DISTRIBUTION OF RATIONAL POINTS ON THE REAL LINE

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1. Introduction

It is well known that no rational number is approximable to order higher than 1. Roth [3] showed that an algebraic number is not approximable to order greater than 2. On the other hand it is easy to construct numbers, the Liouville numbers, which are approximable to any order (see [2], p. 162). We are led to the question, "Let $N_n(\alpha,\beta)$ denote the number of distinct rational points with denominators $\leqq n$ contained in an interval (α,β). What is the behaviour of $N_n(\alpha,\alpha+1/n)$ as α varies on the real line?" We shall prove that

$$0 \leqq N_n\left(\alpha,\alpha+\frac{1}{n}\right) \leqq \tfrac{1}{2}(n+1)$$

and that there are "compressions" and "rarefactions" of rational points on the real line.

Given a real number α, define the *density of rational points at* α, denoted by $D(\alpha)$, by

$$D(\alpha) = \lim_{n\to\infty} \frac{1}{n} N_n\left(\alpha-\frac{1}{2n},\alpha+\frac{1}{2n}\right).$$

We shall prove that $D(\alpha)$ is a constant for irrational α and that $D(p/q)$, where $(p,q)=1$, is a function of q only.

We now state the results. Throughout this paper $[\alpha]$ denotes the greatest integer less or equal to α, and the constant implied by the 0-notation is an absolute constant.

THEOREM 1. *For any real* α, $N_n(\alpha,\alpha+1/n)\leqq\frac{1}{2}(n+1)$.

THEOREM 2. *Given any integers m and n satisfying* $0\leqq m\leqq\frac{1}{2}(n+1)$, *there exists an* α (*indeed a rational* α) *such that* $N_n(\alpha,\alpha+1/n)=m$.

THEOREM 3. *If* m, $n>0$ *are integers, then*

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$$(1.1)\qquad N_n\left(\frac{m}{n}, \frac{m+1}{n}\right) = \begin{cases} 0 \text{ if } m = 0 \\ \dfrac{n}{m(m+1)} \sum_{r=1}^{m} \phi(r) + O(m \log m) \text{ otherwise,} \end{cases}$$

where $\phi(r)$ is Euler's ϕ-function.

It is easy to prove that

$$N_n\left(\frac{m}{n}, \frac{m+1}{n}\right) = N_n\left(\frac{-m-1}{n}, \frac{-m}{n}\right)$$

and that, if $m \equiv m'(\bmod n)$, then

$$N_n\left(\frac{m}{n}, \frac{m+1}{n}\right) = N_n\left(\frac{m'}{n}, \frac{m'+1}{n}\right).$$

It now follows that if $N_n(m/n, (m+1)/n)$ is known for $m = 0, 1, 2, \cdots [\frac{1}{2}(n-1)]$, then $N_n(m/n, (m+1)/n)$ is known for all m.

Corollary 3.1 *If $m > 1$, then*

$$(1.2)\qquad \frac{1}{n} N_n\left(\frac{m}{n}, \frac{m+1}{n}\right) = \frac{3}{\pi^2} + O\left(\frac{\log m}{m}\right) + O\left(\frac{m \log m}{n}\right).$$

The next two theorems enable us to estimate $N_n(\alpha, \alpha + 1/n)$ if we can find a rational point with "small" denominator near α.

Theorem 4. *If $0 < \nu \leqq 1$, $(p,q) = 1$, $q > 0$, then*

$$(1.3)\quad N_n\left(\frac{p}{q}, \frac{p}{q} + \frac{\nu}{n}\right) = \begin{cases} 0 \text{ if } [\nu q] = 0 \\ \dfrac{n}{q} \sum_{r=1}^{[\nu q]} \left(1 - \dfrac{r}{\nu q}\right) \dfrac{\phi(r)}{r} + O(\nu q \log \nu q) \text{ otherwise.} \end{cases}$$

