

TOPOLOGICAL STABILITY OF SOLENOIDAL AUTOMORPHISMS

NOBUO AOKI

§ 0. Introduction

In [10] A. Morimoto proved that every topologically stable homeomorphism of a compact manifold M has the pseudo-orbit tracing property in the case $\dim(M) \geq 2$. Further, in studying relation between the topological stability and other stability of diffeomorphisms, he showed the following

THEOREM A. *Let R^r be the r -dimensional vector group and φ be a group automorphism of R^r . Then the following conditions are mutually equivalent;*

- (i) φ is hyperbolic,
- (ii) φ is expansive,
- (iii) φ is structurally stable,
- (iv) φ has the pseudo-orbit tracing property,
- (v) φ is topologically stable.

The statement further is true for toral automorphisms.

We know (cf. see § 1) that every toral automorphism is contained in the class of solenoidal automorphisms. Thus it will be natural to ask what kind of solenoidal automorphisms have the pseudo-orbit tracing property. Our aim is to investigate this problem by using results in [2] and A. Morimoto [9, 10, 11].

§ 1. A main result and preparatory lemmas

Let $f: X \leftarrow \rightarrow$ be a homeomorphism of a compact metric space (X, d) . We denote by $\mathcal{H}(X)$ the group of all homeomorphisms of X . Then $\mathcal{H}(X)$ becomes a complete topological group with the topology given by the metric $d(f, g) = \max \{d(f(x), g(x)), d(f^{-1}(x), g^{-1}(x)): x \in X\}$ ($f, g \in \mathcal{H}(X)$). We

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call f to be *topologically stable* iff for every $\varepsilon > 0$ there is $\delta > 0$ with the property that for every $g \in \mathcal{H}(X)$ with $d(f, g) < \delta$ there is a continuous map $h: X \leftarrow$ such that

$$\text{i) } h \circ g = f \circ h, \quad \text{ii) } d(h(x), x) < \varepsilon \ (x \in X).$$

A sequence of points $\{x_i\}_{i \in (a, b)}$ ($-\infty \leq a < b \leq +\infty$) is called a δ -pseudo-orbit of f iff $d(f(x_i), x_{i+1}) < \delta$. Given $\varepsilon > 0$, a δ -pseudo-orbit $\{x_i\}$ is called to be ε -traced by a point $y \in X$ iff $d(f^i(y), x_i) < \varepsilon$ for every $i \in (a, b)$. We call f to have the *pseudo-orbit tracing property* (abbrev. P.O.T.P.) iff for every $\varepsilon > 0$ there is $\delta > 0$ such that every δ -pseudo-orbit of f can be ε -traced by some point $y \in X$. We denote by $\text{Orb}^\delta(f)$ the set of all (finite or infinite) δ -pseudo-orbits of f and by $\text{Tr}^\varepsilon(\{x_i\}, f) = \text{Tr}^\varepsilon(\{x_i\})$ the set of all $y \in X$ such that $\{x_i\}$ is ε -traced by y . We call (X, f) to have *weak specification* iff for every $\varepsilon > 0$ there is $M(\varepsilon) > 0$ such that for every $k \geq 1$ and k points $x_1, \dots, x_k \in X$ and for every set of integers $a_1 \leq b_1 < a_2 \leq b_2 \dots < a_k \leq b_k$ with $a_{i+1} - b_i \geq M(\varepsilon)$ ($1 \leq i \leq k - 1$) there is $x \in X$ with $d(f^n(x), f^n(x_i)) < \varepsilon$ ($a_i \leq n \leq b_i$, $1 \leq i \leq k$).

We say that X is *solenoidal* iff X is a compact connected finite-dimensional abelian group. Every finite-dimensional torus is clearly solenoidal.

Hereafter X will be an r -dimensional solenoidal group and σ will be an automorphism of X . Our main result is the following

THEOREM 1. *The following (A) and (B) are equivalent;*

(A) (X, σ) is topologically stable,

(B) (X, σ) has the P.O.T.P..

Further there exist solenoidal automorphisms with P.O.T.P. such that one of the following conditions holds:

(C) (X, σ) is not expansive,

(D) (X, α) is not densely periodic.

The second statement of Theorem 1 will follow from Remark 1 mentioned below.

