

TORSION AND PROTORSION MODULES OVER FREE IDEAL RINGS

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1. Introduction

Free ideal rings (or firs, cf. [2, 3] and § 2 below) form a noncommutative analogue of principal ideal domains, to which they reduce in the commutative case, and in [3] a category T_R of right R -modules was defined, over any fir R , which forms an analogue of finitely generated torsion modules. The category T_R was shown to be abelian, and all its objects have finite composition length; moreover, the corresponding category ${}_R T$ of left R -modules is dual to T_R .

A natural problem at this point is to find an analogue of torsion modules that are not necessarily finitely generated. There are a number of ways of forming the completion of an abelian category (cf. e.g. Gabriel [5]); in our case the problem is simplified by the fact that T_R is a full subcategory of M_R , the category of all right R -modules. On the other hand, we shall want an explicit description of the torsion modules and this is not provided by the general theory. We shall therefore proceed somewhat differently: We begin by defining general torsion modules over a fir (§ 2), and show that these modules form the objects of a full subcategory T_R^\dagger of M_R which is abelian and has exact direct limits and a generator, i.e. it is a Grothendieck category. It is obtained from T_R by forming direct limits in M_R .

A dual procedure leads to the notion of a *protorsion module*, and in this way we obtain a category T_R^\downarrow with exact inverse limits and a cogenerator. Further, when the categories of left R -modules ${}_R T^\dagger$, ${}_R T^\downarrow$ are defined correspondingly, we find that there is a duality between T_R^\dagger and ${}_R T^\downarrow$. This duality allows us to dispense with the verification that T_R^\downarrow is again an abelian category. Using a result of Gabriel [5], we can interpret T_R^\downarrow as the category of all pseudo-compact modules over a certain pseudo-compact ring E . In the special case of a principal ideal domain, E can be constructed as a completion of R (e.g. when $R = \mathbf{Z}$, $E = \hat{\mathbf{Z}}$), but this construction does not extend to general firs.

2. The category of general torsion modules over a fir

All rings are understood to have a unit-element 1, modules are unital and homomorphisms preserve 1. By a *free right ideal ring*, or *right fir* for short, one un-

derstands a ring in which all right ideals are free, of unique rank; *left firs* are defined correspondingly. Throughout this note, we shall only be dealing with rings that are left and right firs.

Given a fir R , and an R -module M with a finite presentation

$$0 \rightarrow R^m \rightarrow R^n \rightarrow M \rightarrow 0,$$

the *characteristic* of M is defined by

$$(1) \quad \chi(M) = n - m.$$

If M is finitely generated but not finitely related, we put $\chi(M) = -\infty$, and if M is not finitely generated, $\chi(M) = \infty$. This defines $\chi(M)$ for every R -module, and it is clear that the definition is independent of the presentation chosen for M . In [3] it was shown that for any short exact sequence

$$(2) \quad 0 \rightarrow M' \rightarrow M \rightarrow M'' \rightarrow 0$$

of modules over a fir, we have

$$(3) \quad \chi(M) = \chi(M') + \chi(M''),$$

with the usual conventions about ∞ and the rules

(α) if $\chi(M'') = \infty$, then $\chi(M) = \infty$,

(β) if $\chi(M'') = -\infty$, and $\chi(M') = \infty$, no conclusion can be drawn.

In particular, (3) is meaningful whenever M' and M'' are finitely generated.

A *torsion module* over R was defined in [3] as an R -module M such that

$$(i) \quad \chi(M) = 0$$

and either of the following conditions (equivalent when (i) holds):

(ii) for every submodule M' of M , $\chi(M') \geq 0$,

(iii) for every quotient module M'' of M , $\chi(M'') \leq 0$.

Any torsion module in the sense of this definition is necessarily finitely generated; we shall generalize this notion by making the following

DEFINITION. A right R -module M is said to be a *torsion module* if every finite subset of M is contained in a finitely generated torsion module.

We denote the full subcategory of M_R whose objects are the torsion modules by T_R^1 and as in [3] denote by T_R the subcategory of finitely generated torsion modules. We remark that general torsion modules can be defined without referring back to the finitely generated case by the conditions

(i') every finite subset is contained in a submodule T such that $\chi(T) = 0$, and

(ii') for every finitely generated submodule M' , $\chi(M') \geq 0$.

Clearly (ii') actually holds for all submodules, finitely generated or not.