Corollary 4.1 *If $0 < \nu \leqq 1$, $(p,q) = 1$, $q > 0$, then*

$$(1.4)\qquad \frac{1}{n} N_n\left(\frac{p}{q}, \frac{p}{q} + \frac{\nu}{n}\right) = \frac{3\nu}{\pi^2} + O\left(\frac{\log q}{q}\right) + O\left(\frac{\nu p \log \nu q}{n}\right).$$

Corollary 4.2 *If $\mu > 0$, $\nu > 0$ and $\mu + \nu = 1$, then*

$$\frac{1}{n} N_n\left(\frac{p}{q} - \frac{\mu}{n}, \frac{p}{q} + \frac{\nu}{n}\right) = \frac{3}{\pi^2} + O\left(\frac{\log q}{q}\right) + O\left(\frac{q \log q}{n}\right).$$

The next theorem helps us to estimate $N_n(\alpha, \alpha + 1/n)$ when no rational point in the interval $(\alpha, \alpha + 1/n)$ has a small denominator of order $O(n^\varepsilon)$, $\varepsilon < \frac{1}{2}$.

THEOREM 5. *If* $\mu > 0$, *then*

$$(1.5)\quad N_n\left(\frac{p}{q}+\frac{\mu}{n},\ \frac{p}{q}+\frac{\mu+1}{n}\right) = An + Bn + O\{(\mu+1)q\log(\mu+1)q\},$$

where

$$(1.6)\qquad A = \frac{1}{q^2\mu(\mu+1)}\sum_{r=1}^{[\mu q]}\phi(r)$$

and

$$(1.7)\qquad B = \frac{1}{q}\sum_{r=[\mu q+1]}^{[\mu q+q]}\left(1-\frac{r}{(\mu+1)q}\right)\frac{\phi(r)}{r}.$$

The following two theorems are on the *density of rational points* at a point on the real line.

THEOREM 6. *If* $(p,q) = 1$, *then*

$$D\left(\frac{p}{q}\right) = \frac{1}{q}\sum_{r=1}^{[\frac{1}{2}q]}\left(1-\frac{2r}{q}\right)\frac{\phi(r)}{r}.$$

and, for large q,

$$D\left(\frac{p}{q}\right) = \frac{3}{\pi^2} + O\left(\frac{\log q}{q}\right).$$

THEOREM 7. *If* α *is irrational, then*

$$D(\alpha) = \frac{3}{\pi^2}.$$

2. Proof of theorems 1 and 2

PROOF OF THEOREM 1. Suppose that

$$\frac{x_1}{y_1},\ \frac{x_2}{y_2}\cdots\frac{x_r}{r_r},\ \frac{x_1'}{y_1'},\ \frac{x_2'}{y_2'}\cdots\frac{x_s'}{y_s'}$$

are the distinct rational points in $(\alpha, \alpha+1/n)$ satisfying

$$1 \leqq y_1 \leqq y_2 \cdots \leqq y_r \leqq \tfrac{1}{2}n < y_1' \leqq \cdots \leqq y_s' \leqq n.$$

For every $y_i \leqq \frac{1}{2}n$, there exists integers c_i, y_{s+i}', x_{s+i}' such that

$$c_i x_i = x_{s+i}' \text{ and } \tfrac{1}{2}n < c_i y_i = y_{s+i}' \leqq n.$$

It is easy to see that no two of

$$y_1', y_2' \cdots y_s',\ y_{s+1}', \cdots y_{r+s}'$$

are equal, for $y_j' = y_k'$ implies

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$$\left|\frac{x'_j}{y'_j} - \frac{x'_k}{y'_k}\right| \geqq \frac{1}{y'_j} \geqq \frac{1}{n}.$$

This contradicts that the open interval $(\alpha, \alpha + 1/n)$ is of length $1/n$. Hence

$$N_n\left(\alpha, \alpha + \frac{1}{n}\right) = r + s \leqq n - [\tfrac{1}{2}n] \leqq \tfrac{1}{2}(n+1).$$