Denote by (G, γ) the dual of (X, σ) ($(\gamma g)(x) = g(\sigma x)$, $g \in G$ and $x \in X$). Let \bar{G} be a minimal divisible extension of G (p. 168, [7]). Since G is torsion free, \bar{G} is so and $\text{rank}(\bar{G}) = \text{rank}(G) = r < \infty$ (p. 34, [7]). It is well known that γ induces an automorphism $\bar{\gamma}$ of \bar{G} . We shall write $\gamma = \bar{\gamma}$ for the sake of simplicity.

Let $\mathcal{Q}[x, x^{-1}]$ be the ring of polynomials in x and x^{-1} with coefficients

in \mathbf{Q} (the notation \mathbf{Q} means the rational field). Since \bar{G} is divisible and torsion free, for every $\bar{f} \in \bar{G}$ and every natural number n there is a unique $\bar{g} \in \bar{G}$ such that $n\bar{g} = \bar{f}$. So we consider \bar{g} to be $(1/n)\bar{f}$ and $\mathbf{Q}[x, x^{-1}]$ to act on \bar{G} by $(\sum_{j=-m}^n b_j x^j)\bar{g} = \sum_{j=-m}^n b_j \gamma^j \bar{g}$ ($b_j \in \mathbf{Q}$ and $\bar{g} \in \bar{G}$). Then \bar{G} becomes a $\mathbf{Q}[x, x^{-1}]$ -module. Since $\mathbf{Q}[x, x^{-1}]$ is a principal ideal domain, it follows (cf. p. 397, [6]) that there is in G a finite sequence $\{g_1, \dots, g_s\}$ such that \bar{G} splits into a direct sum $\bar{G} = \bar{G}_{g_1} \oplus \dots \oplus \bar{G}_{g_s}$ of γ -invariant subgroups \bar{G}_{g_i} where $\bar{G}_{g_i} = \{f \in \bar{G} : mf \in \text{gp}\{\gamma^j g_i : j \in \mathbf{Z}\} \text{ for some } m \neq 0\}$ for $1 \leq i \leq s$ (the notation $\text{gp } E$ means the subgroup generated by a set E). Since $\gamma \bar{G}_{g_i} = \bar{G}_{g_i}$ for $1 \leq i \leq s$, we can find a polynomial $q_i(x) \in \mathbf{Z}[x]$ with minimal degree r_i such that $q_i(\gamma)g_i = 0$ holds, so that $\theta = \{g_1, \dots, \gamma^{r_1-1}g_1, \dots, g_s, \dots, \gamma^{r_s-1}g_s\}$ is linearly independent (the notation $\mathbf{Z}[x]$ means the ring of polynomials with integer coefficients). Hence the factor group $G/\text{gp } \theta$ is a torsion group; i.e. $\text{gp } \theta$ is full in G . Numbering the elements of θ as $\theta = \{e_1, \dots, e_r\}$, every $0 \neq g \in G$ is expressed as $ag = a_1e_1 + \dots + a_re_r$ for some $a \neq 0$ and some a_1, \dots, a_r with $(a_1, \dots, a_r) \neq (0, \dots, 0)$. Since the existence of $(a_1/a, \dots, a_r/a)$ is unique, we can define an into isomorphism $\varphi: G \rightarrow \mathbf{Q}^r$ by $\varphi(g) = (a_1/a, \dots, a_r/a)$. To simplify the notations, we identify g with $(a_1/a, \dots, a_r/a)$ under φ . Then θ is the canonical basis of \mathbf{Z}^r (i.e. $e_1 = (1, 0, \dots, 0), \dots, e_r = (0, \dots, 0, 1)$), so that $\text{gp } \theta = \mathbf{Z}^r \subset G \subset \bar{G} = \mathbf{Q}^r \subset \mathbf{R}^r$. We extend γ on \mathbf{R}^r by the natural way, and denote it by the same symbol. For $t = (t_1, \dots, t_r) \in \mathbf{R}^r$, define $\psi(t)g = t_1a_1/a + \dots + t_ra_r/a$ (addition mod 1) for all $g = (a_1/a, \dots, a_r/a) \in G$. Then we get $\psi(t) \in X$ (p. 251, [12]). In fact, $\psi: \mathbf{R}^r \rightarrow X$ is an into homomorphism. The adjoint map $\hat{\gamma}$ of γ is defined by $\psi(\hat{\gamma}t)g = \psi(t)\gamma g = (\sigma\psi(t))g$ ($t \in \mathbf{R}^r, g \in G$). Since $\hat{\gamma}$ and γ are isomorphic, we denote $\hat{\gamma}$ by γ again, and say (\mathbf{R}^r, γ) to be the *lifting system* of (X, σ) .