Conditions (i') and (ii') again imply a condition on the finitely generated quotient modules, but it is not clear whether this can actually replace (ii'). Before proving it we note a general property of finitely related modules which is used repeatedly in what follows.

LEMMA 2.1. *Let R be any ring and M an R -module with a presentation*

$$0 \rightarrow G \rightarrow F \rightarrow M \rightarrow 0$$

where F is free and G is finitely generated. Then M is the direct sum of a finitely presented module and a free module.

This result follows by taking a finite generating set of G , expressing it in terms of a basis of F and noting that only finitely many elements of the basis of F are involved.

PROPOSITION 2.2. *Let M be any torsion module over a fir. Then*

(iii') *for every finitely generated or finitely related quotient M'' of M , $\chi(M'') \leq 0$.*

PROOF. When M is finitely generated, $\chi(M) = 0$ and the result is clear by (3). Now let M be a general torsion module and $M'' = M/M'$ a finitely generated quotient. Then M contains a finitely generated submodule N such that $N + M' = M$. Enlarging N if necessary we may assume that N is a finitely generated torsion module and hence satisfies (iii'). Now $M'' = (N + M')/M' \cong N/(N \cap M')$, and so $\chi(M'') \leq 0$. Thus every finitely generated quotient of M has nonpositive characteristic. If M'' is finitely related but not finitely generated, then by the lemma, $M'' = M_0 \oplus F$, where F is free of infinite rank. Hence M'' has a finitely generated quotient of positive rank, and so does M , but this contradicts what has just been proved.

In order to show that T_R^\dagger is an abelian category it is enough to verify

(a) *in any short exact sequence (2) of R -modules, if any two terms are torsion modules, so is the third, and*

(b) *for any homomorphism f between torsion modules, both $\ker f$ and $\text{coker } f$ are torsion modules.*

PROOF OF (a). Let M', M'' be torsion modules and X a finite subset of M . The image of X in M'' is contained in a finitely generated torsion submodule T'' of M'' . Pick a finitely generated submodule L of M which contains X and maps onto T'' . Let L' be the kernel of the mapping $L \rightarrow T''$ (induced by the mapping $M \rightarrow M''$), then $\chi(L') = \chi(L) - \chi(T'') = \chi(L)$. The latter is finite, so L' is finitely generated, and is therefore contained in a finitely generated torsion submodule T' of M' . Let T be the submodule of M generated by L and the image of T' under the mapping $M' \rightarrow M$. Then T is an extension of T' by T'' , hence a finitely generated torsion module, and it contains X .

Next assume that M and M' are torsion modules. Given any finite subset

X of M'' , there exists a finite subset of M mapping onto X , and hence a finitely generated torsion module L of M whose image L'' contains X . Let $L' = \ker(L \rightarrow L'')$; thus if M' is identified with a submodule of M , then $L' = L \cap M'$. Since $\chi(L) = 0$ and L'' is finitely generated, L' is finitely related. Suppose first that L' is finitely generated and enlarge it to a finitely generated torsion submodule T' of M' . Then $T = T' + L$ is finitely generated and hence has non-negative characteristic. But as homomorphic image of the torsion module $T' \oplus L$ it has non-positive characteristic, hence $\chi(T) = 0$ and T is a torsion module. Now $\text{coker}(T' \rightarrow T)$ is a finitely generated torsion submodule of M'' containing X . It remains to consider the case when L' is not finitely generated. We know that it is finitely related, so $L' = L_0 \oplus F$, where L_0 is finitely presented and F is free of infinite rank. Since $M' \cap L = L_0 + F$, we can find a finitely generated torsion submodule M_1 of M' which contains $L_0 + F_1$, where F_1 is a direct summand of F , chosen so that $L_0 + F_1$ has positive characteristic. Clearly $M_1 \cap L = L_0 + F_1$ and the exact sequence

$$0 \rightarrow M_1 \cap L \rightarrow M_1 \oplus L \rightarrow M_1 + L \rightarrow 0$$

shows that $M_1 + L$ has negative characteristic. But this contradicts the fact that M is a torsion module. So this case cannot occur, and M'' is therefore a torsion module.