PROOF OF THEOREM 2. Clearly $0 < 1/y < 1/n$ only if $y > n$. So

$$N_n\left(0, \frac{1}{n}\right) = 0.$$

Next we see that if $0 < m \leqq \frac{1}{2}n$, then because $1/(n-m) \leqq 2/n$ the only rational numbers with denominators $\leqq n$ contained in the interval

$$\left(\frac{1}{n-m} - \frac{1}{n}, \ \frac{1}{n-m}\right)$$

are

$$\frac{1}{n}, \frac{1}{n-1}, \cdots, \frac{1}{n-m+1}.$$

Hence

$$N_n\left(\frac{1}{n-m} - \frac{1}{n}, \ \frac{1}{n-m}\right) = m.$$

Lastly if $m = \frac{1}{2}(n+1)$, then n is odd and

$$\frac{2}{n} - \frac{1}{n-m+1} = \frac{2}{n} - \frac{2}{n+1} = \frac{2}{n(n+1)} > 0.$$

Thus

$$N_n\left(\frac{1}{n} - \varepsilon, \frac{2}{n} - \varepsilon)\right) = m \text{ if } \frac{2}{n(n+1)} > \varepsilon > 0.$$

This completes the proof of Theorem 2.

3. Lemmas

In this section we prove the lemmas required for the proofs of Theorems 3–7. Consider the set

$$S_{c,s} = \{c+1, \ c+2, \cdots, c+s\}$$

of s consecutive integers with $c+1$ as the first element. Let $\tau_{c,s}(r)$ denote the number of integers in the set $S_{c,s}$, which are relatively prime to r. We use $d(r)$ to denote the number of divisors of r. Note that if $r \mid s$, then

$$\tau_{c,s}(r) = \frac{s\phi(r)}{r} = \sum_{d \mid r} \mu(d) \frac{s}{d}.$$

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We prove

LEMMA 1. *For all integers c, $s > 0$, we have*

$$\left|\tau_{c,s}(r) - \frac{s\phi(r)}{r}\right| \leqq d(r) - 1.$$

PROOF. By theorem 261 of ([1], p. 234), we deduce for $c \geqq 0$ that

$$\tau_{0,c}(r) = \sum_{d|r} \mu(d)\left[\frac{c}{d}\right]$$

and

$$\tau_{0,c+s}(r) = \sum_{d|r} \mu(d)\left[\frac{c+s}{d}\right].$$

So for $c \geqq 0$,

$$\begin{aligned} \tau_{c,s}(r) &= \tau_{0,s+c} - \tau_{0,c} \\ &= \sum_{d|r} \mu(d)\left\{\left[\frac{s+c}{d}\right] - \left[\frac{c}{d}\right]\right\}. \end{aligned}$$

Hence

$$\begin{aligned} \left|\tau_{c,s}(r) - \frac{s}{r}\phi(r)\right| &\leqq \sum_{d|r} \left|\left[\frac{s+c}{d}\right] - \left[\frac{c}{d}\right] - \frac{s}{d}\right| \\ &\leqq \sum_{d|r} 1 - 1 \\ &= d(r) - 1. \end{aligned}$$

If $c < 0$, then there exists an integer a such that $c' = c + ar > 0$ and $\tau_{c,s}(r) = \tau_{c',s}(r)$.

So

$$\left|\tau_{c,s}(r) - \frac{s}{r}\phi(r)\right| = \left|\tau_{c',s}(r) - \frac{s}{r}\phi(r)\right| \leqq d(r) - 1,$$

as required.

Let $\tau_r(\alpha, \beta)$ be the number of integers, relatively prime to r, which are contained in the open interval (α, β). We prove

LEMMA 2. *For all α and β, $\alpha < \beta$, we have*

$$\left|\tau_r(\alpha, \beta) - (\beta - \alpha)\frac{\phi(r)}{r}\right| \leqq d(r).$$

PROOF. Let s be the number of integers in (α, β), then

$$|(\beta - \alpha) - s| \leqq 1 \text{ and } \tau_r(\alpha, \beta) = \tau_{c,s}(r)$$

for some c. It follows from Lemma 1 that

$$\left|\tau_r(\alpha,\beta)-(\beta-\alpha)\frac{\phi(r)}{r}\right| \leqq \left|\tau_{c,s}(r)-\frac{s\phi(r)}{r}\right| + \left|(\beta-\alpha-s)\frac{\phi(r)}{r}\right|$$

$$\leqq d(r).$$

LEMMA 3. *For all positive integer n*

$$\sum_{r=1}^{n} d(r) = n\log n + (2\gamma-1)n + O(\sqrt{n})$$

where γ is Euler's constant.