LEMMA B ((P.2(i)), [2]). *Under the above notations, $\psi(\mathbf{R}^r)$ is dense in X . If X is a torus then $\psi(\mathbf{R}^r) = X$.*

LEMMA C ((P.2), [2]). *Let F be the annihilator of $\text{gp } \theta$ in X . Then (i) F is totally disconnected and $\psi^{-1}\{\psi(\mathbf{R}^r) \cap F\} = \mathbf{Z}^r$, (ii) $X = \psi(\mathbf{R}^r) + F$ and (iii) there is a small closed neighborhood U of 0 in \mathbf{R}^r such that $\psi(U) \cap F = \{0\}$ and the direct product $U \times F$ is homeomorphic to $\psi(U) + F$. And $\psi(U) + F$ is a closed neighborhood of 0 in X (We write $\psi(U) \oplus F$ such a neighborhood $\psi(U) + F$).*

LEMMA D. *Under the above notations, the followings hold;*

- (i) *there exist a torus V_1 and a vector group V_2 such that ψ induces a 1-1 homomorphism ψ^* from the direct product group $V_1 \oplus V_2$ onto $\psi(\mathbf{R}^r)$,*
- (ii) *$\psi(\mathbf{Z}^r)$ is a closed subgroup in $\psi(\mathbf{R}^r)$,*
- (iii) *$\psi(\mathbf{Z}^r)$ is dense in F .*

Proof. Let K be the kernel of ψ . Then $K \subset \mathbf{Z}^r$ by Lemma C(i), so that there are a torus V_1 and a vector subgroup V_2 such that $\mathbf{R}^r/K = V_1 \oplus V_2$. Therefore ψ induces a 1-1, onto homomorphism $\psi^*: V_1 \oplus V_2 \rightarrow \psi(\mathbf{R}^r)$ in the natural way. (i) was proved. It is clear that $\overline{\psi(\mathbf{Z}^r)} \subset F$. By Lemma C (ii), $\overline{\psi(\mathbf{Z}^r)} \cap \psi(\mathbf{R}^r) \subset F \cap \psi(\mathbf{R}^r) = \psi(\mathbf{Z}^r)$ and so $\overline{\psi(\mathbf{Z}^r)} \cap \psi(\mathbf{R}^r) = \psi(\mathbf{Z}^r)$. This shows (ii). Put $B = \overline{\psi(\mathbf{Z}^r)}$. Then $X/B = \{(\psi(\mathbf{R}^r) + B)/B\} + \{F/B\}$. Since $\psi(\mathbf{R}^r)/\psi(\mathbf{Z}^r)$ is a factor group of $\mathbf{R}^r/\mathbf{Z}^r$, it is a torus and so $(\psi(\mathbf{R}^r) + B)/B$ is also a torus since $(\psi(\mathbf{R}^r) + B)/B \cong \psi(\mathbf{R}^r)/\psi(\mathbf{Z}^r)$. On the other hand, X/B is connected, from which we have $F = B$.

LEMMA E. *Let V_1, V_2 and ψ^* be as in Lemma D. For $\alpha_2 > 0$ small enough, let $B(\alpha_2)$ be a closed neighborhood with the radius α_2 of 0 in $V_1 \oplus V_2$. Then $B(\alpha_2)$ is a closed neighborhood of \mathbf{R}^r and $\psi^*(v) = \psi(v)$ for every $v \in B(\alpha_2)$.*

Proof. This is clear by the proof of Lemma D (i).

LEMMA F((P. 8), [2]). *If $H = \text{ann}(X, \text{gp} \cup_{-\infty}^{\infty} \gamma^j \Theta)$, then there exist subgroups F^- and F^+ of F satisfying the conditions;*

- (i) *$\sigma H = H$ and the topological entropy of σ_H equals zero,*
- (ii) *H contains a sequence $H = H_0 \supset H_1 \supset \dots \supset \bigcap H_n = \{0\}$ of subgroups such that for $n \geq 0$, $\sigma H_n = H_n$ and H/H_n is finite,*
- (iii) *$F^- \supset \sigma^{-1} F^- \supset \dots \supset \bigcap_0^{\infty} \sigma^{-n} F^- = \{0\}$,*
- (iv) *$F^+ \supset \sigma F^+ \supset \dots \supset \bigcap_0^{\infty} \sigma^n F^+ = \{0\}$,*
- (v) *$\sigma F^-/F^-$ and $F^+/\sigma F^+$ are finite,*
- (vi) *$F = F^- \oplus F^+ \oplus H$.*

By Lemmas C (iv) and F (vi), we have $X = \psi(\mathbf{R}^r) + \{F^- \oplus F^+ \oplus H\}$. Since X is connected, it follows that $X = \psi(\mathbf{R}^r) + \{F^- \oplus F^+\}$ when H is finite.

