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A q-EXTENSION OF FELDHEIM'S BILINEAR SUM FOR JACOBI POLYNOMIALS AND SOME APPLICATIONS

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1. Introduction. The main objective of this paper is to find useful q-extensions of Feldheim's [**6**] bilinear formula for Jacobi polynomials, namely,

$$F_4\left(\alpha_1, \alpha_2; \alpha + 1; \beta + 1; \frac{\rho(1 - x)(1 - y)}{4}, \frac{\rho(1 + x)(1 + y)}{4}\right)$$

$$(1.1)\qquad = \sum_{k=0}^{\infty} \frac{k!(\alpha + \beta + 2)_k(2k + \alpha + \beta + 1)}{(\alpha + 1)_k(\beta + 1)_k(\alpha + \beta + 2)_{2k}(k + \alpha + \beta + 1)} (\alpha_1)_k(\alpha_2)_k\rho^k$$

$$\cdot {}_2F_1(\alpha_1 + k, \alpha_2 + k; \alpha + \beta + 2 + 2k; \rho)P_k^{(\alpha,\beta)}(x)P_k^{(\alpha,\beta)}(y),$$

where the Appel function F_4 is defined by

$$(1.2)\qquad F_4(a, b; c, d; x, y) = \sum_{m,n=0}^{\infty} \frac{(a)_{m+n}(b)_{m+n}}{m!n!(c)_m(d)_n} x^m y^n;$$

α_1, α_2, ρ are arbitrary complex parameters such that the series on both sides of (1.1) are convergent, and

$$(1.3)\qquad P_k^{(\alpha,\beta)}(x) = \frac{(\alpha + 1)_k}{k!} {}_2F_1\left(-k, k + \alpha + \beta + 1; \frac{1 - x}{2}\right)$$

is the Jacobi polynomial of degree k, $(a)_k$ being the usual shifted factorial.

A very general q-analogue of the Jacobi polynomials, known as q-Wilson polynomials, has been recently discovered by Askey and Wilson [**2**]

$$(1.4)\qquad p_n(x; a, b, c, d) = {}_4\Phi_3\left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\theta}, ae^{-i\theta} \\ ab, ac, ad \end{matrix}; q, q\right],$$

$x = \cos\theta$, $0 \leqq \theta \leqq \pi$, $n = 0, 1, 2, \ldots$, and the parameters a, b, c, d are usually assumed to be real and numerically less than 1. The symbol on the

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right hand side of (1.4) represents a basic hypergeometric series defined generally by

$$(1.5)\qquad {}_{r+1}\Phi_r\left[\begin{matrix}a_1, a_2, \ldots, a_{r+1}\\ b_1, b_2, \ldots, b_r\end{matrix}; q, z\right] = \sum_{k=0}^{\infty} \frac{(a_1)_k(a_2)_k \ldots (a_{r+1})_k}{(q)_k(b_1)_k \ldots (b_r)_k} z^k,$$

where the shifted factorials on the right hand side no longer mean the same thing as in (1.1), (1.2) or (1.3), rather, they are the so-called q-shifted factorials defined by

$$(1.6)\qquad (a)_k \equiv (a; q)_k = \begin{cases} 1 & , \quad \text{if } k = 0 \\ (1 - a)(1 - aq) \ldots (1 - aq^{k-1}), & k = 1, 2, \ldots . \end{cases}$$

Obviously $(a; q)_k$ is a more distinctive notation for this, but considerations of economy suggest the adoption of the shorthand notation provided its meaning is clear in the context. Throughout the paper we shall use $(a)_k$ to mean $(a; q)_k$ unless otherwise mentioned. The parameter q is usually taken to be numerically less from 1 and the other parameters in (1.5) must satisfy the convergence requirements of the series.

Askey and Wilson [**2**] proved the orthogonality of $p_n(x; a, b, c, d)$

$$(1.7)\qquad \int_{-1}^{1} w(x; a, b, c, d)p_m(x; a, b, c, d)p_n(x; a, b, c, d)dx = h_n\delta_{m,n},$$

where the weight function is given by

$$(1.8)\qquad w(x; a, b, c, d) = (1 - x^2)^{-1/2}\frac{h(x; 1)h(x; -1)h(x; \sqrt{q})h(x; -\sqrt{q})}{h(x; a)h(x; b)h(x; c)h(x; d)},$$

with

$$(1.9)\qquad h(x; a) = \prod_{r=0}^{\infty} (1 - 2axq^r + a^2q^{2r}) = (ae^{i\theta})_\infty(ae^{-i\theta})_\infty,$$

$$x = \cos\theta, \quad (A)_\infty = \lim_{n\to\infty} (A)_n,$$

and the normalization constant h_n is given by

$$(1.10)\qquad h_n = h_0\frac{(q)_n(1 - abcdq^{-1})(cd)_n(bd)_n(bc)_n}{(abcdq^{-1})_n(1 - abcdq^{2n-1})(ab)_n(ac)_n(ad)_n}a^{2n},$$

$$(1.11)\qquad h_0 = \frac{2\pi(abcd)_\infty}{(q)_\infty(ab)_\infty(ac)_\infty(ad)_\infty(bc)_\infty(bd)_\infty(cd)_\infty},$$

subject to the restriction that

(1.12) $\max(|q|, |a|, |b|, |c|, |d|) < 1.$

The basic result that we shall prove in Section 2 is the bilinear formula

$$F(x, y|q) = \sum_{n=0}^{\infty} \frac{\lambda_n}{h_n} p_n(x; a, b, c, d) p_n(y; a, b, c, d), \tag{1.13}$$

where

$$\lambda_n = h_0 \frac{(bc)_n (ad)_n}{(abcd)_{2n}} (-\rho ad)^n q^{\binom{n}{2}} \sum_{k=0}^{\infty} \frac{A_{n+k} (adq^n)_k (adq^n)_k}{(q)_k (abcdq^{2n})_k} \rho^k, \tag{1.14}$$

$\binom{n}{2} = n(n-1)/2$, and

$$\begin{aligned} F(x, y|q) &= \sum_{m=0}^{\infty} \frac{A_m (de^{i\theta})_m (de^{-i\theta})_m (de^{i\Phi})_m (de^{-i\Phi})_m}{(q)_m (bd)_m (cd)_m (d/a)_m} \rho^m \\ &\quad {}_{10}W_9\Big(ad^{-1}q^{-m}; q^{1-m}/bd, q^{1-m}/cd, q^{-m}, \\ &\qquad ae^{i\theta}, ae^{-i\theta}, ae^{i\Phi}, ae^{-i\Phi}; q, \frac{bcq}{ad}\Big) \end{aligned} \tag{1.15}$$

with $x = \cos\theta$, $y = \cos\Phi$, $0 \leqq \theta, \Phi \leqq \pi$, ρ an arbitrary parameter and $\{A_m\}_{m=0}^{\infty}$ an arbitrary complex sequence such that the infinite series on both sides of (1.13) converge. Following [**11**] we have used the shorthand notation ${}_{10}W_9$ for a very well-poised ${}_{10}\Phi_9$ series, that is,

$$\begin{aligned} &{}_{r+3}W_{r+2}(a; b_1, b_2, \dots, b_r; q, z) \\ &\equiv {}_{r+3}\Phi_{r+2}\left[\begin{matrix} a, q\sqrt{a}, -q\sqrt{a}, b_1, b_2, \dots, b_r \\ \sqrt{a}, -\sqrt{a}, aq/b_1, aq/b_2, \dots, aq/b_r \end{matrix}; q, z\right]. \end{aligned} \tag{1.16}$$

Note that the argument of the ${}_{10}W_9$ series in (1.15) is bcq/ad which reduces to q in the special case $ad = bc$. This is an important special case since the corresponding ${}_{10}\Phi_9$ series is now both very well-poised and balanced and therefore transformable to another ${}_{10}\Phi_9$ series of the same kind by virtue of Bailey's transformation [**4**]. Furthermore, as we shall prove in Section 3, it is in this case that there is another closely related bilinear formula

$$G(x, y|q) = \sum_{n=0}^{\infty} \frac{\mu_n}{h_n} p_n(x; a, b, c, d) p_n(y; a, b, c, d), \tag{1.17}$$

where

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$$(1.18)\quad \mu_n = h_0 \frac{(bc)_n(ad/t)_n}{(ad)_n(bct)_n} t^n \sum_{k=0}^{\infty} \frac{B_k}{(q)_k(bctq^n)_k(tq^{1-n}/ad)_n},$$

$$(1.19)\quad ad = bc, \quad \text{and}$$

$$(1.20)\quad G(x, y|q) =$$

$$\frac{(abcd)_\infty(ac^{-1}t)_\infty(t)_\infty|(ae^{i\theta})_\infty(be^{i\Phi})_\infty(cte^{i\Phi})_\infty(dte^{i\theta})_\infty|^2}{(ab)_\infty(ac)_\infty(ad)_\infty(bd)_\infty(bct)_\infty(cdt)_\infty(ac^{-1})_\infty|(te^{i\theta+i\Phi})_\infty(te^{i\theta-i\Phi})_\infty|^2}$$

$$\times \sum_{k=0}^{\infty} \frac{B_k(cdt)_k|(te^{i\theta+i\Phi})_k(te^{i\theta-i\Phi})_k|^2}{(q)_k(t)_k(qt/ad)_k(ac^{-1}t)_k|(cte^{i\Phi})_k(dte^{i\theta})_k|^2}$$

$$\times {}_{10}W_9(cdtq^{k-1}; tq^k, bctq^{k-1}, ca^{-1}tq^k, ce^{i\theta}, ce^{-i\theta}, de^{i\Phi}, de^{-i\Phi}; q, q)$$

$$+ \frac{(abcd)_\infty(ca^{-1}t)_\infty(t)_\infty|(ce^{i\theta})_\infty(de^{i\Phi})_\infty(ate^{i\Phi})_\infty(bte^{i\theta})_\infty|^2}{(bd)_\infty(cd)_\infty(bc)_\infty(ac)_\infty(abt)_\infty(adt)_\infty(ca^{-1})_\infty|(te^{i\theta+i\Phi})_\infty(te^{i\theta-i\Phi})_\infty|^2}$$

$$\times \sum_{k=0}^{\infty} \frac{B_k(abt)_k|(te^{i\theta+i\Phi})_k(te^{i\theta-i\Phi})_k|^2}{(q)_k(t)_k(qt/bc)_k(ca^{-1}t)_k|(ate^{i\Phi})_k(bte^{i\theta})_k|^2}$$

$$\times {}_{10}W_9(abtq^{k-1}; tq^k, adtq^{k-1}, ac^{-1}tq^k, ae^{i\theta}, ae^{-i\theta}, be^{i\Phi}, be^{-i\Phi}; q, q).$$

t is assumed to be an arbitrary real parameter and $\{B_k\}_{k=0}^{\infty}$ an arbitrary complex sequence subject to the requirement that the series in (1.17), (1.18) and (1.20) are all convergent.

In Section 4 we shall deal with the relationship between the bilinear formulas (1.13) and (1.17), and derive what we consider a proper q-analogue of (1.1). In Section 5 we shall show how this q-analogue leads to a Poisson kernel for the q-Wilson polynomials subject only to the restriction (1.19).