PROOF. This is proved in [2], p. 264.

LEMMA 4. *(c.f.*[1], *p. 131. l.23–24) Let n be a positive integer. Then*

(a) $$\sum_{r=1}^{n}\phi(r) = \frac{3n^2}{\pi^2} + O(n\log n)$$

(b) $$\sum_{r=1}^{n}\frac{\phi(r)}{r} = \frac{6n}{\pi^2} + O(\log n).$$

PROOF. (a) is proved in [2], p. 268
(b) can be proved similarly.

4. Proofs of theorems 3, 4 and 5

PROOF OF THEOREM 3. We have shown in the proof of Theorem 2 that

$$N_n\left(\frac{0}{n},\ \frac{1}{n}\right) = 0.$$

Given $r > 0$, let

$$S_r = \left\{\frac{r}{y} : (r,y)=1,\ \frac{m+1}{n} > \frac{r}{y} > \frac{m}{n},\ n \geqq y \geqq 1\right\}.$$

Obviously S_r is empty if $r > m$. Moreover if $m \geqq r \geqq 1$, then

$$\frac{r}{y}\in S_r \text{ if and only if } \frac{rn}{m} > y > \frac{rn}{m+1} \text{ and } (r,y)=1.$$

We deduce that the number of rational points in S_r is $\tau_r(rn/(m+1), rn/m)$. By Lemma 2,

$$\tau_r\left(\frac{rn}{m+1},\ \frac{rn}{m}\right) = \left(\frac{rn}{m} - \frac{nr}{m+1}\right)\frac{\phi(r)}{r} + \eta_r$$

$$= \frac{n}{m(m+1)}\phi(r) + \eta_r,$$

where

$$|\eta_r| \leqq d(r).$$

Therefore

$$N_n\left(\frac{m}{n}, \frac{m+1}{n}\right) = \frac{n}{m(m+1)} \sum_{r=1}^{m} \phi(r) + \sum_{r=1}^{m} \eta_r.$$

By Lemma 3,

$$\left|\sum_{r=1}^{m} \eta_r\right| \leqq \sum_{r=1}^{m} d(r) = O(m \log m).$$

This proves Theorem 3.

Proof of Corollary 3.1 Using Lemma 4, we see that

$$\begin{aligned}\frac{1}{m(m+1)} \sum_{r=1}^{m} \phi(r) &= \frac{1}{m(m+1)} \frac{3m^2}{\pi^2} + O\left(\frac{m \log m}{m(m+1)}\right) \\ &= \frac{3}{\pi^2} + O\left(\frac{\log m}{m}\right).\end{aligned}$$

The proof is complete.

Proof of Theorem 4. To determine $N_n(p/q, p/q + v/n)$, $0 < v \leqq 1$, we look for rational numbers x/y such that

$$(x, y) = 1, \; n \geqq y \geqq 1$$

and

$$0 < \frac{x}{y} - \frac{p}{q} = \frac{xq - yp}{yq} < \frac{v}{n}.$$

Since at most one of m/n, m an integer, is in the interval $(p/q, p/q + v/n)$, we shall neglect the rational points with denominator n and let

$$S_r = \left\{\frac{x}{y}: xq - yp = r, (x, y) = 1, n > y \geqq 1, \frac{v}{n} > \frac{r}{yq} > 0\right\}.$$

Here S_r is empty if $r > [vq]$. We assume $[vq] \geqq r \geqq 1$.