LEMMA G((P. 4), [2]). (i) *$H = \{0\}$ iff $G = \text{gp} \cup_{-\infty}^{\infty} \gamma^j \Theta$, (ii) H is finite iff (G, γ) is finitely generated under γ ; i.e. there is a finite set Λ in G such that $G = \text{gp} \cup_{-\infty}^{\infty} \gamma^j \Lambda$.*

The space \mathbf{R}^r splits into a direct sum $\mathbf{R}^r = E^u \oplus E^s \oplus E^c$ of γ -invariant

subspaces E^u , E^s and E^c such that the eigenvalues of γ_{E^u} , γ_{E^s} and γ_{E^c} have modulus > 1 , < 1 and 1 respectively. We call that (R^r, γ) is *hyperbolic* iff $E^c = \{0\}$. It is easily proved that there are norms $\|\cdot\|_u$ and $\|\cdot\|_s$ of E^u and E^s , respectively and $\lambda_0 \in (0, 1)$ such that $\|\gamma^n x\|_u \leq \lambda_0^{-n} \|x\|_u$ ($x \in E^u$ and $n \leq 0$) and $\|\gamma^n x\|_s \leq \lambda_0^n \|x\|_s$ ($x \in E^s$ and $n \geq 0$). If $E^c \neq \{0\}$, by using Jordan normal form in the real field for (E^c, γ) we get a finite direct sum $E^c = E^{c_0} \oplus \dots \oplus E^{c_k}$ of the subspaces E^{c_i} satisfying the following conditions; for $0 \leq i \leq k$, the dimension of E^{c_i} is 1 or 2, and

$$\gamma_{E^c} = \begin{bmatrix} \gamma_0 & I_1 & & 0 \\ & \gamma_1 & \cdot & \\ & 0 & \cdot & I_k \\ & & & \gamma_k \end{bmatrix}$$

where $\gamma_i: E^{c_i} \leftarrow$ is an isometry under some norm $\|\cdot\|_{c_i}$ of E^{c_i} and each $I_i: E^{c_i} \rightarrow E^{c_{i-1}}$ is either a zero map or a map corresponding to the identity matrix. We call that (R^r, γ) is *central spin* iff $E^c \neq \{0\}$ and each I_i is a zero map. If the characteristic polynomial $p(x)$ of γ is irreducible over \mathbb{Q} and $p(x)$ has roots of modulus one, then (R^r, γ) is central spin. Define a norm $\|\cdot\|_c$ of E^c by $\|x\|_c = \max \{\|x^i\|_{c_i} : 0 \leq i \leq k\}$ ($x = x^0 + \dots + x^k \in \bigoplus_0^k E^{c_i}$). Then we get easily that $d_0(x, y) = \max \{\|x^u - y^u\|_u, \|x^s - y^s\|_s, \|x^c - y^c\|_c\}$ is a metric of R^r satisfying the following conditions; (i) d_0 is translation invariant, (ii) there is $\lambda_0 \in (0, 1)$ with

$$d_0(\gamma^n x, 0) \leq \begin{cases} \lambda_0^{-n} d_0(x, 0) & (x \in E^u, n \leq 0), \\ \lambda_0^n d_0(x, 0) & (x \in E^s, n \geq 0), \end{cases}$$

and (iii) if $E^c \neq \{0\}$ then each of γ_i (under the above notations) is d_0 -isometry. We see that there are $\alpha_1 > 0$ such that for every $\alpha \in (0, \alpha_1]$, if $B(\alpha) = \{x \in R^r : d_0(x, 0) \leq \alpha\}$ then $\psi B(\alpha) \oplus F^- \oplus F^+ \oplus H$ is a closed neighborhood of X .