Thirdly, assume M, M'' to be torsion modules and let X be a finite subset of M' . Let us take M' to be a submodule of M , then there exists a finitely generated torsion submodule T of M containing X . Put $T' = T \cap M'$ and $T'' = \text{coker}(T' \rightarrow T)$, then T'' is finitely generated, a quotient of T and a submodule of M'' , and hence is a torsion module. Therefore T' is also a torsion module and it contains X . This shows M' to be a torsion module, and it completes the proof of (a).

PROOF OF (b). Let $f: M \rightarrow N$ be a homomorphism of torsion modules, and put $K = \ker f$. Any finite subset X' of M/K is the image of a finite subset X of M . Let T be a finitely generated torsion submodule of M containing X and consider the isomorphism

$$(4) \quad (T + K)/K \cong T/(T \cap K).$$

The left-hand side contains X' and is isomorphic to a submodule of N , hence $\chi((T + K)/K) \geq 0$. But (4) shows that $\chi(T/(T \cap K)) \leq 0$, hence $(T + K)/K$ is a torsion module containing X' , and we have shown that M/K is a torsion module. Now part (a), just proved, shows that $\ker f$, $\text{im } f$ and $\text{coker } f$ are also torsion modules. Thus T_R^1 is an abelian category and we have established part of

THEOREM 2.3. *The category T_R^1 of torsion modules over a fir R is a Grothendieck category, i.e. an abelian category with exact direct limits and a generator.*

It is clear that T_R^1 is closed under direct unions, and the AB.5 axiom

$$(5) \quad \bigcup (M_\alpha \cap N) = (\bigcup M_\alpha) \cap N$$

follows because we have a subcategory of M_R . To obtain a generator we simply form the direct sum of one copy each of all the simple finitely generated torsion modules.

3. Protorsion modules

DEFINITION. A *protorsion module* over a fir R is a module of the form

$$M = \varinjlim M_i,$$

where (M_i) is an inverse system of finitely generated torsion modules.

Every protorsion module carries a natural topology, obtained by taking finitely generated torsion modules to be discrete and regarding M as a submodule of the topological product $\prod M_i$. We shall write T_R^\downarrow for the category consisting of all protorsion modules and all continuous homomorphisms between them. By arguments similar to those of § 2 it can be shown that T_R^\downarrow is an abelian category satisfying the dual of AB.5; this will also follow directly from the next result which establishes a duality between ${}_R T^\downarrow$ and T_R^\uparrow . Note that this extends the duality between ${}_R T$ and T_R described in [3], Th. 5.2. We shall abbreviate Ext_R^\downarrow by Ext in what follows; this is unambiguous because R is a fixed ring, always a fir, and so $\text{Ext}_R^2 = 0$.

THEOREM 3.1. *Over any fir R , the category T_R^\uparrow of right torsion modules is dual to the category ${}_R T^\downarrow$ of left protorsion modules.*

PROOF. Let $M \in T_R^\uparrow$ say $M = \varinjlim M_i$, where $M_i \in T_R$. Then, on writing $DM = \text{Ext}(M, R)$ (for any R -module M), we have

$$DM = \text{Ext}(M, R) = \text{Ext}(\varinjlim M_i, R) = \varinjlim \text{Ext}(M_i, R) = \varinjlim DM_i.$$

By Th. 5.2 of [3], $DM_i \in {}_R T$, hence $DM \in {}_R T^\downarrow$. Thus

$$(1) \quad D : M \mapsto \text{Ext}(M, R)$$

is a contravariant functor from T_R^\uparrow to ${}_R T^\downarrow$. We complete the proof by showing that D is a duality. In the first place we observe that D preserves short exact sequences, because $\text{Ext}_R^2 = 0$ and $\text{Hom}(M, R) = 0$ for any $M \in T_R^\uparrow$. Thus D is exact.

Next we show that every object P of ${}_R T^\downarrow$ has the form DQ , for some $Q \in T_R^\uparrow$. Let $P = \varinjlim P_i$, where (P_i) is an inverse system of modules in ${}_R T$, and write $Q_i = DP_i$, then (Q_i) is a direct system in T_R and hence $Q = \varinjlim Q_i \in T_R^\uparrow$, so

$$DQ = D(\varinjlim Q_i) = \varinjlim DQ_i = \varinjlim P_i = P.$$

It only remains to show that for any $M, N \in T_R^\uparrow$, the natural mapping

$$(2) \quad \text{Hom}(M, N) \rightarrow \text{Hom}(DN, DM)$$

is a bijection. It is injective, for if $f : M \rightarrow N$ is non-zero, take $M' \subseteq M$, finitely

generated, such that $fM' \neq 0$ and choose a finitely generated submodule N' of N such that $fM' \subseteq N'$, then f induces a mapping $f' : M' \rightarrow N'$ which is non-zero; we have the commutative diagram with exact rows

$$\begin{array}{ccccc} DM & \longrightarrow & DM' & \longrightarrow & 0 \\ \uparrow Df & & \uparrow Df' & & \\ DN & \longrightarrow & DN' & \longrightarrow & 0 \end{array}$$

By the duality in the finite case, $Df' \neq 0$ and hence $Df \neq 0$.