Since $(p, q) = 1$, there exist integers x_0, y_0 such that

$$x_0 q - y_0 p = 1.$$

Moreover, all integral solutions of

$$xq - yp = r \tag{4.1}$$

are given by

$$x = rx_0 + pt, \; y = ry_0 + qt. \tag{4.2}$$

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Now (4.2) implies $(r, t) \mid (x, y)$ and hence $(r, t) = 1$. It follows that

$$\frac{x}{y} \in S_r \text{ if and only if } (r, t) = 1 \text{ and } n > ry_0 + qt > \frac{rn}{vq},$$

which can be reduced to

$$\frac{n}{q} - \frac{ry_0}{q} > t > \frac{rn}{vq^2} - \frac{ry_0}{q}.$$

Thus the number of elements in S_r is equal to

$$\tau_r\left(\frac{rn}{vq^2} - \frac{ry_0}{q}, \frac{n}{q} - \frac{ry_0}{q}\right) = \frac{n}{q}\left(1 - \frac{r}{vq}\right)\frac{\phi(r)}{r} + \eta_r \tag{4.3}$$

where $|\eta_r| \leqq d(r)$ by Lemma 2. So by Lemma 3,

$$N_n\left(\frac{p}{q}, \frac{p}{q} + \frac{v}{n}\right) = \frac{n}{q}\sum_{r=1}^{[vq]}\left(1 - \frac{r}{vq}\right)\frac{\phi(r)}{r} + O(vq \log vq),$$

which is (1.3) and the proof is complete.

Proof of Corollary 4.1 It follows easily from Lemma 4 that

$$\sum_{r=1}^{[vq]} \frac{\phi(r)}{r} = \frac{6[vq]}{\pi^2} + O(\log[vq])$$

and that

$$\sum_{r=1}^{[vq]} \phi(r) = \frac{3[vq]^2}{\pi^2} + O([vq]\log[vq]).$$

So

$$\frac{1}{q}\sum_{r=1}^{[vq]}\left(1 - \frac{r}{vq}\right)\frac{\phi(r)}{r} = \frac{6[vq]}{q\pi^2} - \frac{3[vq]^2}{vq^2\pi^2} + O\left(\frac{\log[vq]}{q}\right) = \frac{3v}{\pi^2} + O\left(\frac{\log q}{q}\right)$$

Proof of Corollary 4.2 This follows from

$$N_n\left(\frac{p}{q} - \frac{\mu}{n}, \frac{p}{q}\right) = N_n\left(\frac{-p}{q}, \frac{-p}{q} + \frac{\mu}{n}\right) \text{ and Corollary 4.1.}$$

Proof of Theorem 5. If x/y is in $(p/q + \mu/n,\ p/q + (\mu+1)/n)$ and $1 \leqq y \leqq n$, then

$$\frac{\mu}{n} < \frac{x}{y} - \frac{p}{q} = \frac{xq - yp}{yq} < \frac{\mu+1}{n}.$$

Putting $xq - yp = r$, we obtain from the above inequalities,

$$\frac{rn}{\mu q} > y > \frac{rn}{(\mu+1)q}.$$

As in Theorem 4, neglecting rational points with denominator n, we let

$$S_r = \left\{\frac{x}{y} : xq - yp = r, (x, y) = 1, n > y \geqq 1, \frac{rn}{\mu q} > y > \frac{rn}{(\mu+1)q}\right\}.$$

Here S_r is empty if $r > (\mu+1)q$. Moreover if $x/y \in S_r$ then

$$\frac{rn}{\mu q} > y > \frac{rn}{(\mu+1)q} \quad \text{if } [\mu q] \geqq r \geqq 1;$$

and

$$n > y > \frac{rn}{(\mu+1)q} \quad \text{if } [\mu q + q] \geqq r \geqq [\mu q + 1].$$