Let λ_0 be as above. Then the functions

$$d_-(x, y) = \begin{cases} \lambda_0^n & \text{if } x - y \in \sigma^{-n} F^- \text{ and } x - y \notin \sigma^{-(n+1)} F^- \\ 0 & \text{if } x = y, \end{cases}$$

$$d_+(x, y) = \begin{cases} \lambda_0^n & \text{if } x - y \in \sigma^n F^+ \text{ and } x - y \notin \sigma^{n+1} F^+ \\ 0 & \text{if } x = y \end{cases}$$

are metrics generating the original topology of F^- and F^+ respectively. For $x = x_0 + x_1 + x_2 + x_3 \in \psi B(\alpha_1) \oplus F^- \oplus F^+ \oplus H$, put

$$\rho(x) = \min \{ \alpha_1, \max \{ d_0(\psi^{-1}x_0, 0), d_-(x_1, 0), d_+(x_2, 0), d(x_3, 0) \} \} .$$

Then the metric of X defined by

$$d(x, y) = \begin{cases} \rho(x - y) & \text{if } x - y \in \psi B(\alpha_1) \oplus F^- \oplus F^+ \oplus H \\ \alpha_1 & \text{otherwise} \end{cases}$$

is compatible with the original topology. It follows that for $\varepsilon \in (0, \alpha_1)$, $B(\varepsilon) = B^u(\varepsilon) \oplus B^s(\varepsilon) \oplus B^c(\varepsilon)$ where $B^u(\varepsilon) = B(\varepsilon) \cap E^u$, $B^s(\varepsilon) = B(\varepsilon) \cap E^s$ and $B^c(\varepsilon) = B(\varepsilon) \cap E^c$.

In proving our results, it is important that closed neighborhoods are chosen to be proper subsets of X , so that we take and fix a number α_0 such that

$$(*) \quad 0 < \alpha_0 < \min \{ \alpha_1, \alpha_2 \} .$$

Here α_2 is the number chosen in Lemma E. For $\varepsilon \in (0, \alpha_0]$, a closed neighborhood $W(\varepsilon) = \{x \in X: d(x, 0) \leq \varepsilon\}$ is expressed as

$$(**) \quad W(\varepsilon) = W^u(\varepsilon) \oplus W^s(\varepsilon) \oplus W^c(\varepsilon)$$

where $W^u(\varepsilon) = W(\varepsilon) \cap \{\psi B^u(\varepsilon) \oplus F^-\}$, $W^s(\varepsilon) = W(\varepsilon) \cap \{\psi B^s(\varepsilon) \oplus F^+\}$ and $W^c(\varepsilon) = W(\varepsilon) \cap \{\psi B^c(\varepsilon) \oplus H\}$. Let d be a metric of X defined as above. For $x = x^u + x^s + x^c \in W^u(\alpha_0) \oplus W^s(\alpha_0) \oplus W^c(\alpha_0)$, we have

$$\begin{aligned} d(x, 0) &= \max \{ d(x^u, 0), d(x^s, 0), d(x^c, 0) \} \quad \text{and} \\ (***) \quad d(\sigma^n x, 0) &\leq \begin{cases} \lambda_0^{-n} d(x, 0) & (x \in W^u(\alpha_0), n \leq 0) \\ \lambda_0^n d(x, 0) & (x \in W^s(\alpha_0), n \geq 0) . \end{cases} \end{aligned}$$

LEMMA H. *If (X, σ) is ergodic under the Haar measure, then $\psi(E^c)$ is dense in X .*

Proof. There is in E^c a γ -invariant subspace E^{c_1} such that $\gamma_{E^{c_1}}$ is d_0 -isometry. Hence E^c is expressed as $E^c = E^{c_1} \oplus E^{c_2}$ where E^{c_2} is a subspace. Assume that $\psi(E^c)$ is not dense in X , and put $A = \overline{\psi(E^{c_1})}$. Then A is a σ -invariant connected subgroup of X . Obviously, σ_A is d -isometry; i.e. the topological entropy of σ_A is zero ($\text{ent}(\sigma_A) = 0$). As before let (G, γ) be the dual of (X, σ) and G_A be the annihilator of A in G . Since $G_A \subset G \subset \mathbf{R}^r$, we denote by V_A the smallest vector subgroup of \mathbf{R}^r containing G_A . Then \mathbf{R}^r is expressed as $\mathbf{R}^r = V_A \oplus V'$ where V' is a subspace. We see that $\mathbf{R}^r/V_A \cong V'$ is the smallest subspace containing $\dot{G}_A = G/G_A$. Since G is finitely generated under γ , so is \dot{G}_A . Let $p(x)$ be the characteristic

polynomial of $\gamma_{\hat{\sigma}_A}$. Then the Kolmogorov entropy of σ_A equals $h(\sigma_A) = \sum_{|\lambda|>1} \log |\lambda| + \log \Delta$ where λ 's are the eigenvalues of $\gamma_{\hat{\sigma}_A}$ and Δ is the smallest positive integer such that $\Delta p(x)$ has the integer coefficients (see [13]). Since $\text{ent}(\sigma_A) = h(\sigma_A) = 0$, we have $\Delta = 1$ (hence $p(x) \in \mathbb{Z}[x]$) and all the roots of $p(x)$ are modulus one. It follows from a result in the number theory that they are the roots of unity since $p(x) \in \mathbb{Z}[x]$. On the other hand, since (X, σ) is ergodic, all the roots of the characteristic polynomial of γ are not the roots of unity. This is a contradiction.