2. Proof of (1.13). We shall start by computing the integral

$$I_{k,l} \equiv \int_{-1}^{1} w(y; a, b, c, d)p_n(y; a, b, c, d)\,|(ae^{i\Phi})_k(de^{i\Phi})_l|^2 dy.$$

Since by Sears' **[14]** transformation formula for a balanced and terminating ${}_4\Phi_3$ series

$$(2.1)\quad \begin{aligned} &p_n(y; a, b, c, d) \\ &= {}_4\Phi_3\left[\begin{matrix} q^{-n}, abcdq^{n-1}, ae^{i\Phi}, ae^{-i\Phi} \\ ab, ac, ad \end{matrix}; q, q\right] \\ &= \frac{(bd)_n(bc)_n}{(ac)_n(ad)_n}(ab^{-1})^n\, {}_4\Phi_3\left[\begin{matrix} q^{-n}, abcdq^{n-1}, be^{i\Phi}, be^{-i\Phi} \\ ba, bc, bd \end{matrix}; q, q\right], \end{aligned}$$

we get

$$I_{k,l} = \frac{(bd)_n(bc)_n}{(ac)_n(ad)_n}(ab^{-1})^n \sum_{j=0}^{n} \frac{(q^{-n})_j(abcdq^{n-1})_j q^j}{(q)_j(ba)_j(bc)_j(bd)_j} \times \int_{-1}^{1} w(y;\, aq^k,\, bq^j,\, c,\, dq^{n-j})dy. \tag{2.2}$$

Setting $m = n = 0$ in (1.7) and replacing a, b, d, by aq^k, bq^j and dq^{n-j}, respectively, and simplifying, we obtain

$$I_{k,l} = h_0 \frac{(bc)_n(ad)^n}{(ad)_n(abcd)_{n+k+l}}(ab)_k(ac)_k(bd)_l(cd)_l(ad)_{k+l} \times (q^{-k-l})_n q^{n(k+l)} \times {}_4\Phi_3\left[\begin{matrix} q^{-n},\, abcdq^{n-1},\, q^{-k},\, ad^{-1}q^{-l} \\ ab,\, ac,\, q^{-k-l} \end{matrix}; q, q\right], \tag{2.3}$$

provided $0 \leqq n \leqq k + l$, and, 0, otherwise. It is, of course, assumed that

$$|q| < 1 \quad \text{and} \quad \max(\, |a|, |b|, |c|, |d|\,) < 1.$$

A simple transformation of the double sum on the right hand side of (1.15) gives

$$F(x, y|q) = \sum_{k=0}^{\infty} \sum_{l=0}^{\infty} A_{k+l} \frac{1 - ad^{-1}q^{k-l}}{1 - ad^{-1}} \rho^{k+l} q^l |\, (ae^{i\theta})_k (de^{i\theta})_l|^2 \times |\, (ae^{i\Phi})_k (de^{i\Phi})_l|^2 / (q)_k (q)_l (ab)_k (ac)_k \times (aq/d)_k (bd)_l (cd)_l (dq/a)_l. \tag{2.4}$$

Using (2.3) and simplifying we get

$$\int_{-1}^{1} w(y;\, a, b, c, d) F(x, y|q) p_n(y;\, a, b, c, d) dy$$
$$= h_0 \frac{(bc)_n (q)_n}{(abcd)_{2n}} (-q\rho ad)^n q^{\binom{n}{2}} \sum_{m=0}^{\infty} (\rho q)^m \frac{A_{n+m}(adq^n)_m (q^{n+1})_m}{(q)_m (abcdq^{2n})_m}$$
$$\times \sum_{j=0}^{n} \frac{(q^{-n})_j (abcdq^{n-1})_j (-1)^j q^{-\binom{j}{2}-j}}{(q)_j (ab)_j (ac)_j (q^{-m-n})_j (aq/d)_j}$$
$$\times \frac{|\, (ae^{i\theta})_j (de^{i\theta})_{m+n-j}|^2 (ad^{-1}q^{j-m-n})_{j+1}}{(q)_{m+n-j} (dq/a)_{m+n-j} (1 - ad^{-1})} \tag{2.5}$$
$$\times {}_6\Phi_5\left[\begin{matrix} ad^{-1}q^{2j-m-n},\, q\sqrt{\ },\, -q\sqrt{\ },\, ae^{i\theta}q^j,\, ae^{-i\theta}q^j,\, q^{j-m-n} \\ \sqrt{\ },\, -\sqrt{\ },\, d^{-1}e^{-i\theta}q^{1+j-m-n},\, d^{-1}e^{i\theta}q^{1+j-m-n},\, ad^{-1}q^{1+j} \end{matrix};\, q,\, q^{1-j}/ad\right],$$

where the square root is over $ad^{-1}q^{2j-m-n}$.

The ${}_6\Phi_5$ series is summable by [**16**, IV.9] with sum

$$\frac{(ad^{-1}q^{1+2j-m-n})_{m+n-j}(a^{-1}d^{-1}q^{1-m-n})_{m+n-j}}{(d^{-1}e^{i\theta}q^{1+j-m-n})_{m+n-j}(d^{-1}e^{-i\theta}q^{1+j-m-n})_{m+n-j}}$$
$$= \frac{(da^{-1}q^{-j})_{m+n-j}(adq^{j})_{m+n-j}}{|(de^{i\theta})_{m+n-j}|^2}.$$

A few more steps of straightforward computation leads to the connection relation

$$\int_{-1}^{1} w(y;\, a, b, c, d)F(x, y|q)p_n(y;\, a, b, c, d)dy = \lambda_n p_n(x;\, a, b, c, d), \tag{2.6}$$

where λ_n is given by (1.14).

Let us assume that all the parameters are real and that the sequence $\{A_n\}$ is such that

$$\sum_{n=0}^{\infty} \lambda_n^2 < \infty,$$

$F(x, y|q)$ is square-integrable, non-negative and even continuous, at least in any interval

$$1 - 2\epsilon_1 \leqq x, y \leqq 1 - 2\epsilon_2, \quad \epsilon_1, \epsilon_2 > 0.$$

Then Mercer's theorem immediately leads to the bilinear formula (1.13).

(1.13) is, of course, much too general to be useful, so we consider some important special cases.

Case I.

$$A_n = (q^{-r})_n(abcdq^{r-1})_n/(ad)_n(ad)_n,$$

$\rho = q$; r a non-negative integer.

Here

$$\lambda_n = h_0\frac{(q^{-r})_n(abcdq^{r-1})_n(bc)_n}{(ad)_n(abcd)_{2n}} \times (-qad)^n q^{\binom{n}{2}}{}_2\Phi_1\left[\begin{matrix} q^{n-r},\ abcdq^{n+r-1} \\ abcdq^{2n}\end{matrix};\, q,\, q\right]$$
$$= h_0\frac{(q^{-r})_n(abcdq^{r-1})_n(bc)_n}{(ad)_n(abcd)_{n+r}}(-qad)^n q^{\binom{n}{2}}(q^{1+n-r})_{r-n}.$$

This vanishes unless $n = r$. Simplifying, we get

$$\lambda_n = h_0\frac{(q)_n(1 - abcdq^{-1})(bc)_n}{(abcdq^{-1})_n(1 - abcdq^{2n-1})(ad)_n}(ad)^n\delta_{n,r}. \tag{2.7}$$

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This leads to Rahman's product formula [**8**, **10**]

$$p_r(x;\, a, b, c, d)p_r(y;\, a, b, c, d)$$

$$= \frac{(bd)_r(cd)_r}{(ac)_r(ab)_r}(a/d)^r$$

$$\times \sum_{n=0}^{r} \frac{(q^{-r})_n(abcdq^{r-1})_n|(de^{i\theta})_n(de^{i\Phi})_n|^2 q^n}{(q)_n(ad)_n(ad)_n(bd)_n(cd)_n(da^{-1})_n} \tag{2.8}$$

$$\times\ {}_{10}W_9(ad^{-1}q^{-n};\, q^{1-n}/bd,\, q^{1-n}/cd,\, q^{-n},\, ae^{i\theta},\, ae^{-i\theta},$$

$$ae^{i\Phi},\, ae^{-i\Phi};\, q,\, bcq/ad).$$

Case II.

$$A_n = (q^{-r})_n(\lambda\mu\nu\tau q^{r-1})_n(-ab)_n(cd)_n$$

$$/(ad)_n(ad)_n(-\lambda\mu)_n(\nu\tau)_n,\ \rho = q;$$

r a non-negative integer, λ, μ, ν, τ arbitrary constants.
Here we have

$$\lambda_n = h_0\frac{(q^{-r})_n(\lambda\mu\nu\tau q^{r-1})_n(bc)_n(cd)_n(-ab)_n}{(ad)_n(-\lambda\mu)_n(\nu\tau)_n(abcd)_{2n}}(-qad)^n q^{\binom{n}{2}}$$

$$\times\ {}_4\Phi_3\left[\begin{matrix} q^{n-r},\ \lambda\mu\nu\tau q^{n+r-1},\ -abq^n,\ cdq^n \\ abcdq^{2n},\ -\lambda\mu q^n,\ \nu\tau q^n \end{matrix};\, q,\, q\right] \tag{2.9}$$

$$= h_0\frac{(-\nu\tau)_r(\lambda\mu)_r}{(\nu\tau)_r(-\lambda\mu)_r}(-1)^r$$

$$\times \frac{(q^{-r})_n(\lambda\mu\nu\tau q^{r-1})_n(bc)_n(cd)_n(-ab)_n}{(ad)_n(\lambda\mu)_n(-\nu\tau)_n(abcd)_{2n}}(qad)^n q^{\binom{n}{2}}$$

$$\times\ {}_4\Phi_3\left[\begin{matrix} q^{n-r},\ \lambda\mu\nu\tau q^{n+r-1},\ abq^n,\ -cdq^n \\ abcdq^{2n},\ \lambda\mu q^n,\ -\nu\tau q^n \end{matrix};\, q,\, q\right],$$

by use of Sears' formula (2.1). So the corresponding bilinear formula is

$$\sum_{m=0}^{r} \frac{(q^{-r})_m(\lambda\mu\nu\tau q^{r-1})_m(-ab)_m|(de^{i\theta})_m(de^{i\Phi})_m|^2 q^m}{(q)_m(ad)_m(ad)_m(bd)_m(-\lambda\mu)_m(\nu\tau)_m(d/a)_m}$$

$$\times\ {}_{10}W_9(ad^{-1}q^{-m};\, q^{1-m}/bd,\, q^{1-m}/cd,\, q^{-m},\, ae^{i\theta},\, ae^{-i\theta},$$

$$ae^{i\Phi},\, ae^{-i\Phi};\, q,\, bcq/ad)$$

$$= \frac{(\lambda\mu)_r(-\nu\tau)_r}{(\nu\tau)_r(-\lambda\mu)_r}(-1)^r$$

$$\times \sum_{n=0}^{r} \frac{(abcdq^{-1})_n(1 - abcdq^{2n-1})(ab)_n(ac)_n(-ab)_n}{(q)_n(1 - abcdq^{-1})(\lambda\mu)_n(bd)_n(-\nu\tau)_n}(qd/a)^n \tag{2.10}$$

$$q^{\binom{n}{2}}\frac{(q^{-r})_n(\lambda\mu\nu\tau q^{r-1})_n}{(abcd)_{2n}}$$

$$\times {}_4\Phi_3\left[\begin{matrix} q^{n-r}, \lambda\mu\nu\tau q^{n+r-1}, abq^n, -cdq^n \\ abcdq^{2n}, \lambda\mu q^n, -\nu\tau q^n \end{matrix}; q, q\right]$$

$$\times p_n(x; a, b, c, d)p_n(y; a, b, c, d).$$

It wouldn't appear there is anything special about this formula until we consider the following special case

$$\begin{matrix} a = -d = \sqrt{q}, b = q^{\alpha+1/2}, c = -q^{\beta+1/2} \\ \lambda = -\tau = \sqrt{q}, \mu = q^{a+1/2}, \nu = -q^{b+1/2} \end{matrix} \tag{2.11}$$

and set $e^{i\Phi} = a$.