Using the same argument in Theorem 4, we deduce that the number of elements in S_r is equal to

$$\text{(4.4)} \qquad \frac{n}{\mu(\mu+1)q^2}\phi(r) + \eta_r \quad \text{if } [\mu q] \geqq r \geqq 1,$$

and is equal to

$$\text{(4.5)} \qquad \frac{n}{q}\left(1 - \frac{r}{q(\mu+1)}\right)\frac{\phi(r)}{r} + \eta_r \quad \text{if } [\mu q + q] \geqq r \geqq [\mu q + 1],$$

where $|\eta_r| \leqq d(r)$. It now follows from (4.4) and (4.5) that

$$N_n\left(\frac{p}{q} + \frac{\mu}{n}, \frac{p}{q} + \frac{\mu+1}{n}\right) = An + Bn + \sum_{r=1}^{[\mu q + q]} \eta_r,$$

where A, B are given in (1.6) and (1.7). Using Lemma 3, we obtain (1.5). This proves Theorem 5.

5. The function $D(\alpha)$

In this section, we prove the theorems on the density of rational points.

PROOF OF THEOREM 6. Putting $\nu = \frac{1}{2}$, we obtain from (1.3) that

$$N_n\left(\frac{p}{q}, \frac{p}{q} + \frac{1}{2n}\right) = \frac{n}{q}\sum_{r=1}^{[\frac{1}{2}q]}\left(1 - \frac{2r}{q}\right)\frac{\phi(r)}{r} + O(q \log q) \text{ for } q \geqq 2.$$

It now follows from this equation and

$$N_n\left(\frac{p}{q} - \frac{1}{2n}, \frac{p}{q}\right) = N_n\left(\frac{-p}{q}, \frac{-p}{q} + \frac{1}{2n}\right)$$

that

$$N_n\left(\frac{p}{q} - \frac{1}{2n}, \frac{p}{q} + \frac{1}{2n}\right) = \frac{2n}{q}\sum_{r=1}^{[\frac{1}{2}q]}\left(1 - \frac{2r}{q}\right)\frac{\phi(r)}{r} + O(q \log q).$$

Hence

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$$D\left(\frac{p}{q}\right) = \frac{2}{q}\sum_{r=1}^{[\frac{1}{2}q]}\left(1-\frac{2r}{q}\right)\frac{\phi(r)}{r}$$

as required.

For large q, using Lemma 4, we see that

$$D\left(\frac{p}{q}\right) = \frac{2}{q}\left\{\frac{6}{\pi^2}(\tfrac{1}{2}q)+O(\log q)\right\} - \frac{4}{q^2}\left\{\frac{3}{\pi^2}(\tfrac{1}{2}q)^2+O(q\log q)\right\}$$
$$= \frac{3}{\pi^2}+O\left(\frac{\log q}{q}\right),$$

and the proof of Theorem 6 is complete.

PROOF OF THEOREM 7. Given an irrational real number α, let $[a_0, a_1, a_2, \cdots]$ be the infinite simple continued fraction representation of α and let

$$\frac{p_s}{q_s} = [a_0, a_1, \cdots, a_s], \qquad s = 0, 1, 2, \cdots,$$

denote the convergents. It is well known that

$$\left|\frac{p_s}{q_s}-\alpha\right| < \frac{1}{q_s q_{s+1}}$$

and that $q_s < q_{s+1}$ if $s > 0$.

We may suppose then that

$$(5.1) \qquad \left|\frac{p_s}{q_s}-\alpha\right| = \frac{1}{q^{\kappa}{}_s},\ \kappa > 2,\ q_s^{\kappa-1} > q_{s+1}.$$

Now for every n, there exists an s such that

$$(5.2) \qquad q_s^2 < 2n \leqq q_{s+1}^2.$$

We consider separately the cases

$$q_s^2 < 2n \leqq q_s^{\kappa},\ q_s^{\kappa} < 2n \leqq \kappa q_s^{\kappa},\ \kappa q_s^{\kappa} < 2n \leqq q_{s+1}^2;$$

if either of q_s^{κ}, κq_s^{κ} is greater than q_{s+1}^2, then one or more of the cases does not arise. We prove that if (5.2) holds, then

$$(5.3) \qquad \frac{1}{n}N_n\left(\alpha-\frac{1}{2n}, \alpha+\frac{1}{2n}\right) = \frac{3}{\pi^2}+O\left(\frac{\log q_s}{q_s}\right).$$