Remark 1. Let M be a compact manifold and φ be a diffeomorphism of M . It is proved in [10] that the set of all periodic points is dense in the non-wandering set when (M, φ) is topologically stable. In general this is not true for homeomorphisms on compact metric spaces. For example, let γ be an automorphism of \mathbb{Q}^r . Consider to \mathbb{Q}^r be an abelian group imposed with the discrete topology. If (\mathbb{R}^r, γ) is hyperbolic, then $(\gamma^j - I)\mathbb{Q}^r = \mathbb{Q}^r$ for every $j > 0$. From this we get that the dual (X, σ) of (\mathbb{Q}^r, γ) has no periodic points except 0; i.e. (X, σ) is not densely periodic. By Theorem 2 in the next section, (X, σ) has the P.O.T.P., and (X, σ) is topologically stable by the first statement of Theorem 1.

Since \mathbb{Q}^r is not finitely generated under γ , (X, σ) is not expansive (by Theorem 1, [2]). Therefore it will follow that there is a solenoidal automorphism which has the P.O.T.P., but is not expansive.

Remark 2. The set $\mathcal{C}(X)$ of all non-empty closed subsets of X is a compact metric space by the Hausdorff metric \bar{d} . Denote by $\widetilde{\text{Orb}}^s(\sigma)$ the set of all $A \in \mathcal{C}(X)$ for which there is $\{x_i\} \in \text{Orb}^s(\sigma)$ such that $A = \overline{\{x_i : i \in \mathbb{Z}\}}$. Let $E(\sigma)$ denote the set of all $A \in \mathcal{C}(X)$ such that for every $\epsilon > 0$ there is $A_\epsilon \in \widetilde{\text{Orb}}^s(\sigma)$ with $\bar{d}(A, A_\epsilon) < \epsilon$. Then $E(\sigma)$ is closed in $\mathcal{C}(X)$. We define $O(\sigma) = \{O_\epsilon(x) : x \in X\} \subset \mathcal{C}(X)$ where $O_\epsilon(x) = \{\sigma^i(x) : i \in \mathbb{Z}\}$ for $x \in X$. Obviously $O(\sigma) \subset E(\sigma)$. We call σ to have the OE-property iff $O(\sigma) = E(\sigma)$. A. Morimoto asks in [11] whether the lifting system (\mathbb{R}^r, γ) of (X, σ) is hyperbolic if σ has the OE-property. In [3] it is proved that for every automorphism β of a compact metric group, β has the OE-property iff β has the P.O.T.P.. From this result together with Theorem 2 in the next section, we shall see that Morimoto's problem is completely solved.

From Theorem 1 and Remark 2 we get the following

COROLLARY. *The following (A)', (B)' and (C)' are equivalent;*

- (A') (X, σ) is topologically stable,
 (B') (X, σ) has the P.O.T.P.,
 (C') (X, σ) satisfies the OE-property.

Hereafter, the restriction and the factor of σ will be denoted by the same symbol if there is no confusion.

§ 2. An auxiliary result

In this section we shall prove the following

THEOREM 2. *The lifting system (R^r, γ) of (X, σ) is hyperbolic iff (X, σ) has the P.O.T.P..*

For the proof we need the following lemmas.

LEMMA 2.1. *(X, σ) has the P.O.T.P., then (X, σ) is topologically mixing.*

Proof. By (Theorem 2, [1]), X contains σ -invariant subgroups X_1 and X_2 such that (X_1, σ) has zero entropy, (X_2, σ) is ergodic and X splits into a sum $X = X_1 + X_2$. Since X/X_2 is a factor group of X_1 , $(X/X_2, \sigma)$ has zero entropy. As we saw in the proof of Lemma H, X/X_2 is a torus and $(X/X_2, \sigma)$ is not hyperbolic. It is easy to see that $(X/X_2, \sigma)$ has the P.O.T.P., so that we must have $X_1 = \{0\}$ by Theorem A. Therefore (X, σ) is ergodic and hence (X, σ) is topologically mixing.

