ON THE DOMINANT RELATIVISTIC TERMS IN THE LUNAR THEORY

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ABSTRACT: For a simplified 3-body (Earth,Moon,Sun) problem it is shown how the usual Einstein-Infeld-Hoffmann equations for the lunar motion reduce to the Jacobi-equations after the transformation to the proper reference frame. The dominant relativistic contributions to the lunar laser ranging observables are then obtained in a Hill-Brown calculation. It is argued that in the proper reference frame all post-Newtonian variational terms are proportional to $m = n'/(n-n')$ [$n(n')$ = mean motion of Moon (Sun)].

1. INTRODUCTION

The problem of post-Newtonian celestial mechanics in the solar system generally falls into two parts: 1) solving for the post-Newtonian equations of motion for the gravitating bodies in some suitably chosen coordinate system and 2) in the derivation of observables. The first part is usually treated by solving the (PPN modified) Einstein-Infeld-Hoffmann equations of motion in the harmonic (isotropic) gauge. The second part of the problem is related to the introduction of an observer's reference frame in some way or another. This can be done globally e.g. by using spatial tangent vectors to light rays originating from distant stars or extragalactic radio sources as spatial reference directions and using the well-known relation between proper time indicated by an atomic clock and coordinate time in the solar system barycentric frame of reference (Moyer 1981; Brumberg 1981). Locally, a frame can be defined e.g. by means of gyroscopes and atomic clocks that in an operational sense locally yield the observer's tetrad basis. The idea to derive observables by projecting tensors onto locally defined tetrad vectors originally goes back to Pirani(1957) and Synge(1960). This approach has been first applied to the relativistic lunar motion by Mashhoon(1985) and will be persued a bit further in the present article.

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2. THE LUNAR COORDINATE MOTION

The restricted post-Newtonian 3-body problem for Earth, Moon and Sun in harmonic coordinates with circular motion for Sun has been first treated in detail in the Hill-Brown formalism by Brumberg(1958). If we simplify the problem further and neglect lunar eccentricity, latitude and solar parallax, his expessions for radius vector r(coordinate distance) and true longitude v of the Moon read:

$$r/\bar{a}_o \cong 1 + \sigma\left[-\frac{9}{4} + m - \frac{47}{32}m^2 + ..\right] + \left[-m^2 + .. + \sigma\left(\frac{1}{4} + \frac{17}{4}m^2 + ..\right)\right]\cos 2D + ..$$

$$v \cong n(t-t_o) + \left[\frac{11}{8}m^2 + .. + \sigma\left(-\frac{1}{4} - \frac{11}{4}m^2 + ..\right)\right]\sin 2D + ..$$

with D = mean longitude of (Moon-Sun),

$$\bar{a}_o = a_o\left(1 - \frac{1}{6}m^2 + ..\right) \qquad n^2 a_o^3 = GM_\oplus \qquad m = \frac{n'}{(n-n')}$$

$$n' = \frac{(GM_\odot)^{1/2}}{a'^{3/2}}\left(1 - \frac{3}{2}\sigma\right) \equiv n'_o\,\sigma_{-3/2}$$

and the parameter $\sigma \equiv GM_\odot/(c^2 a') \simeq 10^{-8}$ indicates the various relativistic contributions. It can be seen that in this harmonic coordinate motion of the Moon the dominant relativistic range oscillation has an amplitude of $(\sigma/4)\bar{a}_o \simeq 100$cm and a period of half a synodic month. Various authors like e.g. Baierlein(1967) believed this term to be measurable in Lunar Laser Ranging experiments; later, however, it was shown that this is indeed not the case since the amplitude is strongly dependent upon the choice of coordinates.

3. TETRADS

The ambiguity introduced by the choice of the coordinates disappears (locally) if one uses local coordinates that have a well defined direct physical meaning such as e.g. Fermi normal coordinates. In Fermi normal coordinates X^μ centered at the Earth-Moon barycenter the value for the directly observed Earth-Moon distance is essentially given by:

$$R \cong (\delta_{ij}X^i X^j)^{1/2}$$

It is illustrative to construct such a Fermi frame for our simplified problem out of the canonical tetrad frame induced by the isotropic Schwarzschild coordinates of the Sun ($\theta = \pi/2$): ($\sigma_\pm \equiv 1 \pm \sigma$)

$$e_{(t)} = \sigma_+ \frac{\partial}{\partial ct} \quad ; \quad e_{(r)} = \sigma_- \frac{\partial}{\partial r} \quad ; \quad e_{(\phi)} = \frac{\sigma_-}{a'}\frac{\partial}{\partial \phi} \quad ; \quad e_{(\theta)} = \frac{\sigma_-}{a'}\frac{\partial}{\partial \theta}$$