Case 1. Suppose that $q_s^2 \leqq 2n \leqq q_s^{\kappa}$. Then

$$\frac{p_s}{q_s} \in \left(\alpha-\frac{1}{2n}, \alpha+\frac{1}{2n}\right)$$

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and there exist positive numbers μ, ν satisfying $\mu + \nu = 1$ such that

$$\alpha - \frac{1}{2n} = \frac{p_s}{q_s} - \frac{\mu}{n}$$

and

$$\alpha + \frac{1}{2n} = \frac{p_s}{q_s} + \frac{\nu}{n}.$$

It now follows from Corollary 4.2 that

$$\frac{1}{n}N_n\left(\alpha - \frac{1}{2n}, \alpha + \frac{1}{2n}\right) = \frac{1}{n}N_n\left(\frac{p_s}{q_s} - \frac{\mu}{n}, \frac{p_s}{q_s} + \frac{\nu}{n}\right)$$
$$= \frac{3}{\pi^2} + O\left(\frac{\log q_s}{q_s}\right), \text{ since } n > \tfrac{1}{2}q_s^2.$$

Case 2. Suppose that $\kappa q_s^\kappa < 2n \leqq q_{s+1}^2$.

Here

$$\left|\frac{p_{s+1}}{q_{s+1}} - \alpha\right| < \frac{1}{q_{s+1}\, q_{s+2}} < \frac{1}{2n}.$$

So

$$\frac{p_{s+1}}{q_{s+1}} \in \left(\alpha - \frac{1}{2n}, \alpha + \frac{1}{2n}\right)$$

and there exist positive numbers μ, ν such that $\mu + \nu = 1$,

$$\alpha - \frac{1}{2n} = \frac{p_{s+1}}{q_{s+1}} - \frac{\mu}{n},$$

$$\alpha + \frac{1}{2n} = \frac{p_{s+1}}{q_{s+1}} + \frac{\nu}{n}.$$

By Corollary 4.2,

$$\frac{1}{n}N_n\left(\alpha - \frac{1}{2n}, \alpha + \frac{1}{2n}\right) = \frac{3}{\pi^2} + O\left(\frac{\log q_{s+1}}{q_{s+1}}\right) + O\left(\frac{q_{s+1}\log q_{s+1}}{n}\right)$$
$$= \frac{3}{\pi^2} + O\left(\frac{\log q_s}{q_s}\right)$$

because $q_s < q_{s+1} < q_s^{\kappa-1}$ and $\kappa q_s^\kappa < 2n$.

Case 3. Suppose that $q_s^\kappa < 2n \leqq \kappa q_s^\kappa$. Writing p for p_s, q for q_s and putting

$$2n = (2\mu + 1)q^\kappa,$$

we obtain

$$1 < 2\mu + 1 \leqq \kappa \text{ and } \frac{1}{q^\kappa} = \frac{2\mu + 1}{2n}. \tag{5.4}$$

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In (5.1), if $\alpha - p/q = 1/q^\kappa$, then by (5.4),

$$(5.5)\quad N_n\left(\alpha - \frac{1}{2n}, \alpha + \frac{1}{2n}\right) = N_n\left(\frac{p}{q} + \frac{1}{q^\kappa} - \frac{1}{2n}, \frac{p}{q} + \frac{1}{q^\kappa} + \frac{1}{2n}\right)$$

$$= N_n\left(\frac{p}{q} + \frac{\mu}{n}, \frac{p}{q} + \frac{\mu+1}{n}\right).$$

On the other hand if $\alpha - p/q = -1/q^\kappa$, then

$$(5.6)\quad N_n\left(\alpha - \frac{1}{2n}, \alpha + \frac{1}{2n}\right) = N_n\left(\frac{p}{q} - \frac{1}{q^\kappa} - \frac{1}{2n}, \frac{p}{q} - \frac{1}{q^\kappa} + \frac{1}{2n}\right)$$

$$= N_n\left(\frac{p}{q} - \frac{\mu+1}{n}, \frac{p}{q} - \frac{\mu}{n}\right)$$

$$= N_n\left(\frac{-p}{q} + \frac{\mu}{n}, \frac{-p}{q} + \frac{\mu+1}{n}\right).$$