LEMMA 2.2. *If (X, σ) has the P.O.T.P., then (X, σ) satisfies weak specification.*

Proof. By Lemma 2.1, (X, σ) is topologically mixing. Let $\varepsilon > 0$ be given. Choose $\delta = \delta(\varepsilon) > 0$ as in the definition of the P.O.T.P. Cover X by a finite family \mathcal{U} of δ -balls. For any two $U_i, U_j \in \mathcal{U}$ there is $M_{ij} > 0$ such that $\sigma^n U_i \cap U_j \neq \emptyset$ for $n \geq M_{ij}$. Put $M = \max \{M_{ij} : i, j\} < \infty$. Let x_1, \dots, x_k be points in X and $a_1 \leq b_1 < \dots < a_k \leq b_k$ be integers with $a_j - b_{j-1} \geq M$ for $2 \leq j \leq k$. For $z \in X$ we denote by $U(z)$ some $U \in \mathcal{U}$ with $z \in U$. For $1 \leq j \leq k$ there is a point $y_j \in U(\sigma^{b_j} x_j)$ such that $\sigma^{a_{j+1}-b_j} y_j \in U(\sigma^{a_{j+1}} x_{j+1})$. Consider the δ -pseudo-orbit $\{z_i : a_1 \leq i \leq a_k\}$ defined by $z_i = \sigma^i x_j$ for $a_j \leq i \leq b_j$ and $z_i = \sigma^{i-b_j}(y_j)$ for $b_j < i < a_{j+1}$. Then there is a point $x \in X$ which ε -traces the orbit. From this we get $d(\sigma^i(x), \sigma^i(x_j)) < \varepsilon$ for $a_j \leq i \leq b_j$ ($1 \leq j \leq k$).

LEMMA 2.3. *Let σ be an automorphism of a compact metric group Y*

and K be a completely σ -invariant normal subgroup of $Y(\sigma(K) = K)$. If both $(Y/K, \sigma)$ and (K, σ) have the P.O.T.P., then so is (Y, σ) .

Proof. By assumption, for every $\varepsilon > 0$ there is a $\delta > 0$ with $\delta < \varepsilon$ such that for every δ -pseudo-orbit in K , a point in K $\varepsilon/2$ -traces the orbit. Choose η with $0 < \eta < \delta/3$ such that the following conditions hold;

(a) $d(\sigma(x), \sigma(y)) < \delta/3$ when $d(x, y) < \eta$ and

(b) for an arbitrary η -pseudo-orbit $\{x_i: a < i < b\}$ of Y , Y/K contains a point $x + K \in Y/K$ with $d(\sigma^i(x + K), x_i + K) < \delta/3 (a < i < b)$ (here d is a metric on X/K defined by $d(x + K, y + K) = \inf \{d(x + k, y + k'): k, k' \in K\}$).

By (b), for $a < i < b$ there is $k_i \in K$ such that $d(\sigma^i(x) + k_i, x_i) < \eta$.

By (a), $d(\sigma^{i+1}(x) + \sigma(k_i), \sigma(x_i)) < \delta/3$. We calculate that for $a < i < b - 1$

$$\begin{aligned} d(\sigma(k_i), k_{i+1}) &= d(\sigma^{i+1}(x) + \sigma(k_i), \sigma^{i+1}(x) + k_{i+1}) \\ &\leq d(\sigma^{i+1}(x) + \sigma(k_i), \sigma(x_i)) + d(\sigma(x_i), x_{i+1}) + d(x_{i+1}, \sigma^{i+1}(x) + k_{i+1}) \\ &< \delta, \end{aligned}$$

from which there is a point $k \in K$ $\varepsilon/2$ -tracing the orbit $\{k_i: a < i < b\}$. Since

$$d(\sigma^i(x + k), x_i) \leq d(\sigma^i(x + k), \sigma^i(x) + k_i) + d(\sigma^i(x) + k_i, x_i) < \varepsilon,$$

the point $x + k$ ε -traces the orbit $\{x_i: a < i < b\}$ in Y and the proof is completed.