Transition to a comoving frame is obtained by a Lorentz-boost in $e_{(\phi)}$ direction:

$$e'_{(t)} = \frac{n'_o}{n'}\frac{\partial}{\partial ct} + \frac{n'_o}{c}\frac{\partial}{\partial\phi} = \frac{\partial}{\partial\tau}$$

$$e'_{(\phi)} = \frac{n'_o a'}{c}\frac{\partial}{\partial ct} + \frac{\sigma_{-1/2}}{a'}\frac{\partial}{\partial\phi}$$

where τ is proper time an atomic clock fixed to the Earth-Moon barycenter would measure. Rotation of spatial axes about $e_{(\theta)}$ with $\Phi = n'(t-t_o)$ yields a fixed star oriented frame with axes $\bar{e}_{(\alpha)}$. The Fermi-frame axes ${}^F e_{(i)}$, obeying the Fermi-Walker transport law are then given by:

$${}^F e_{(i)} = \bar{e}_{(i)}\left[\Phi \rightarrow \Phi_F\right]$$

where

$$\Phi_F = n' \; (\tau-\tau_o) = \Phi + \Omega_{GP} \; (\tau-\tau_o)$$

I.e. the Fermi-frame axes simply undergo the usual de Sitter - Fokker (geodetic) precession about the fixed star oriented axes with frequency:

$$\Omega_{GP} = n' - n'_o = -\frac{3}{2}\sigma$$

Now, in the Fermi-frame with normal coordinates the Jacobi-equation reads:

$$\ddot{X}^i - (\nabla\Phi_\oplus)^i + K^{(i)}{}_{(j)}X^j = 0 \qquad (\dot{} = \frac{d}{d\tau})$$

with:

$$K^{(i)}{}_{(j)} = R_{(o)(i)(o)(j)} = R_{\mu\nu\rho\sigma}\,{}^F e^\mu_{(o)}\,{}^F e^\nu_{(i)}\,{}^F e^\rho_{(o)}\,{}^F e^\sigma_{(j)}$$

if we add the gravitational potential of the Earth. For our problem the Jacobi equations w.r.t. $e'_{(\alpha)}$ read:

$$\ddot{X} - 2n'\dot{Y} + (GM_\oplus/R^3)X - 3n'^2X = \sigma[3n'^2X]$$

$$\ddot{Y} + 2n'\dot{X} + (GM_\oplus/R^3)Y = 0$$

$$\ddot{Z} + (GM_\oplus/R^3)Z + n'^2Z = \sigma[-3n'^2Z]$$

4. THE HILL-BROWN THEORY IN THE PROPER REFERENCE FRAME

The corresponding EIH equations of motion in instantaneous geocentric coordinates read:

$$\frac{d^2x}{dt^2} - 2n'\frac{dy}{dt} + \frac{GM_\oplus}{r^3}x - 3n'^2x = \sigma\left[-2n'\frac{dy}{dt} - 6n'^2x + 6\frac{GM_\oplus}{r^3}x + \frac{3GM_\oplus xy^2}{2\ r^5}\right]$$

$$\frac{d^2y}{dt^2} + 2n'\frac{dx}{dt} + \frac{GM_\oplus}{r^3}y = \sigma\left[4n'\frac{dx}{dt} + 6\frac{GM_\oplus}{r^3}y + \frac{3GM_\oplus y^3}{2\ r^5}\right]$$

$$\frac{d^2z}{dt^2} + \frac{GM_\oplus}{r^3}z + n'^2z = \sigma\left[6\frac{GM_\oplus}{r^3}z + \frac{3GM_\oplus zy^2}{2\ r^5}\right]$$

It is easy to see that they reduce to the Jacobi equations after the transformation: $(t,x,y,z) \rightarrow (\tau,X,Y,Z)$

$$\tau = \sigma_{-3/2}\,t \quad ; \quad X = \sigma_+\,x \quad ; \quad Y = \sigma_+\sigma_{+1/2}\,y \quad ; \quad Z = \sigma_+\,z$$

consisting of a) a Lorentz-boost and b) a rescaling of rulers. It now turns out that this transformation to proper coordinates is intimately related to Brumberg's original transformation to auxiliary variables that simplifies the Hill-Brown calculation drastically and that the Hill-Brown expressions for the measurable quantities in the proper reference frame can essentially be read off his paper. For the measurable Earth-Moon distance and longitude one finds:

$$R/\bar{a}_o \cong 1 + \sigma\left[-\frac{1}{2}m^2 + ..\right] + \left[-m^2 + ..(1+\sigma)\right]\cos 2D_\tau + ...$$

$$V \cong n(\tau - \tau_o) + \left[\frac{11}{8}m^2 + ..(1+\sigma)\right]\sin 2D_\tau + ...$$

where $D_\tau = (n-n')(\tau - \tau_0)$. It can be seen that the dominant relativistic range oscillation is again a 2D term but with amplitude $m^2\bar{a}_0 \sim 2$cm (see also Mashhoon 1985). In the proper reference frame all post-Newtonian variational terms are proportional to m and all m-independent terms in the EIH-coordinate motion arise from the heliocentric notion of simultaneity (see also Nordtvedt 1973) used in this coordinate picture.

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DISCUSSION

Nobili : how do you take into account the long-term effects which accumulate in time ?

Soffel : I was only interested in short periodic effects. Long periodic and secular effects are difficult to assess because of the lack of precision of the initial conditions and of the dynamical parameters.

Alley : it should be emphasized that the Lorentz contraction at the Earth-Moon distance is of the order of 100 cm and presently the r.m.s residual of lunar laser ranging is about 18 cm. This is the only explicit measurement of this effect I know.

Bertotti : how did you define the central world line in your Fermi frame ?

Soffel : it should represent the Earth-Moon barycenter.

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