Then the left hand side of (2.10) reduces to

$${}_4\Phi_3\left[\begin{matrix} q^{-r}, q^{r+a+b+1}, -\sqrt{q}e^{i\theta}, -\sqrt{q}e^{-i\theta} \\ -q, -q^{a+1}, q^{b+1} \end{matrix}; q, q\right]$$

$$= \frac{(-q^{b+1})_r(q^{a+1})_r}{(-q^{a+1})_r(q^{b+1})_r}(-1)^r$$

$$\times {}_4\Phi_3\left[\begin{matrix} q^{-r}, q^{r+a+b+1}, \sqrt{q}e^{i\theta}, \sqrt{q}e^{-i\theta} \\ q^{a+1}, -q^{b+1}, -q \end{matrix}; q, q\right] \tag{2.12}$$

$$= \frac{(q)_r(-q)_r}{(-q^{a+1})_r(q^{b+1})_r}(-1)^r P_r^{(a,b)}(x; q),$$

where

$$P_r^{(a,b)}(x; q) = \frac{(q^{a+1})_r(-q^{b+1})_r}{(q)_r(-q)_r} \times {}_4\Phi_3\left[\begin{matrix} q^{-r}, q^{r+a+b+1}, \sqrt{q}e^{i\theta}, \sqrt{q}e^{-i\theta} \\ q^{a+1}, -q^{b+1}, -q \end{matrix}; q, q\right] \tag{2.13}$$

is the continuous q-Jacobi polynomial introduced by the author in [**11**]. The right hand side of (2.10) simplifies to

$$\frac{(q)_r(-q)_r}{(q^{b+1})_r(-q^{a+1})_r}(-1)^r \sum_{n=0}^{r} g(n, r; q)P_n^{(\alpha,\beta)}(x; q)$$

where

$$g(n, r; q) = \frac{(q^{a+1})_r(-q^{b+1})_r}{(q)_r(-q)_r} \frac{(q^{\alpha+\beta+1})_n(-q)_n}{(q^{\alpha+\beta+1})_{2n}}$$

$$(2.14) \quad \cdot \frac{(q^{r+a+b+1})_n q^{n^2-nr}}{(q)_{r-n}(q^{a+1})_n(-q^{b+1})_n}$$

$$\times {}_4\Phi_3\left[\begin{matrix} q^{n-r}, q^{n+r+a+b+1}, q^{\alpha+1+n}, -q^{\beta+1+n} \\ q^{\alpha+\beta+2+2n}, q^{a+1+n}, -q^{b+1+n} \end{matrix}; q, q\right].$$

This leads to the projection formula [**2**]

$$(2.15) \quad P_r^{(a,b)}(x; q) = \sum_{n=0}^{r} g(n, r; q) P_n^{(\alpha,\beta)}(x; q)$$

which is, of course, a q-extension of an analogous formula given by Feldheim [**6**] for Jacobi polynomials.

Case III.

$$A_n = \frac{(\alpha_1)_n(\alpha_2)_n(-\alpha_3)_n(-\alpha_4)_n}{(ad)_n(ad)_n(-\alpha_5/abcd)_n(-q\alpha_1\alpha_2\alpha_3\alpha_4/\alpha_5)_n},$$

where $\alpha_1, \ldots, \alpha_5$ are arbitrary parameters.

The bilinear sum (1.13) reduces to

$$\sum_{m=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_m(-\alpha_3)_m(-\alpha_4)_m}{(q)_m(ad)_m(ad)_m(bd)_m(cd)_m}$$

$$\times \frac{|(de^{i\theta})_m(de^{i\Phi})_m|^2\rho^m}{(-\alpha_5/abcd)_m(-q\alpha_1\alpha_2\alpha_3\alpha_4/\alpha_5)_m(d/a)_m}$$

$$\times {}_{10}W_9(ad^{-1}q^{-m}; q^{1-m}/bd, q^{1-m}cd, q^{-m}, ae^{i\theta},$$

$$ae^{-i\theta}, ae^{i\Phi}, ae^{-i\Phi}; q, bcq/ad)$$

$$(2.16) \quad = \sum_{n=0}^{\infty} \frac{(abcdq^{-1})_n(1 - abcdq^{2n-1})(ab)_n(ac)_n}{(q)_n(1 - abcdq^{-1})(cd)_n(bd)_n(-\alpha_5/abcd)_n}$$

$$\times \frac{(\alpha_1)_n(\alpha_2)_n(-\alpha_3)_n(-\alpha_4)_n}{(-q\alpha_1\alpha_2\alpha_3\alpha_4/\alpha_5)_n(abcd)_{2n}} q^{\binom{n}{2}}(-\rho d/a)^n$$

$$\times {}_4\Phi_3\left[\begin{matrix} \alpha_1q^n, \alpha_2q^n, -\alpha_3q^n, -\alpha_4q^n, \\ abcdq^{2n}, -q^n\alpha_5/abcd, -q^{n+1}\alpha_1\alpha_2\alpha_3\alpha_4/\alpha_5 \end{matrix}; q, \rho\right]$$

$$\times p_n(x; a, b, c, d)p_n(y; a, b, c, d),$$

which is valid provided $|\rho| < 1$. One can see, of course, that the cases I and II are both obtainable from this by specializing the parameters $\alpha_1, \ldots, \alpha_5$ and ρ. If we replace α_i by q^{α_i}, $i = 1, \ldots, 5$, specialize a, b, c, d according to

(2.11), and then let $q \to 1$, then (2.16) reduces to Feldheim's formula (1.1). In this very formal sense, therefore, (2.16) is a q-analogue of (1.1). However, it is a pretty useless analogue because, unlike the ${}_2F_1$ function on the right hand side of (1.1), the ${}_4\Phi_3$ series in (2.16) generally cannot be transformed even when we set $\rho = q$ forcing it to be a balanced series, unless it terminates. Bearing in mind how Bailey's Poisson kernel for Jacobi polynomials was deduced [**12**] from (1.1) by taking

$$\alpha_1 = (\alpha + \beta + 2)/2, \alpha_2 = \alpha_1 + 1/2,$$
$$\rho^{-1/2} = 1/2(t^{1/2} + t^{-1/2}), 0 < t < 1,$$

we may set up an analogous situation by putting

$$\text{(2.17)} \quad \begin{aligned} &ad = bc, \rho = q, \alpha_1 = \alpha_3 = bc, \\ &\alpha_2 = \alpha_4 = bc\sqrt{q}, \alpha_5 = -(bc)^3qt, \end{aligned}$$

so that (2.16) gives

$$\sum_{m=0}^{\infty} \frac{(-bc)_m(bc\sqrt{q})_m(-bc\sqrt{q})_m|(bca^{-1}e^{i\theta})_m(bca^{-1}e^{i\Phi})_m|^2}{(q)_m(bc)_m(bc^2a^{-1})_m(b^2ca^{-1})_m(bca^{-2})_m(bcqt)_m(bcq/t)_m} q^m$$
$$\times {}_{10}W_9(a^2b^{-1}c^{-1}q^{-m}; ab^{-2}c^{-1}q^{1-m}, ab^{-1}c^{-2}q^{1-m},$$
$$q^{-m}, ae^{i\theta}, ae^{-i\theta}, ae^{i\Phi}, ae^{-i\Phi}; q, q)$$

$$\text{(2.18)} \quad = \sum_{n=0}^{\infty} \frac{(b^2c^2q^{-1})_n(1 - b^2c^2q^{2n-1})(ab)_n(ac)_n}{(q)_n(1 - b^2c^2q^{-1})(a^{-1}bc^2)_n(a^{-1}b^2c)_n}$$
$$\times a^{-2n}\left\{\frac{(bc\sqrt{q})_n(-bc\sqrt{q})_n}{(bcqt)_n(bcq/t)_n} q^{\binom{n}{2}}(-bcq)^n \right.$$
$$\left. \times {}_4\Phi_3\left[\begin{matrix} bcq^n, -bcq^n, bcq^{n+1/2}, -bcq^{n+1/2} \\ b^2c^2q^{2n}, \quad bctq^{n+1}, bcq^{n+1}/t \end{matrix} ; q, q\right]\right\}$$
$$\times p_n(x; a, b, c, bca^{-1})p_n(y; a, b, c, bca^{-1}).$$

The expression within the curly brackets on the right cannot be expressed as a multiple of the nth power of any parameter for any choice of t, so (2.18) does not give a Poisson kernel for the q-Wilson polynomials $p_n(x; a, b, c, bca^{-1})$, even though the limiting sum, with $q \to 1$, does lead to Bailey's formula [**3**, p. 102].

To obtain a proper q-analogue of (1.1) we first set $\rho = q$ in (2.16) and observe that the ${}_4\Phi_3$ series is closely related to another ${}_4\Phi_3$ series of the same kind through a very useful formula of Bailey [**3**, p. 69]:

$${}_8W_7(a; b, c, d, e, f; q, a^2q^2/bcdef)$$

$$= \frac{(aq)_\infty (aq/de)_\infty (aq/df)_\infty (aq/ef)_\infty}{(aq/d)_\infty (aq/e)_\infty (aq/f)_\infty (aq/def)_\infty}$$

$$\times {}_4\Phi_3\left[\begin{matrix} d, e, f, aq/bc \\ aq/b, aq/c, def/a \end{matrix}; q, q\right]$$

$$(2.19) \quad + \frac{(aq)_\infty (aq/bc)_\infty (d)_\infty (e)_\infty (f)_\infty}{(aq/b)_\infty (aq/c)_\infty (aq/d)_\infty (aq/e)_\infty}$$

$$\times \frac{(a^2q^2/cdef)_\infty (a^2q^2/bdef)_\infty}{(aq/f)_\infty (a^2q^2/bcdef)_\infty (def/aq)_\infty}$$

$$\times {}_4\Phi_3\left[\begin{matrix} aq/de, aq/df, aq/ef, a^2q^2/bcdef \\ a^2q^2/cdef, a^2q^2/bdef, aq^2/def \end{matrix}; q, q\right].$$