To show that (2) is surjective, let $g : DN \rightarrow DM$ be given, and assume that $M = \varinjlim M_i$; then $DM = \varinjlim DM_i$ and by composition we get mappings $g_i : DN \rightarrow DM_i$. Let $DN = \varinjlim P_\alpha$, then by the continuity of g_i there exists $\alpha = \alpha(i)$ and a mapping $u_i : P_\alpha \rightarrow DM_i$ such that the triangle

$$\begin{array}{ccc} DN & \xrightarrow{\quad} & DM_i \\ & \searrow p_\alpha & \nearrow u_i \\ & & P_\alpha \end{array}$$

commutes, where p_α is the natural projection. Write $Q_\alpha = DP_\alpha$, $Q = \varinjlim Q_\alpha$, then $DQ = D(\varinjlim Q_\alpha) = \varinjlim DQ_\alpha = \varinjlim P_\alpha = DN$, hence $N \cong Q$, by what has been shown. Now $Du_i : M_i \rightarrow Q_\alpha$ combined with the natural injection gives a mapping $u_i^* : M_i \rightarrow Q$. If $\varphi_{ji} : M_i \rightarrow M_j$ denotes the canonical mapping ($j > i$), then clearly the u_i^* satisfy

$$u_i^* = u_j^* \varphi_{ji} \quad \text{for } j > i.$$

Hence the u_i^* combine to give a mapping $u^* : M \rightarrow Q$ which satisfies $Du^* = g$.

It now follows (e.g. by the criterion 1.19 of [1]) that D defines a duality between T_R^\dagger and ${}_R T^\dagger$, and this completes the proof.

Since the notion of a (two-sided) fir is left-right symmetric, we also have a duality between ${}_R T^\dagger$ and T_R^\dagger .

Let us return to T_R and consider $S = \text{Spec } T_R^\dagger$, the associated spectral category (cf. [6, 8]). We recall that the natural functor

$$P : T_R^\dagger \rightarrow S = \text{Spec } T_R^\dagger$$

may be described as the solution of the universal problem of making all essential monomorphisms in T_R^\dagger invertible. Further, $PM \cong PN$ if and only if M and N have isomorphic injective hulls¹, or equivalently, if there exists $L \in T_R^\dagger$ with essential monomorphisms $L \rightarrow M$, $L \rightarrow N$. Consider first the case where M, N are both finitely generated, by n elements say. Writing $T = R_n$, the $n \times n$ matrix ring

¹ Taken in T_R^\dagger , of course. The injective hull within M_R is generally larger.

over R , we may regard M and N as cyclic T -modules, say $M = T/cT$, $N = T/dT$, where c, d are elements of T defining torsion modules. We recall from [3] that an element c defines a torsion module precisely if it has no left or right factor which is a zero-divisor; such an element of T will be called *regular*. We note that a T_R -submodule M' of M corresponds to a right factor of c : if $c = ab$, then $aT/abT \cong T/bT = M'$ say, $T/aT = M''$, where the relation between M, M', M'' is given by the exact sequence (2) of § 1. In terms of the matrices a, b, c defining M'', M', M the monomorphism $M' \rightarrow M$ is essential if and only if b is an essential right factor of c in the sense of the following

DEFINITION. Let c be a regular element of a ring T . Then a right factor c' of c is *inessential* if c has a non-unit right factor which is right coprime to c' . Otherwise c' is *essential*.

With this definition two regular $n \times n$ matrices over a fir R correspond to right R -modules with isomorphic injective hulls if and only if they have an essential right factor in common.

Since T_R^\dagger is locally Noetherian, each injective is a direct sum of indecomposable injectives. In general, these summands are not finitely generated, but any indecomposable injective can be obtained as the injective hull of any one of its non-zero submodules, and we know that there always is a finitely generated submodule.

