from the required value. Since the collapsed core solution is gravothermally unstable, the deviation will grow and, therefore, it is meaningless to consult with the similarity solution. This criticism is given in detail in Bettwieser and Sugimoto (1984, M.N.R.A.S. in press).

The question is concerned with your recent solution for general expansion in the post collapse phase. If we compare it with the collapse phase at the same central density, the expansion is slower than the contraction by a factor of say 20 when the central density has fallen appreciably and thus the energy generation rate has fallen appreciably. Since the temperature is decreasing outward everywhere, the system should be gravothermally collapsing if there is no energy source at the same timescale as the collapse at the same age. You imply that it is stopped and reversed. However, the power of reversing is ≈20 times weaker and seems to be able to be neglected as compared with the ponderamotive force to contraction. How can you interpret this expansion physically? Is it related with the fact that you used the variable ln(1 + L) for the flux? It does not all L < -1 but the negative large value of L is essential for the gravothermal expansion. Or does your calculation correspond to a case with high energy generation where the system became a thermal system and makes a general expansion for the energy input? Though you said that you computed the same case as ours (Bettwieser and Sugimoto 1984), such could be the case if our paper should contain any important typographical error in numerical values.

HEGGIE: I agree that the Inagaki & Lynden-Bell solution, with its singular density profile, does not occur in my simulation. I argue that the central singularity is replaced by a non-singular core which produces the same luminosity as the singular core would, and has the same velocity dispersion, v. Thus it is meaningful to compare v with the behaviour predicted by Inagaki & Lynden-Bell, and yet the question whether or not a certain singular density profile is stable is not relevant, since such a profile is not present.

The reason why expansion is observed, and not collapse, even with temperature decreasing outward and a low rate of energy generation, is that a larger part of the system is isothermal than at the corresponding time in the collapse phase. Thus the heat flux across the edge of the isothermal part is smaller than at the corresponding time during collapse, and a smaller rate of energy generation is sufficient not only to prevent collapse but also to sustain expansion.

The question on the rate of energy generation I used, and on typographical errors, are dealt with in my reply to Dr. Cohn. The use of the variable ln(1 + L) is defended in my reply to Dr. Bettwieser.

GOODMAN: I'd like to point out that binaries do not have to supply kinetic energy *directly* to the cusp: by ejecting mass, they decrease the binding energy of the cusp and hence drive its expansion even if the "heat" evolved by binary encounters is lost from the cluster. Also, to balance the ejection, stars have to flow inward, yielding up energy as they do so and heating the cusp. The latter mechanism is qualitatively the same as that by which a black hole heats a cluster.

KING: I share your awe of the accuracy with which Hénon predicted twenty years ago what we would be talking about today. You pointed out that many of his assumptions were wrong; I suggest to you that one of the characteristics of genius is to get the right answer from the wrong assumptions!