The formula corresponding to the ${}_4\Phi_3$ series in (2.16) is, then,

$${}_8W_7(\alpha_5\alpha_4{}^{-1}q^{2n-1}; \alpha_5/\alpha_4 abcd, -abcdq^n/\alpha_4, \alpha_1 q^n, \alpha_2 q^n, -\alpha_3 q^n; q, \alpha_5/\alpha_1\alpha_2\alpha_3)$$

$$(2.20) \quad = \frac{(\alpha_5 q^{2n}/\alpha_4)_\infty (\alpha_5/\alpha_1\alpha_2\alpha_4)_\infty (-\alpha_5/\alpha_1\alpha_3\alpha_4)_\infty (-\alpha_5/\alpha_2\alpha_3\alpha_4)_\infty}{(\alpha_5 q^n/\alpha_1\alpha_4)_\infty (\alpha_5 q^n/\alpha_2\alpha_4)_\infty (-\alpha_5 q^n/\alpha_3\alpha_4)_\infty (-\alpha_5/\alpha_1\alpha_2\alpha_3\alpha_4 q^n)_\infty}$$

$$\times {}_4\Phi_3\left[\begin{matrix} \alpha_1 q^n, \alpha_2 q^n, -\alpha_3 q^n, -\alpha_4 q^n \\ abcdq^{2n}, -q^n\alpha_5/abcd, -q^{n+1}\alpha_1\alpha_2\alpha_3\alpha_4/\alpha_5 \end{matrix}; q; q\right]$$

$$+ \frac{(\alpha_5 q^{2n}/\alpha_4)_\infty (\alpha_1 q^n)_\infty (\alpha_2 q^n)_\infty (-\alpha_3 q^n)_\infty}{(abcdq^{2n})_\infty \left(-\frac{\alpha_5 q^n}{abcd}\right)_\infty \left(\frac{\alpha_5 q^n}{\alpha_1\alpha_4}\right)_\infty \left(\frac{\alpha_5 q^n}{\alpha_2\alpha_4}\right)_\infty}$$

$$\times \frac{(-\alpha_4 q^n)_\infty \left(-\frac{\alpha_5 abcdq^n}{\alpha_1\alpha_2\alpha_3\ \alpha_4}\right)_\infty \left(\frac{\alpha_5{}^2}{abcd\alpha_1\alpha_2\alpha_3\alpha_4}\right)_\infty}{\left(-\frac{\alpha_5 q^n}{\alpha_3\alpha_4}\right)_\infty \left(\frac{\alpha_5}{\alpha_1\alpha_2\alpha_3}\right)_\infty \left(-\frac{\alpha_1\alpha_2\alpha_3\alpha_4 q^n}{\alpha_5}\right)_\infty}$$

$$\times {}_4\Phi_3\left[\begin{matrix} \frac{\alpha_5}{\alpha_1\alpha_2\alpha_4}, -\frac{\alpha_5}{\alpha_1\alpha_3\alpha_4}, -\frac{\alpha_5}{\alpha_2\alpha_3\alpha_4}, \frac{\alpha_5}{\alpha_1\alpha_2\alpha_3} \\ \frac{\alpha_5{}^2}{abcd\alpha_1\alpha_2\alpha_3\alpha_4}, -\frac{\alpha_5 abcdq^n}{\alpha_1\alpha_2\alpha_3\alpha_4}, -\frac{\alpha_5 q^{1-n}}{\alpha_1\alpha_2\alpha_3\alpha_4} \end{matrix}; q, q\right],$$

provided $|\alpha_5/\alpha_1\alpha_2\alpha_3| < 1$ when the series are non-terminating. While neither of the ${}_4\Phi_3$ series on the right is transformable, the ${}_8\Phi_7$ series on the left can be transformed to another ${}_8\Phi_7$ by a limiting case of Bailey's formula [**3**, 8.5(1)]:

$$
\begin{aligned}
&{}_8W_7(a;\, b,\, c,\, d,\, e, f;\, q,\, a^2q^2/bcdef)\\
(2.21)\quad &= \frac{(aq)_\infty(aq/ef)_\infty(a^2q^2/bcdf)_\infty(a^2q^2/bcde)_\infty}{(aq/e)_2(aq/f)_\infty(a^2q^2/bcdef)_\infty(a^2q^2/bcd)_\infty}\\
&\times\, {}_8W_7(a^2q/bcd;\, aq/cd,\, aq/bd,\, aq/bc,\, e, f;\, q,\, aq/ef).
\end{aligned}
$$

This is the key to the fact that (2.16) alone cannot be regarded as a q-analogue of (1.1); we need another bilinear sum with a ${}_4\Phi_3$ which is of the type that appears in the second term on the right of (2.20). This, then, is the motivation of considering the kernel $G(x, y|q)$ introduced in (1.17) and (1.18).

Before we proceed to the next section to do this computation it may be of interest to point out a special case of (2.16) where the ${}_4\Phi_3$ series can be summed. Set

$$\alpha_3 = \alpha_5, \quad \rho = abcd/\alpha_1\alpha_2,$$

and then let $\alpha_3, \alpha_4 \to 0$. The ${}_4\Phi_3$ series becomes a ${}_2\Phi_1$ which can be summed by Heine's formula [**3**, 8.4(3)], provided

$$|abcd/\alpha_1\alpha_2| < 1,$$

or the series terminates. Simplifying the result we get

$$
\begin{aligned}
&\sum_{m=0}^{\infty} \frac{(\alpha_1)_m(\alpha_2)_m|\,(de^{i\theta})_m(de^{i\Phi})_m|^2}{(q)_m(ad)_m(ad)_m(bd)_m(cd)_m(d/a)_m}\left(\frac{abcd}{\alpha_1\alpha_2}\right)^m\\
&\times\, {}_{10}W_9(ad^{-1}q^{-m};\, q^{1-m}/bd,\, q^{1-m}/cd,\, q^{-m},\\
&\qquad\qquad ae^{i\theta},\, ae^{-i\theta},\, ae^{i\Phi},\, ae^{-i\Phi};\, q,\, bcq/ad)\\
&= \frac{(abcd/\alpha_1)_\infty(abcd/\alpha_2)_\infty}{(abcd)_\infty(abcd/\alpha_1\alpha_2)_\infty}\\
&\times \sum_{n=0}^{\infty} \frac{(abcdq^{-1})_n(1 - abcdq^{2n-1})(ab)_n(ac)_n(\alpha_1)_n(\alpha_2)_n}{(q)_n(1 - abcdq^{-1})(cd)_n(bd)_n\left(\frac{abcd}{\alpha_1}\right)_n\left(\frac{abcd}{\alpha_2}\right)_n}\\
&\times\, q^{\binom{n}{2}}(-qd/a)^n p_n(x;\, a,\, b,\, c,\, d)p_n(y;\, a,\, b,\, c,\, d).
\end{aligned}
$$

This is a q-analogue of [**12**, (3.7)].

3. Proof of (1.17). The first step is to use the q-integral representation of $p_n(x;\, a,\, b,\, c,\, d)$ suggested by Al-Salam and Verma [**1**] and elaborated by Gasper and Rahman [**7**, (4.2)]

$$p_n(x;\, a,\, b,\, c,\, d) =$$

$$A(\theta)\frac{(cd)_n}{(ab)_n}\int_{e^{i\theta}q/b}^{e^{-i\theta}q/b}\frac{(abcdu/q)_\infty(bue^{i\theta})_\infty(bue^{-i\theta})_\infty}{(bau/q)_\infty(bcu/q)_\infty(bdu/q)_\infty} \times \frac{(q/u)_n}{(abcdu/q)_n}\left(\frac{abu}{q}\right)^n d_qu, \tag{3.1}$$

where

$$A(\theta) = \frac{2ib}{q(1-q)(q)_\infty(ac)_\infty(ad)_\infty(cd)_\infty|(be^{i\theta})_\infty|^2 w(x;\, a,\, b,\, c,\, d)},$$

and the q-integral is defined by

$$\int_a^b f(u)d_qu = \int_0^b f(u)d_qu - \int_0^a f(u)d_qu,$$
$$\int_0^a f(u)d_qu = a(1-q)\sum_{n=0}^{\infty} f(aq^n)q^n. \tag{3.2}$$

Using the symmetry of $p_n(x;\, a,\, b,\, c,\, d)$ in b, c and d we now express the product of the q-Wilson polynomials in terms of a q-integral:

$$\begin{aligned} &p_n(x;\, a,\, b,\, c,\, d)p_n(y;\, a,\, b,\, c,\, d) \\ &= B(\theta,\, \Phi)\frac{(cd)_n(bd)_n}{(ab)_n(ac)_n}\int_{qe^{i\theta}/b}^{qe^{-i\theta}/b}\frac{(abcdu/q)_\infty(bue^{i\theta})_\infty(bue^{-i\theta})_\infty}{(bau/q)_\infty(bcu/q)_\infty(bdu/q)_\infty} \\ &\times \int_{qe^{i\Phi}/c}^{qe^{-i\Phi}/c}\frac{(abcdv/q)_\infty(cve^{i\Phi})_\infty(cve^{-i\Phi})_\infty}{(cav/q)_\infty(cbv/q)_\infty(cdv/q)_\infty} \\ &\times \frac{(q/u)_n(q/v)_n}{(abcdu/q)_n\,(abcdv/q)_n}\left(\frac{a^2bcuv}{q^2}\right)^n d_qud_qv, \end{aligned} \tag{3.3}$$

where

$$\begin{aligned} B(\theta,\, \Phi) = &-4bc[\,(q(1-q)(q)_\infty(ad)_\infty)^2(ab)_\infty(ac)_\infty(bd)_\infty(cd)_\infty \\ &|(be^{i\theta})_\infty(ce^{i\Phi})_\infty|^2 w(x;\, a,\, b,\, c,\, d)w(y;\, a,\, b,\, c,\, d)\,]^{-1}. \end{aligned} \tag{3.4}$$

Assuming uniform convergence of the series on the right hand side of (1.17) we have

$$G(x,\, y|q) = B(\theta,\, \Phi)\sum_{k=0}^{\infty}\frac{B_k}{(q)_k(bct)_k(qt/ad)_k}H_k(x,\, y;\, a,\, b,\, c,\, d), \tag{3.5}$$

where

$$
\begin{aligned}
&H_k(x, y; a, b, c, d) \\
&= \int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} \frac{(abcdu/q)_\infty (bue^{i\theta})_\infty (bue^{-i\theta})_\infty}{(bau/q)_\infty (bcu/q)_\infty (bdu/q)_\infty} d_q u \\
(3.6) \quad &\int_{qe^{i\Phi}/c}^{qe^{-i\Phi}/c} d_q v \frac{(abcdv/q)_\infty (cve^{i\Phi})_\infty (cve^{-i\Phi})_\infty}{(cav/q)_\infty (cbv/q)_\infty (cdv/q)_\infty} \\
&\times {}_6W_5(abcdq^{-1}; q/u, q/v, adq^{-k}/t; q, tbcuvq^{k-2}).
\end{aligned}
$$

Using the sum of the ${}_6\Phi_5$ series [**16**, IV.9] in (3.6) and simplifying, we get

$$
\begin{aligned}
&H_k(x, y; a, b, c, d) = \\
(3.7) \quad &\frac{(abcd)_\infty}{(bctq^k)_\infty} \int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} d_q u \frac{(bctuq^{k-1})_\infty (bue^{i\theta})_\infty (bue^{-i\theta})_\infty}{(bau/q)_\infty (bcu/q)_\infty (bdu/q)_\infty} \\
&\times \int_{qe^{i\Phi}/c}^{qe^{-i\Phi}/c} d_q v \frac{(bctvq^{k-1})_\infty (cve^{i\Phi})_\infty (cve^{-i\Phi})_\infty (abcduv/q^2)_\infty}{(cav/q)_\infty (cbv/q)_\infty (cdv/q)_\infty (bctuvq^{k-2})_\infty}.
\end{aligned}
$$

However, using Al-Salam and Verma's [**1**, **7**] formula for expressing the ${}_8W_7$ in (2.19) as a q-integral, the integral over v in (3.7) turns out to be

$$
\begin{aligned}
&\frac{q(1-q)}{2ic \sin \Phi} \frac{(q)_\infty (ab)_\infty (ad)_\infty (bd)_\infty}{|(ae^{i\Phi})_\infty (be^{i\Phi})_\infty (de^{i\Phi})_\infty|^2} \\
(3.8) \quad &\times \frac{|(e^{2i\Phi})_\infty|^2 (btq^k e^{-i\Phi})_\infty (abdue^{-i\Phi}/q)_\infty}{(abde^{-i\Phi})_\infty (btuq^{k-1}e^{-i\Phi})_\infty} \\
&\times {}_8W_7(abdq^{-1}e^{-i\Phi}; q/u, adt^{-1}q^{-k}, ae^{-i\Phi}, \\
&\qquad be^{-i\Phi}, de^{-i\Phi}; q; btuq^{k-1}e^{i\Phi}).
\end{aligned}
$$