Using (1.5) we obtain, from (5.5) or (5.6),

$$\frac{1}{n} N_n\left(\alpha - \frac{1}{2n}, \alpha + \frac{1}{2n}\right) = A + B + O\left\{\frac{(\mu+1)q\log(\mu+1)q}{n}\right\},$$

where A, B are given in (1.6) and (1.7). Clearly, as $\kappa > 2$,

$$O\left\{\frac{(\mu+1)q\log(\mu+1)q}{n}\right\} = O\left\{\frac{(2\mu+2)q\log(\mu+1)q}{(2\mu+1)q^\kappa}\right\}$$

$$= O\left\{\frac{\log \kappa q}{q^{\kappa-1}}\right\}$$

$$= O\left\{\frac{\log q}{q}\right\}.$$

We now prove

$$A + B = \frac{3}{\pi^2} + O\left\{\frac{\log q}{q}\right\}.$$

First suppose $\mu \geqq q$. Then by Lemma 4,

$$A = \frac{1}{q^2\mu(\mu+1)} \sum_{r=1}^{[\mu q]} \phi(r)$$

$$= \frac{1}{q^2\mu(\mu+1)}\left\{\frac{3}{\pi}(\mu q)^2\right\} + O\left\{\frac{\log \mu q}{(\mu+1)q}\right\}$$

$$= \frac{3}{\pi^2} + O\left(\frac{\log q}{q}\right),$$

$$|B| = \frac{1}{(\mu+1)q} \left| \sum_{r=[\mu q+1]}^{[\mu q+q]} \left(\frac{(\mu+1)q - r}{q} \right) \frac{\phi(r)}{r} \right|$$
$$\leqq \frac{q}{(\mu+1)q} = O\left(\frac{1}{q}\right).$$

Thus (5.3) is true if $\mu \geqq q$.

Next we suppose $0 < \mu < 1/q$. Then $[\mu q] = 0$. So $A = 0$ and

$$B = \frac{1}{q} \sum_{r=1}^{q} \left(1 - \frac{r}{(\mu+1)q}\right) \frac{\phi(r)}{r}$$
$$= \frac{1}{q}\left\{\frac{6q}{\pi^2} + O(\log q)\right\} - \frac{1}{(\mu+1)q^2}\left\{\frac{3q^2}{\pi^2} + O(q\log q)\right\}$$
$$= \frac{3}{\pi^2} + O\left(\frac{\log q}{q}\right).$$

Thus (5.3) is true if $0 < \mu < 1/q$.

Lastly suppose $1/q \leqq \mu < q$. Then using Lemma 4, we see that

$$A = \frac{3\mu}{(\mu+1)\pi^2} + O\left(\frac{\log q}{q}\right)$$

and

$$B = \frac{1}{q}\sum_{r=1}^{[\mu q+q]}\left(1 - \frac{r}{(\mu+1)q}\right)\frac{\phi(r)}{r} - \frac{1}{q}\sum_{r=1}^{[\mu q]}\left(1 - \frac{r}{(\mu+1)q}\right)\frac{\phi(r)}{r}$$
$$= \frac{6}{q\pi^2}\{(\mu q+q) - (\mu q)\} - \frac{3}{q^2(\mu+1)\pi^2}\{(\mu q+q)^2 - (\mu q)^2\} + O\left(\frac{\log q}{q}\right)$$
$$= \frac{6}{\pi^2} - \frac{3(2\mu+1)}{(\mu+1)\pi^2} + O\left(\frac{\log q}{q}\right).$$

Thus

$$A + B = \frac{3}{\pi^2} + O\left(\frac{\log q}{q}\right).$$

We have shown that (5.3) is true if

$$q_s^2 < 2n \leqq q_{s+1}^2.$$

This proves Theorem 7.

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