We recall that an object PM of S is simple if and only if M is coirreducible in T_R , i.e. if M has a unique minimal non-zero submodule M' , and clearly $PM \cong PM'$. Thus the simple S -objects P are just the injective hulls of the T -simple torsion modules, so each is determined by an atom in some R_n ($n = 1, 2, \dots$). Let \mathbf{P} be a system of representatives for the simple S -objects, then the mappings $M \rightarrow \text{Hom}_S(P, M) (P \in \mathbf{P})$ provide an equivalence between S and a product category

$$S \rightarrow \text{IIM}_{K_P}, \text{ where } K_P = \text{End}_S(P) \text{ is a skew field.}$$

Now in T_R^\dagger any direct sum of injectives is again injective (cf. e.g. [1], 6.50). Let U be the direct sum of one copy of each indecomposable injective in T_R^\dagger . This module U may also be formed obtained by taking one copy of each T -simple module and forming the injective hull. Let E denote the endomorphism ring of U :

$$(3) \quad E = \text{End}_T(U).$$

The ring E may be regarded as a pseudo-compact ring in a natural way ([5], Prop. IV.4.13); by Th. IV.4.4 of [5], ${}_R T^\dagger$ is dual to $PC(E)$, the category of pseudo-compact right E -modules, hence by applying Th. 3.1 above we obtain

THEOREM 3.2. *Let $PC(E)$ be the category of pseudo-compact right E -modules, where E is given by (3), then T_R^\dagger is equivalent to $PC(E)$.*

By way of illustration let us briefly consider the case of a principal ideal domain. By [4] the category T_R^\dagger , as subcategory of M_R , admits submodules if and only

if R is a principal right ideal domain. In that case the duality of Th. 3.1 reduces to the situation familiar in abelian groups:

$$\text{Ext}(M, R) = \text{Hom}(M, K),$$

where $K = I/R$, I being the injective hull of R as right R -module. More particularly, assume that R is a principal ideal domain (not necessarily commutative). Then every simple object in T_R^\dagger is a cyclic module and hence of the form R/pR (p an atom of R). Now I is just the skew field of fractions of R (this is true for any right Ore domain; it follows e.g. from Prop. 3, p. 95 of [7]). The R -module K , as image of I , is again injective.

We can write I in the form $I = \varinjlim a^{-1}R$; then

$$\text{End}(K_R) = \text{Hom}(\varinjlim a^{-1}R/R, K) = \varinjlim \text{Hom}(a^{-1}R/R, K) = \varinjlim R/Ra,$$

hence

$$(4) \quad \text{End}(K_R) = \varinjlim R/Ra = \hat{R}.$$

Clearly K is an injective cogenerator for T_R^\dagger ; hence \hat{R} , defined by (4), is a projective generator for ${}_R T^\dagger$. In fact \hat{R} is just the ring E defined in (3), hence ${}_R T^\dagger$ may be regarded as the category of *all* pseudo-compact left \hat{R} -modules.

Since K is a direct sum of indecomposable injectives (cf. e.g. [1], 6.50), \hat{R} is a direct product of indecomposable projectives. These correspond to the different atoms of R and the indecomposable projective corresponding to a given atom p of R has the form

$$(5) \quad R_p = \varinjlim R/Ra,$$

where a runs over all elements of R containing p as essential right factor. In the commutative case (5) of course simplifies to

$$R_p = \varinjlim R/Rp^n.$$

Returning to the case of a general fir R , we still have an injective cogenerator K for T_R^\dagger which is a direct sum of indecomposable injectives. Its dual DK is a direct product of indecomposable projectives and is a projective generator for ${}_R T^\dagger$. However, there is now no natural way of constructing DK (analogously to \hat{R}); firstly because not every simple object of T_R is defined by an atom in R (we generally need R_n for all n), and secondly, because even in R itself the family of principal right ideals generated by atoms is not closed under finite intersections. Even the formula (5) for an indecomposable projective may fail in this case: e.g., if $R = k\langle x, y, z \rangle$ is the free k -algebra on x, y, z (k a field), then yx, zx are two elements with x as essential right factor, but $Ryx \cap Rzx = 0$. This means that the injective hull of R/xR cannot be expressed as a direct limit of cyclic modules, and correspondingly the projective cover of R/Rx is not an inverse limit of the form (5).

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