The next important step is to use Nassrallah and Rahman's [**9**] integral representation for an ${}_8\Phi_7$:

$$
\begin{aligned}
&{}_8W_7(abdq^{-1}e^{-i\Phi}; q/u, adt^{-1}q^{-k}, ae^{-i\Phi}, be^{-i\Phi}, de^{-i\Phi}; \\
&\qquad q, btuq^{k-1}e^{i\Phi}) \\
&= \frac{(q)_\infty (ae^{-i\Phi})_\infty (be^{-i\Phi})_\infty (de^{-i\Phi})_\infty (bau/q)_\infty (bdu/q)_\infty}{2\pi (ad)_\infty (ab)_\infty (bd)_\infty (abduq^{-1}e^{-i\Phi})_\infty} \\
&\times \frac{(adu/q)_\infty (abde^{-i\Phi})_\infty (tq^k)_\infty (ba^{-1}tq^k)_\infty (bd^{-1}tq^k)_\infty}{(bte^{-i\Phi}q^k)_\infty} \\
(3.9) \cdot &\int_{-1}^{1} w\left(z; \sqrt{\frac{ab}{d}}\, e^{-i\Phi/2}, \sqrt{\frac{ad}{b}}\, e^{-i\Phi/2}, \sqrt{\frac{bd}{a}}\, e^{-i\Phi/2}, \frac{\sqrt{abd}}{q}\, ue^{i\Phi/2}\right) \\
&\times \frac{h(z; \sqrt{abd}\, e^{i\Phi/2})}{h(z; \sqrt{b/ad}\, tq^k e^{i\Phi/2})} dz,
\end{aligned}
$$

provided

$$\max\left(\sqrt{\frac{ab}{d}}, \sqrt{\frac{ad}{b}}, \sqrt{\frac{bd}{a}}\right) < 1.$$

Using (3.8) and (3.9) in (3.7) and simplifying, we get, with $z = \cos \psi$,

$$\begin{aligned} &H_k(x, y; a, b, c, d) \\ &= \frac{(abcd)_\infty}{(bctq^k)_\infty} \frac{q(1 - q)(q)_\infty^2}{2ic \sin \Phi} \\ &\times \frac{|(e^{2i\Phi})_\infty|^2 (tq^k)_\infty (ba^{-1}tq^k)_\infty (bd^{-1}tq^k)_\infty}{2\pi(ae^{i\Phi})_\infty (be^{i\Phi})_\infty (de^{i\Phi})_\infty} \\ &\times \int_{-1}^{1} w\left(z; \sqrt{\frac{ab}{d}} e^{-i\Phi/2}, \sqrt{\frac{ad}{b}} d^{-i\Phi/2}, \sqrt{\frac{bd}{a}} e^{-i\Phi/2}, \sqrt{\frac{b}{ad}} tq^k e^{i\Phi/2}\right) dz \\ &\times \int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} \frac{(bctuq^{k-1})_\infty (bue^{i\theta})_\infty}{(bcu/q)_\infty (btue^{-i\Phi}q^{k-1})_\infty} \\ &\times \frac{(bue^{-i\theta})_\infty (adu/q)_\infty}{(\sqrt{adb}\, ue^{i\psi + i\Phi/2}/q)_\infty (\sqrt{abd}\, ue^{i\Phi/2 - i\psi}/q)_\infty} d_q u. \end{aligned} \tag{3.10}$$

If we now assume $ad = bc$ the last q-integral becomes the sum of appropriate multiples of the balanced and non-terminating ${}_3\Phi_2$ series which is summable by Sears' formula [**15**]. Thus

$$\begin{aligned} &\int_{qe^{i\theta}/b}^{qe^{-i\theta}/b} \frac{(bctuq^{k-1})_\infty (bue^{i\theta})_\infty (bue^{-i\theta})_\infty}{(btue^{-i\theta}q^{k-1})_\infty (b\sqrt{c}ue^{i\Phi/2 + i\psi}/q)_\infty (b\sqrt{c}ue^{i\Phi/2}q^{-i\psi})_\infty} d_q u \\ &= \frac{q(1 - q)(q)_\infty}{2ib \sin \theta} \frac{|(e^{2i\theta})_\infty|^2 (ce^{i\Phi})_\infty h(z; \sqrt{c}tq^k e^{-i\Phi/2})}{h(x; tq^k e^{-i\Phi}) h(z; \sqrt{c}e^{i\theta + i\Phi/2}) h(z; \sqrt{c}e^{i\Phi/2 - i\theta})}. \end{aligned} \tag{3.11}$$

Use of (3.11) in (3.10) yields

$$\begin{aligned} &H_k(x, y; a, b, c, bca^{-1}) \\ &= -\frac{q^2(1 - q)^2 (q)_\infty^3 (b^2c^2)_\infty |(e^{2i\theta})_\infty (e^{2i\Phi})_\infty|^2 (ce^{i\Phi})_\infty}{8\pi bc \sin \theta \sin \Phi (ae^{i\Phi})_\infty (be^{i\Phi})_\infty (bca^{-1}e^{i\Phi})_\infty} \\ &\times \frac{(tq^k)_\infty (ba^{-1}tq^k)_\infty (ac^{-1}tq^k)_\infty}{(bctq^k)_\infty (tq^k e^{i\theta - i\Phi})_\infty (tq^k e^{-i\theta - i\Phi})_\infty} \\ &\times \int_{-1}^{1} w\left(z; \sqrt{c}e^{-i\Phi/2}, \frac{b\sqrt{c}}{a} e^{-i\Phi/2}, \sqrt{c}e^{i\theta + i\Phi/2}, \sqrt{c}e^{i\Phi/2 - i\theta}\right) \end{aligned} \tag{3.12}$$

$$\times \frac{h(z;\, b\sqrt{c}e^{i\Phi/2})h(z;\, \sqrt{c}te^{-i\Phi/2}q^k)}{h\left(z;\, \frac{a}{\sqrt{c}}e^{-i\Phi/2}\right)h\left(z;\, \frac{tq^k}{\sqrt{c}}e^{i\Phi/2}\right)}dz.$$

The integral above is of the type

$$\int_{-1}^{1} \frac{h(z;\, 1)h(z;\, -1)h(z;\, \sqrt{q})}{h(z;\, \lambda)h(z;\, \mu)h(z;\, \nu)}$$

$$\times \frac{h(z;\, -\sqrt{q})h(z;\, \omega)h(z;\, \lambda\mu\nu\rho\sigma\tau/\omega)}{h(z;\, \rho)h(z;\, \sigma)h(z;\, \tau)} \frac{dz}{\sqrt{1-z^2}}.$$

Note that the integrand has an overall balance in the parameters; this property enabled the author recently to express the integral as the sum of two balanced, very well-poised and non-terminating ${}_{10}\Phi_9$ series [**13**].

Use of this formula for the integral in (3.12) gives

$$\frac{2\pi(bc^2a^{-1})_\infty(ctq^ke^{i\Phi})_\infty(bca^{-1}tq^ke^{i\Phi})_\infty}{(q)_\infty(ce^{i\Phi})_\infty(bca^{-1}e^{-i\Phi})_\infty|(ce^{i\theta})_\infty(bca^{-1}e^{i\theta})_\infty|^2}$$

$$\times \frac{|(bca^{-1}tq^ke^{i\theta})_\infty|^2}{(tq^k)_\infty(ba^{-1}tq^k)_\infty h(x;\, tq^ke^{i\Phi})}$$

$$\times \left\{\frac{(bc)_\infty(be^{i\Phi})_\infty(tq^k)_\infty(ctq^ke^{-i\Phi})_\infty}{(ac^{-1})_\infty(ae^{-i\Phi})_\infty(bc^2a^{-1}tq^k)_\infty(bca^{-1}tq^ke^{i\Phi})_\infty}\right.$$

$$\times {}_{10}W_9(bc^2a^{-1}tq^{k-1};\, tq^k,\, bctq^{k-1},\, ca^{-1}tq^k,$$

$$ce^{i\theta},\, ce^{-i\theta},\, bca^{-1}e^{i\Phi},\, bca^{-1}e^{-i\Phi};\, q,\, q)$$

$$+ \frac{(ab)_\infty(bc)_\infty|(ce^{i\theta})_\infty(bca^{-1}e^{i\Phi})_\infty}{(bc^2a^{-1})_\infty(ca^{-1})_\infty|(ae^{i\theta})_\infty(bca^{-1}tq^ke^{i\theta})_\infty|^2}$$

$$\times \frac{(atq^ke^{i\Phi})_\infty(btq^ke^{i\theta})_\infty|^2(tq^k)_\infty(ca^{-1}tq^k)_\infty}{(ae^{-i\Phi})_\infty(be^{-i\Phi})_\infty(ctq^ke^{i\Phi})_\infty(bca^{-1}tq^ke^{i\Phi})_\infty(ac^{-1}tq^k)_\infty(abtq^k)_\infty}$$

$$\times {}_{10}W_9(abtq^{k-1};\, tq^k,\, bctq^{k-1},\, ac^{-1}tq^k,$$

$$\left. ae^{i\theta},\, ae^{-i\theta},\, be^{i\Phi},\, be^{-i\Phi};\, q,\, q)\right\}.$$

Substituting this for the integral in (3.12) and using the subsequent expression for H_k in (3.5) immediately leads to (1.20). This completes the proof of (1.17).

Note that the restrictions that are required for the existence of the Riemann integral in (3.12), namely, that

$$\max(|c|,\, |b^2ca^{-2}|,\, |a^2c^{-1}|,\, |t^2/c|) < 1$$

can be removed, by analytic continuation, and do not need to be applied to (1.17) and (1.20) which are subject only to requirement of uniform convergence.

4. A proper q-analogue of Feldheim's formula. As we observed in Section 2, eq. (2.16) provides only a formal analogue of Feldheim's sum (1.1), but it cannot be directly applied to any problem of interest, unless we find a companion formula that connects the ${}_4\Phi_3$ in (2.16) to its natural companion in (2.20). Accordingly, we set

$$(4.1)\qquad B_k = q^k(\alpha_1 t/bc)_k(\alpha_2 t/bc)_k(-\alpha_3 t/bc)_k(-\alpha_4 t/bc)_k / (\alpha_1\alpha_2\alpha_3\alpha_4 t^2/b^4c^4)_k$$

in (1.17) and (1.19) to get

$$\sum_{n=0}^{\infty} \frac{(b^2c^2q^{-1})_n(1 - b^2c^2q^{2n-1})(ab)_n(ac)_n(bc/t)_n}{(q)_n(1 - b^2c^2q^{-1})(bc^2a^{-1})_n(b^2ca^{-1})_n(bct)_n}(t/a^2)^n$$
$$\times {}_4\Phi_3\left[\begin{matrix}\alpha_1 t/bc,\ \alpha_2 t/bc,\ -\alpha_3 t/bc,\ -\alpha_4 t/bc \\ \alpha_1\alpha_2\alpha_3\alpha_4 t^2/b^4c^4,\ bctq^n,\ tq^{1-n}/bc\end{matrix}; q, q\right]$$
$$\times p_n(x;\ a, b, c, bca^{-1})p_n(y;\ a, b, c, bca^{-1})$$
$$= \frac{(b^2c^2)_\infty(ac^{-1}t)_\infty(t)_\infty}{(ab)_\infty(ac)_\infty(bc)_\infty(ac^{-1})_\infty}$$
$$\times \frac{|(ae^{i\theta})_\infty(be^{i\Phi})_\infty(cte^{i\Phi})_\infty(bca^{-1}te^{i\theta})_\infty|^2}{(b^2ca^{-1})_\infty(bct)_\infty(bc^2a^{-1}t)_\infty|(te^{i\theta+i\Phi})_\infty(te^{i\theta-i\Phi})_\infty|^2}$$
$$\times \sum_{k=0}^{\infty} \frac{(\alpha_1 t/bc)_k(\alpha_2 t/bc)_k(-\alpha_3 t/bc)_k(-\alpha_4 t/bc)_k(bc^2a^{-1}t)_k}{(q)_k(t)_k(qt/bc)_k(ac^{-1}t)_k(\alpha_1\alpha_2\alpha_3\alpha_4 t^2/b^4c^4)_k}$$
$$\times \frac{|(te^{i\theta+i\Phi})_k(te^{i\theta-i\Phi})_k|^2}{|(cte^{i\Phi})_k(bca^{-1}te^{i\theta})_k|^2}$$
$$\times q^k {}_{10}W_9(bc^2a^{-1}tq^{k-1};\ tq^k,\ bctq^{k-1},\ ca^{-1}tq^k,\ ce^{i\theta},\ ce^{-i\theta},\ bca^{-1}e^{i\Phi},\ bca^{-1}e^{-i\Phi};\ q, q)$$
$$+ \frac{(b^2c^2)_\infty(ca^{-1}t)_\infty(t)_\infty}{(ac)_\infty(bc)_\infty(b^2ca^{-1})_\infty}$$
$$\times \frac{|(ce^{i\theta})_\infty(bca^{-1}e^{i\theta})_\infty(ate^{i\Phi})_\infty(bte^{i\theta})_\infty|^2}{(bc^2a^{-1})_\infty(ca^{-1})_\infty(abt)_\infty(bct)_\infty|(te^{i\theta+i\Phi})_\infty(te^{i\theta-i\Phi})_\infty|^2}$$
$$(4.2)\qquad \times \sum_{k=0}^{\infty} \frac{(\alpha_1 t/bc)_k(\alpha_2 t/bc)_k(-\alpha_3 t/bc)_k(-\alpha_4 t/bc)_k(abt)_k}{(q)_k(t)_k(qt/bc)_k(ca^{-1}t)_k(\alpha_1\alpha_2\alpha_3\alpha_4 t^2/b^4/c^4)_k}$$

$$\times \frac{|(te^{i\theta+i\Phi})_k(te^{i\theta-i\Phi})_k|^2}{|(ate^{i\Phi})_k(bte^{i\theta})_k|^2}$$

$$\times q^k {}_{10}W_9(abtq^{k-1};\ tq^k,\ bctq^{k-1},\ ac^{-1}tq^k,$$

$$ae^{i\theta},\ ae^{-i\theta},\ be^{i\Phi},\ be^{-i\Phi};\ q,\ q).$$

Setting $ad = bc$ and $\alpha_5 = -\alpha_1\alpha_2\alpha_3\alpha_4 t/bc$ in (2.20) we get

$$\frac{(b^2c^2q^{2n})_\infty\left(\frac{\alpha_1\alpha_2\alpha_3\alpha_4 tq^n}{b^3c^3}\right)_\infty\left(\frac{\alpha_1\alpha_2 tq^n}{bc}\right)_\infty}{\left(-\frac{\alpha_1\alpha_2\alpha_3 tq^{2n}}{bc}\right)_\infty(\alpha_1q^n)_\infty(\alpha_2q^n)_\infty(-\alpha_3q^n)_\infty}$$

$$\times \frac{\left(-\frac{\alpha_1\alpha_3 tq^n}{bc}\right)_\infty\left(-\frac{\alpha_2\alpha_3 tq^n}{bc}\right)_\infty\left(-\frac{\alpha_4 t}{bc}\right)_\infty\left(\frac{bc}{t}\right)_\infty}{(-\alpha_4q^n)_\infty(bct)_\infty\left(\frac{\alpha_1\alpha_2\alpha_3\alpha_4 t^2}{b^4c^4}\right)_\infty} t^n$$

$$\times {}_8W_7\left(-\frac{\alpha_1\alpha_2\alpha_3 tq^{2n-1}}{bc};\ -\frac{\alpha_1\alpha_2\alpha_3 t}{b^3c^3},\ -\frac{b^2c^2q^n}{\alpha_4},\right.$$

$$\left.\alpha_1q^n,\ \alpha_2q^n,\ -\alpha_3q^n;\ q,\ -\alpha_4 t/bc\right)$$

(4.3) $$= t^n\frac{(bc/t)_n}{(bct)_n}{}_4\Phi_3\left[\begin{matrix}\alpha_1t/bc,\ \alpha_2t/bc,\ -\alpha_3t/bc,\ -\alpha_4t/bc\\ \alpha_1\alpha_2\alpha_3\alpha_4t^2/b^4c^4,\ bctq^n,\ tq^{1-n}/bc\end{matrix};\ q,\ q\right]$$

$$+ \frac{(\alpha_1t/bc)_\infty(\alpha_2t/bc)_\infty(-\alpha_3t/bc)_\infty(-\alpha_4t/bc)_\infty}{(\alpha_1)_\infty(\alpha_2)_\infty(-\alpha_3)_\infty(-\alpha_4)_\infty(bct)_\infty(t/bc)_\infty}$$

$$\times \frac{(bc/t)_\infty(\alpha_1\alpha_2\alpha_3\alpha_4t/b^3c^3)_\infty(b^2c^2)_\infty}{(\alpha_1\alpha_2\alpha_3\alpha_4t^2/b^4c^4)_\infty}(-1)^nq^{\binom{n}{2}}(-qbc)^n$$

$$\times \frac{(\alpha_1)_n(\alpha_2)_n(-\alpha_3)_n(-\alpha_4)_n}{(bcq/t)_n\left(\frac{\alpha_1\alpha_2\alpha_3\alpha_4 t}{b^3c^3}\right)_n(b^2c^2)_{2n}}$$

$$\times {}_4\Phi_3\left[\begin{matrix}\alpha_1q^n,\ \alpha_2q^n,\ -\alpha_3q^n,\ -\alpha_4q^n\\ b^2c^2q^{2n},\ \frac{\alpha_1\alpha_2\alpha_3\alpha_4tq^n}{b^3c^3},\ bcq^{n+1}/t\end{matrix};\ q,\ q\right].$$

Now we add the appropriate multiple of (2.16) as suggested by (4.3) to (4.2) after setting

$$d = bca^{-1} \quad \text{and} \quad \alpha_5 = -\alpha_1\alpha_2\alpha_3\alpha_4 t/bc,$$

to obtain

$$\sum_{n=0}^{\infty} \frac{(b^2c^2q^{-1})_n(1 - b^2c^2q^{2n-1})(ab)_n}{(q)_n(1 - b^2c^2q^{-1})(bc^2a^{-1})_n(b^2ca^{-1})_n}$$

$$\times \frac{(ac)_n(\alpha_1)_n(\alpha_2)_n(-\alpha_3)_n(-\alpha_n)_n\left(-\frac{\alpha_1\alpha_2\alpha_3 t}{bc}\right)_{2n}}{\left(-\frac{\alpha_2\alpha_3 t}{bc}\right)_n\left(-\frac{\alpha_1\alpha_3 t}{bc}\right)_n\left(\frac{\alpha_1\alpha_2 t}{bc}\right)_n\left(\frac{\alpha_1\alpha_2\alpha_3\alpha_4 t}{b^3c^3}\right)_n(b^2c^2)_{2n}}$$

$$\times (t/a^2)^n {}_8W_7(-\frac{\alpha_1\alpha_2\alpha_3 tq^{2n-1}}{bc}; -\frac{\alpha_1\alpha_2\alpha_3 t}{b^3c^3}, -\frac{b^2c^2q^n}{\alpha_4},$$

$$\alpha_1 q^n, \alpha_2 q^n, -\alpha_3 q^n; q, -\alpha_4 t/bc)$$

$$\times p_n(x; a, b, c, bca^{-1})p_n(y; a, b, c, bca^{-1})$$

$$= \frac{(\alpha_1 t/bc)_\infty(\alpha_2 t/bc)_\infty(-\alpha_3 t/bc)_\infty(-\alpha_1\alpha_2\alpha_3 t/bc)_\infty}{(\alpha_1\alpha_2 t/bc)_\infty(-\alpha_1\alpha_3 t/bc)_\infty(-\alpha_2\alpha_3 t/bc)_\infty(t/bc)_\infty}$$

$$\times \sum_{k=0}^{\infty} \frac{(\alpha_1)_k(\alpha_2)_k(-\alpha_3)_k(-\alpha_4)_k}{(q)_k(bc)_k(bc)_k(b^2ca^{-1})_k(bc^2a^{-1})_k}$$

$$\times \frac{|(bca^{-1}e^{i\theta})_k(bca^{-1}e^{i\Phi})_k|^2 q^k}{(\alpha_1\alpha_2\alpha_3\alpha_4 t/b^3c^3)_k(bcq/t)_k(bca^{-2})_k}$$

$$\times {}_{10}W_9(a^2b^{-1}c^{-1}q^{-k}; aq^{1-k}/b^2c, aq^{1-k}/bc^2, q^{-k},$$

$$ae^{i\theta}, ae^{-i\theta}, ae^{i\Phi}, ae^{-i\Phi}; q, q)$$

$$+ \frac{(\alpha_1)_\infty(\alpha_2)_\infty(-\alpha_3)_\infty(-\alpha_4)_\infty(-\alpha_1\alpha_2\alpha_3 t/bc)_\infty}{(\alpha_1\alpha_2 t/bc)_\infty(-\alpha_1\alpha_3 t/bc)_\infty(-\alpha_2\alpha_3 t/bc)_\infty}$$

$$\times \frac{(\alpha_1\alpha_2\alpha_3\alpha_4 t^2/b^4c^4)_\infty(t)_\infty(ac^{-1}t)_\infty}{(-\alpha_4 t/bc)_\infty(\alpha_1\alpha_2\alpha_3\alpha_4 t/b^3c^3)_\infty(ab)_\infty(ac)_\infty(ac^{-1})_\infty}$$

$$\times \frac{|(ae^{i\theta})_\infty(be^{i\Phi})_\infty(bca^{-1}te^{i\theta})_\infty(cte^{i\Phi})_\infty|^2}{(bc)_\infty(b^2ca^{-1})_\infty(bc^2a^{-1}t)_\infty(bc/t)_\infty|(te^{i\theta+i\Phi})_\infty(te^{i\theta-i\Phi})_\infty|^2}$$

$$\times \sum_{k=0}^{\infty} \frac{(\alpha_1 t/bc)_k(\alpha_2 t/bc)_k(-\alpha_3 t/bc)_k(-\alpha_4 t/bc)_k(bc^2a^{-1}t)_k}{(q)_k(t)_k(qt/bc)_k(ac^{-1}t)_k(\alpha_1\alpha_2\alpha_3\alpha_4 t^2/b^4c^4)_k}$$

$$\times \frac{|(te^{i\theta+i\Phi})_k(te^{i\theta-i\Phi})_k|^2}{|(cte^{i\Phi})_k(bca^{-1}te^{i\theta})_k|^2} q^k {}_{10}W_9(bc^2a^{-1}tq^{k-1};$$

$$tq^k, bctq^{k-1}, ca^{-1}tq^k, ce^{i\theta}, ce^{-i\theta}, bca^{-1}e^{i\Phi}, bca^{-1}e^{-i\Phi}; q, q)$$

$$+ \frac{(\alpha_1)_\infty(\alpha_2)_\infty(-\alpha_3)_\infty(-\alpha_4)_\infty(-\alpha_1\alpha_2\alpha_3 t/bc)_\infty}{(\alpha_1\alpha_2 t/bc)_\infty(-\alpha_1\alpha_3 t/bc)_\infty(-\alpha_2\alpha_3 t/bc)_\infty}$$

$$\times \frac{(\alpha_1\alpha_2\alpha_3\alpha_4 t^2/b^4c^4)_\infty (t)_\infty (ca^{-1}t)_\infty}{(-\alpha_4 t/bc)_\infty (\alpha_1\alpha_2\alpha_3\alpha_4 t/b^3c^3)_\infty (ac)_\infty (b^2ca^{-1})_\infty (bc^2a^{-1})_\infty}$$

$$\times \frac{|(ce^{i\theta})_\infty (bca^{-1}e^{i\Phi})_\infty (bte^{i\theta})_\infty (ate^{i\Phi})_\infty|^2}{(bc)_\infty (ca^{-1})_\infty (abt)_\infty (bc/t)_\infty |(te^{i\theta+i\Phi})_\infty (te^{i\theta-i\Phi})_\infty|^2}$$

$$\sum_{k=0}^{\infty} \frac{(\alpha_1 t/bc)_k (\alpha_2 t/bc)_k}{(q)_k (t)_k}$$

$$(4.4) \qquad \times \frac{(-\alpha_3 t/bc)_k (-\alpha_4 t/bc)_k (abt)_k}{(qt/bc)_k (ca^{-1}t)_k (\alpha_1\alpha_2\alpha_3\alpha_4 t^2/b^4c^4)_k}$$

$$\times \frac{|(te^{i\theta+i\Phi})_k (te^{i\theta-i\Phi})_k|^2}{|(ate^{i\Phi})_k (bte^{i\theta})_k|^2} q^k$$

$${}_{10}W_9(abtq^{k-1};\ tq^k,\ bctq^{k-1},\ ac^{-1}tq^k,$$

$$ae^{i\theta},\ ae^{-i\theta},\ be^{i\Phi},\ be^{-i\Phi};\ q,\ q).$$

This is a formidable-looking formula and on first sight may appear unusable. However, there are a number of properties that make it much more useful than the apparently simpler formula (2.16). First, the ${}_8W_7$ series on the left (which is, of course, the same as an ${}_8\Phi_7$) is transformable to another ${}_8W_7$ of the same kind as in (2.21) or to a pair of balanced ${}_4\Phi_3$'s in a number of ways as in (2.19). Second, the ${}_{10}W_9$ series in the first term on the right hand side is a very well-poised terminating and balanced ${}_{10}\Phi_9$ and as such is transformable to another ${}_{10}\Phi_9$ by Bailey's theorem [**4**]. Third, the pair of generally non-terminating ${}_{10}W_9$ series in the second and third terms on the right hand side can be transformed to a similar pair by Bailey's formula [**5**]. Finally, one can show by suitably choosing the parameters α_1, α_2, α_3, α_4 as powers of q, setting

$$(4.5) \qquad a = q^{\alpha/2+1/4},\ b = q^{\alpha/2+3/4},\ c = -q^{\beta/2+1/4}$$

and taking the limit $q \to 1$ that (4.4) does approach a formula that is reducible to Feldheim's sum (1.1). Accordingly, we shall call (4.4) a q-analogue of (1.1), and we claim that it is a more appropriate one than (2.16). In the next section we shall consider a special case of (4.4) in an attempt to justify this claim.

5. Some special cases: a Poisson kernel. In this section we shall consider some special cases of (4.4); cases in which the ${}_8W_7$ series on the left can be summed. There is one obvious case in which the series becomes a ${}_6\Phi_5$. This happens when, for instance, $\alpha_3 = -bc/t$. Then we get

(5.1)
$$
\begin{aligned}
&{}_8W_7(\ ) \\
&= {}_8\Phi_7\left[\begin{matrix} \alpha_1\alpha_2 q^{2n-1},\ q\sqrt{\ },\ -q\sqrt{\ },\ \alpha_1 q^n,\ \alpha_2 q^n, \\ \sqrt{\ },\ -\sqrt{\ },\ \alpha_2 q^n,\ \alpha_1 q^n, \end{matrix}\right. \\
&\qquad\qquad \left.\begin{matrix} bcq^n/t,\ \dfrac{\alpha_1\alpha_2}{b^2c^2},\ -\dfrac{b^2c^2q^n}{\alpha_4} \\ \alpha_1\alpha_2 tq^n/bc,\ b^2c^2q^{2n},\ -\dfrac{\alpha_1\alpha_2\alpha_4 q^n}{b^2c^2} \end{matrix} ;\ q,\ -\alpha_4 t/bc\right] \\
&= {}_6\Phi_5\left[\begin{matrix} \alpha_1\alpha_2 q^{2n-1},\ q\sqrt{\ },\ -q\sqrt{\ },\ bcq^n/t, \\ \sqrt{\ },\ -\sqrt{\ },\ \alpha_1\alpha_2 tq^n/bc, \end{matrix}\right. \\
&\qquad\qquad \left.\begin{matrix} \alpha_1\alpha_2/b^2c^2,\ -b^2c^2q^n/\alpha_4 \\ b^2c^2q^{2n},\ -\alpha_1\alpha_2\alpha_4 q^n/b^2c^2 \end{matrix} ;\ q,\ -\alpha_4 t/bc\right] \\
&= \frac{(\alpha_1\alpha_2 q^{2n})_\infty (bctq^n)_\infty (-\alpha_1\alpha_2\alpha_4 t/b^3c^3)_\infty (-\alpha_4 q^n)_\infty}{(\alpha_1\alpha_2 tq^n/bc)_\infty (b^2c^2q^{2n})_\infty (-\alpha_1\alpha_2\alpha_4 q^n/b^2c^2)_\infty (-\alpha_4 t/bc)_\infty}
\end{aligned}
$$

provided, of course, $|\alpha_4 t/bc| < 1$. The left hand side of (4.4) then simplifies to

(5.2)
$$
\begin{aligned}
&\frac{(bct)_\infty(-\alpha_4)_\infty(\alpha_1\alpha_2)_\infty(-\alpha_1\alpha_2\alpha_4 t/b^3c^3)_\infty}{(b^2c^2)_\infty(-\alpha_4 t/bc)_\infty(\alpha_1\alpha_2 t/bc)_\infty(-\alpha_1\alpha_2\alpha_4/b^2c^2)_\infty} \\
&\times \sum_{n=0}^{\infty} \frac{(b^2c^2q^{-1})_\infty(1 - b^2c^2q^{2n-1})}{(q)_n(1 - b^2c^2q^{-1})} \\
&\times \frac{(ab)_n(ac)_n(bc/t)_n}{(bc^2a^{-1})_n(b^2ca^{-1})_n(bct)_n}(t/a^2)^n \\
&\times p_n(x;\ a,\ b,\ c,\ bca^{-1})p_n(y;\ a,\ b,\ c,\ bca^{-1}).
\end{aligned}
$$

Because of the factor $(-\alpha_3 t/bc)_\infty$ the first term on the right hand side of (4.4) vanishes. Also, the second and third terms on the right become single series since the factor $(-\alpha_3 t/bc)_k$ is zero unless $k = 0$. When we equate both sides the terms with α_1, α_2, α_4 cancel out and we get the following formula

$$
\begin{aligned}
&\frac{(b^2c^2)_\infty(t)_\infty(ac^{-1}t)_\infty}{(bc)_\infty(ab)_\infty(ac)_\infty(ac^{-1})_\infty} \\
&\times \frac{|(ae^{i\theta})_\infty(be^{i\Phi})_\infty(bca^{-1}te^{i\theta})_\infty(cte^{i\Phi})_\infty|^2}{(b^2ca^{-1})_\infty(bct)_\infty(bc^2a^{-1}t)_\infty|(te^{i\theta+i\Phi})_\infty(te^{i\theta-i\Phi})_\infty|^2}
\end{aligned}
$$

$$\times\ {}_{10}W_9(tbc^2a^{-1}q^{-1};\ t,\ bctq^{-1},\ ca^{-1}t,$$
$$ce^{i\theta},\ ce^{-i\theta},\ bca^{-1}e^{i\Phi},\ bca^{-1}e^{-i\Phi};\ q,\ q)$$

$$(5.3)\quad + \frac{(b^2c^2)_\infty(t)_\infty(ca^{-1}t)_\infty}{(bc)_\infty(ac)_\infty(b^2ca^{-1})_\infty}$$
$$\times \frac{|(ce^{i\theta})_\infty(bca^{-1}e^{i\Phi})_\infty(bte^{i\theta})_\infty(ate^{i\Phi})_\infty|^2}{(ca^{-1})_\infty(bc^2a^{-1})_\infty(bct)_\infty(abt)_\infty|(te^{i\theta+i\Phi})_\infty(te^{i\theta-i\Phi})_\infty|^2}$$
$$\times\ {}_{10}W_9(abtq^{-1};\ t,\ bctq^{-1},\ ac^{-1}t,\ ae^{i\theta},\ ae^{-i\theta},\ be^{i\Phi},\ be^{-i\Phi};\ q,\ q)$$
$$= \sum_{n=0}^{\infty} \frac{(b^2c^2q^{-1})_n(1-b^2c^2q^{2n-1})(ab)_n(ac)_n(bc/t)_n}{(q)_n(1-b^2c^2q^{-1})(bc^2a^{-1})_n(b^2ca^{-1})_n(bct)_n}(t/a^2)^n$$
$$\times\ p_n(x;\ a,\ b,\ c,\ bca^{-1})p_n(y;\ a,\ b,\ c,\ bca^{-1}).$$

This may be viewed as a non-terminating extension of the inverse of the product formula (2.8).

There are other special cases that one might consider, but the one that is of more interest to us corresponds to an evaluation of the series

$${}_8W_7(At^2;\ A,\ t,\ -t,\ t\sqrt{q},\ -t\sqrt{q};\ q,\ Aq)\quad \text{when } |Aq| < 1.$$

Setting $a = At^2$, $b = t$, $c = -t$, $d = t\sqrt{q}$, $e = -t\sqrt{q}$, $f = A$ in (2.19) we find

$${}_8W_7(At^2;\ t,\ -t,\ t\sqrt{q},\ -t\sqrt{q},\ A;\ q,\ Aq)$$
$$= \frac{(Aqt^2)_\infty(-A)_\infty(t\sqrt{q})_\infty(-t\sqrt{q})_\infty}{(At\sqrt{q})_\infty(-At\sqrt{q})_\infty(qt^2)_\infty(-1)_\infty}$$
$${}_4\Phi_3\left[\begin{matrix}A,\ -Aq,\ t\sqrt{q},\ -t\sqrt{q}\\ Atq,\ -Atq,\ -q\end{matrix};\ q,\ q\right]$$
$$+ \frac{(Aqt^2)_\infty(-A)_\infty(t\sqrt{q})_\infty(-t\sqrt{q})_\infty}{(At\sqrt{q})_\infty(-At\sqrt{q})_\infty(qt^2)_\infty(-1)_\infty}\ \frac{1-A}{1+A}$$
$${}_4\Phi_3\left[\begin{matrix}Aq,\ -A,\ t\sqrt{q},\ -t\sqrt{q}\\ Atq,\ -Atq,\ -q\end{matrix};\ q,\ q\right].$$

Now,

$${}_4\Phi_3\left[\begin{matrix}A,\ -Aq,\ t\sqrt{q},\ -t\sqrt{q}\\ Atq,\ -Atq,\ -q\end{matrix};\ q,\ q\right]$$
$$+ \frac{1-A}{1+A}\,{}_4\Phi_3\left[\begin{matrix}Aq,\ -A,\ t\sqrt{q},\ -t\sqrt{q}\\ Atq,\ -Atq,\ -q\end{matrix};\ q,\ q\right]$$

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$$= \frac{1-A}{1+A} \sum_{k=0}^{\infty} \frac{(t\sqrt{q})_k(-t\sqrt{q})_k q^k}{(q)_k(-q)_k(Atq)_k(-Atq)_k}$$

(5.5) $$\times \left\{ (A)_k(-Aq)_k \frac{1+A}{1-A} + (-A)_k(Aq)_k \right\}$$

$$= \frac{2}{1+A} \sum_{k=0}^{\infty} \frac{(A)_k(-A)_k(t\sqrt{q})_k(-t\sqrt{q})_k}{(q)_k(-q)_k(Atq)_k(-Atq)_k} q^k$$

$$= \frac{2}{1+A} {}_2\Phi_1\left[\begin{matrix} A^2, qt^2 \\ A^2t^2q^2 \end{matrix}; q^2, q\right]$$

since

(5.6) $$(a; q)_k(-a; q)_k = (a^2; q^2)_k$$

and

$$(A)_k(-Aq)_k \frac{1+A}{1-A} + (-A)_k(Aq)_k$$

$$= (Aq)_{k-1}(-A)_{k+1} + (-A)_k(Aq)_k$$

$$= 2(-A)_k(Aq)_{k-1} = \frac{2(A)_k(-A)_k}{1-A}.$$

However, by Heine's formula [**3**, 8.4 (3)]

(5.7) $${}_2\Phi_1\left[\begin{matrix} A^2, qt^2 \\ A^2t^2q^2 \end{matrix}; q^2, q\right] = \frac{(A^2q; q^2)_\infty(t^2q^2; q^2)_\infty}{(A^2t^2q^2; q^2)_\infty(q; q^2)_\infty}$$
$$= \frac{(A\sqrt{q}; q)_\infty(-A\sqrt{q}; q)_\infty(qt; q)_\infty(-qt; q)_\infty}{(Atq; q)_\infty(-Atq; q)_\infty(\sqrt{q}; q)_\infty(-\sqrt{q}; q)_\infty},$$

by (5.6).

Using (5.7) in (5.5) and (5.4), we obtain

(5.8) $${}_8W_7(At^2; t, -t, t\sqrt{q}, -t\sqrt{q}, A; q, Aq)$$
$$= \frac{(A^2q)_\infty(Aqt^2)_\infty}{(Aq)_\infty(A^2qt^2)_\infty}.$$

In deriving the final expression on the right hand side we made use of the identity

(5.9) $$(a; q)_{2n} = (a; q^2)_n(aq; q^2)_n$$
$$= (\sqrt{a}; q)_n(-\sqrt{a}; q)_n(\sqrt{aq}; q)_n(-\sqrt{aq}; q)_n.$$

To see how the parameters $\alpha_1, \ldots, \alpha_4$ may be chosen to relate the ${}_8W_7$ in (4.4) with the ${}_8W_7$ in (5.8) we first transform it by (2.21):

$$\begin{aligned}
&{}_8W_7(-\alpha_1\alpha_2\alpha_3 tq^{2n-1}/bc;\ \alpha_1 q^n,\ \alpha_2 q^n,\ -\alpha_3 q^n,\\
&\qquad -\alpha_1\alpha_2\alpha_3 t/b^3c^3,\ -b^2c^2q^n/\alpha_4;\ q,\ -\alpha_4 t/bc)\\
(5.10)\quad &= \frac{(-\alpha_1\alpha_2\alpha_3 tq^{2n}/bc)_\infty(-\alpha_4 q^n)_\infty}{(b^2c^2q^{2n})_\infty(\alpha_1\alpha_2\alpha_3\alpha_4 tq^n/b^3c^3)_\infty}\\
&\times \frac{(\alpha_1\alpha_2\alpha_3\alpha_4 t^2/b^4c^4)_\infty(bctq^n)_\infty}{(-\alpha_4 t/bc)_\infty(-\alpha_1\alpha_2\alpha_3 t^2q^2/b^2c^2)_\infty}\\
&\times {}_8W_7(-\alpha_1\alpha_2\alpha_3 t^2q^{n-1}/b^2c^2;\ \alpha_1 t/bc,\ \alpha_2 t/bc,\ -\alpha_3 t/bc,\\
&\qquad -\alpha_1\alpha_2\alpha_3 t/b^3c^3,\ -b^2c^2q^n/\alpha_4;\ q,\ -\alpha_4 q^n).
\end{aligned}$$

Now we choose $\alpha_1 = \alpha_3 = bc$, $\alpha_2 = \alpha_4 = bc\sqrt{q}$ so that the left hand side of (5.10) equals

$$\begin{aligned}
&\frac{(-b^2c^2tq^{2n+1/2})_\infty(-bcq^{n+1/2})_\infty(qt^2)_\infty(bctq^n)_\infty}{(b^2c^2q^{2n})_\infty(bctq^{n+1})_\infty(-t\sqrt{q})_\infty(-bct^2q^{n+1/2})_\infty}\\
&{}_8W_7(-bct^2q^{n-1/2};\ t,\ -t,\ t\sqrt{q},\ -t\sqrt{q},\ -bcq^{n-1/2};\ q,\ -bcq^{n+1/2})\\
&= \frac{(-b^2c^2tq^{2n+1/2})_\infty(-bcq^{n+1/2})_\infty(qt^2)_\infty(bctq^n)_\infty}{(b^2c^2q^{2n})_\infty(bctq^{n+1})_\infty(-t\sqrt{q})_\infty(-bct^2q^{n+1/2})_\infty}\\
&\times \frac{(b^2c^2q^{2n})_\infty(-bct^2q^{n+1/2})_\infty}{(-bcq^{n+1/2})_\infty(b^2c^2t^2q^{2n})_\infty},
\end{aligned}$$

by (5.8)

$$= \frac{(qt^2)_\infty(bctq^n)_\infty(-b^2c^2tq^{2n+1/2})_\infty}{(bctq^{n+1})(-t\sqrt{q})_\infty(b^2c^2t^2q^{2n})_\infty}.$$

Substituting this for the ${}_8W_7$ on the left hand side of (4.4) and simplifying by using (5.9) we get

$$\begin{aligned}
&\frac{(qt^2)_\infty(bct)_\infty(-b^2c^2t\sqrt{q})_\infty}{(-t\sqrt{q})_\infty(bcqt)_\infty(b^2c^2t^2)_\infty}\\
&\times \sum_{n=0}^{\infty} \frac{(b^2c^2q^{-1})_n(1 - b^2c^2q^{2n-1})(ab)_n(ac)_n}{(q)_n(1 - b^2c^2q^{-1})(bc^2a^{-1})_n(b^2ca^{-1})_n} a^{-2n}t^2\\
&\times p_n(x;\ a,\ b,\ c,\ bca^{-1})p_n(y;\ a,\ b,\ c,\ bca^{-1}).
\end{aligned}$$

Using this in (4.4) and simplifying the coefficients we obtain the Poisson kernel for q-Wilson polynomials

$$\sum_{n=0}^{\infty} \frac{(b^2c^2q^{-1})_n(1 - b^2c^2q^{2n-1})(ab)_n(ac)_n}{(q)_n(1 - b^2c^2q^{-1})(bc^2a^{-1})_n(b^2ca^{-1})_n}$$

$$\times a^{-2n}t^n p_n(x; a, b, c, bca^{-1})p_n(y; a, b, c, bca^{-1})$$

$$= (1 - t^2)\frac{(bcqt)_\infty}{(t/bc)_\infty} \sum_{k=0}^{\infty} \frac{(-bc)_k(bc\sqrt{q})_k(-bc\sqrt{q})_k}{(q)_k(bc^2a^{-1})_k(b^2ca^{-1})_k}$$

$$\times \frac{|(bca^{-1}e^{i\theta})_k(bca^{-1}e^{i\Phi})_k|^2q^k}{(bc)_k(bca^{-2})_k(bcqt)_k(bcq/t)_k}$$

(5.11)
$$\times {}_{10}W_9(a^2b^{-1}c^{-1}q^{-k}; ab^{-2}c^{-1}q^{1-k}, ab^{-1}c^{-2}q^{1-k}, q^{-k}, ae^{i\theta}, ae^{-i\theta}, ae^{i\Phi}, ae^{-i\Phi}; q, q)$$

$$+ \frac{(t)_\infty(ac^{-1}t)_\infty(b^2c^2)_\infty}{(ab)_\infty(ac)_\infty(bc)_\infty(ac^{-1})_\infty}$$

$$\times \frac{|(ae^{i\theta})_\infty(be^{i\Phi})_\infty(cte^{i\Phi})_\infty(bca^{-1}te^{i\theta})_\infty|^2}{(b^2ca^{-1})_\infty(bc^2a^{-1}t)_\infty(bc/t)_\infty|(te^{i\theta+i\Phi})_\infty(te^{i\theta-i\Phi})_\infty|^2}$$

$$\cdot \sum_{k=0}^{\infty} \frac{(-t)_k(t\sqrt{q})_k(-t\sqrt{q})_k(bc^2a^{-1}t)_k|(te^{i\theta+i\Phi})_k(te^{i\theta-i\Phi})_k|^2}{(q)_k(qt/bc)_k(qt^2)_k(ac^{-1}t)_k|(cte^{i\Phi})_k(bca^{-1}te^{i\theta})_k|^2}q^k$$

$$\times {}_{10}W_9(bc^2a^{-1}tq^{k-1}; tq^k, bctq^{k-1}, ca^{-1}tq^k, bca^{-1}e^{i\Phi}, bca^{-1}e^{-i\Phi}, ce^{i\theta}, ce^{-i\theta}; q, q)$$

$$+ \frac{(t)_\infty(ca^{-1}t)_\infty(b^2c^2)_\infty}{(ac)_\infty(bc)_\infty(ca^{-1})_\infty(bc^2a^{-1})_\infty}$$

$$\times \frac{|(ce^{i\theta})_\infty(bca^{-1}e^{i\Phi})_\infty(ate^{i\Phi})_\infty(bte^{i\theta})_\infty|^2}{(b^2ca^{-1})_\infty(abt)_\infty(bc/t)_\infty|(te^{i\theta+i\Phi})_\infty(te^{i\theta-i\Phi})_\infty|^2}$$

$$\times \sum_{k=0}^{\infty} \frac{(-t)_k(t\sqrt{q})_k(-t\sqrt{q})_k(abt)_k|(te^{i\theta+i\Phi})_k(te^{i\theta-i\Phi})_k|^2}{(q)_k(qt/bc)_k(qt^2)_k(ca^{-1}t)_k|(ate^{i\Phi})_k(bte^{i\theta})_k|^2}q^k$$

$$\times {}_{10}W_9(abtq^{k-1}; tq^k, bctq^{k-1}, ac^{-1}tq^k, be^{i\Phi}, be^{-i\Phi}, ae^{i\theta}, ae^{-i\theta}; q, q).$$

This is the same as eq. (6.13) in [7]. The right hand side is clearly non-negative for $\alpha, \beta > -1$ where

$$a = q^{\alpha/2+1/4}, b = a\sqrt{q}, c = -q^{\beta/2+1/4}$$

and $0 < t < 1$, $0 < q < 1$.

Note that the procedure of obtaining the Poisson kernel from (4.4) is analogous to deducing Bailey's formula [**3**, p. 102] from (1.1) in [**12**]. As such, one might even call (4.4) a generalization of the Poisson kernel for continuous q-Jacobi polynomials.

There are other interesting applications of the bilinear formulas derived here which we hope to report in a later paper.

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