LEMMA 2.4. *Let Y and σ be as in Lemma 2.3. If Y contains a sequence $Y = K_0 \supset K_1 \supset \dots \supset \cap K_n = \{e\}$ of normal subgroups such that for $n \geq 0$, $\sigma K_n = K_n$ and Y/K_n is finite, then (Y, σ) has the P.O.T.P.*

Proof. For every $\varepsilon > 0$, there are $n > 0$ and δ with $0 < \delta < \varepsilon$ such that

$$\{x \in Y: d(x, 0) < \delta\} \subset K_n \subset \{x \in Y: d(x, 0) < \varepsilon\}.$$

Let $\{x_i: a < i < b\}$ be an arbitrary δ -pseudo-orbit in Y ; i.e. $d(\sigma x_i, x_{i+1}) < \delta$, $a < i < b - 1$ (without loss of generality we may assume $a + 1 \leq 0$). Then $\sigma x_i - x_{i+1} \in K_n$ ($a < i < b - 1$). Hence $\sigma^i x_0 - x_i \in K_n$ since $\sigma K_n = K_n$; i.e. $d(\sigma^i x_0, x_i)$ ($a < i < b$). This shows that (Y, σ) has the P.O.T.P.)

Proof of Theorem 2.

As we saw in Section 1, \mathbf{R}^r splits into a direct sum $\mathbf{R}^r = E^u \oplus E^s \oplus E^c$ of the γ -invariant subspaces.

Proof of ⇐: By Lemma 2.2, (X, σ) obeys weak specification. Hence (\mathbf{R}^r, γ) is central spin by (Theorem 2, [2]); i.e. if $E^c \neq \{0\}$ then γ on E^c is d_0 -isometry (the metric d_0 is defined as in Section 1). We shall now prove that if $E^c \neq \{0\}$ then (X, σ) has not the P.O.T.P.. To do this, assume that (X, σ) has the P.O.T.P.; i.e. for every $\varepsilon \in (0, \alpha_0)$ there is $\delta > 0$ with $\delta < \varepsilon$ such that every δ -pseudo-orbit is $\varepsilon/2$ -traced by some point of X . Fix $0 \neq v_0 \in E^c$ with $d(v_0, 0) < \delta$ and set $z_j = j\gamma^j(v_0)$ for $j \in \mathbf{Z}$. Then it follows that

$$d(\gamma(z_j), z_{j+1}) = d_0(j\gamma^{j+1}(v_0), (j + 1)\gamma^{j+1}(v_0)) = d_0(0, v_0) < \delta,$$

and so $\{z_j\} \in \text{Orb}^\delta(\gamma)$. Put $V_\varepsilon(v) = \{u \in \mathbf{R}^r : d_0(v, u) < \varepsilon\}$ for $v \in \mathbf{R}^r$. Then there is $k > 0$ such that $V_\varepsilon(kv_0) \cap V_\varepsilon(0) = \phi$. From the relation between the metrics d_0 and d , we have

$$\delta > d_0(\gamma(z_j), z_{j+1}) = d(\sigma\psi(z_j), \psi(z_{j+1}))$$

for $j \in \mathbf{Z}$, and so $\{\psi(z_j)\} \in \text{Orb}^\delta(\sigma)$. By the assumption there is $x \in X$ such that $d(\sigma^i(x), \psi(z_j)) < \varepsilon/2$ for $j \in \mathbf{Z}$. Since $\psi(E^c)$ is dense in X (by Lemma H), we can find $0 \neq y \in \psi(E^c)$ with $\max\{d(\sigma^i(y), \sigma^i(x)) : 0 \leq i \leq k\} \leq \varepsilon/2$. Hence $d(\sigma^j(y), \psi(z_j)) < \varepsilon$ for $0 \leq j \leq k$, from which we have

$$\begin{aligned} d(y, \psi(z_0)) &= d(y, 0) = d_0(\psi^{-1}(y), 0) < \varepsilon \quad \text{and} \\ d(\sigma^k(y), \psi(z_k)) &= d_0(\gamma^k\psi^{-1}(y), z_k) = d_0(\gamma^k\psi^{-1}(y), \gamma^k(kv_0)) \\ &= d_0(\psi^{-1}(y), kv_0) < \varepsilon. \end{aligned}$$

Therefore $\psi^{-1}(y) \in V_\varepsilon(0) \cap V_\varepsilon(kv_0)$, which is a contradiction.

Proof of ⇒: Since H is the annihilator of $\text{gp} \bigcup_{-\infty}^{\infty} \gamma^n \theta$ in X (see Lemma F), $(\text{gp} \bigcup_{-\infty}^{\infty} \gamma^n \theta, \gamma)$ is the dual of $(X/H, \sigma)$ and $G/\text{gp} \bigcup_{-\infty}^{\infty} \gamma^n \theta$ is a torsion group. Hence the dimension of $\dot{X} = X/H$ is equal to that of X since H is zero-dimensional. Since $\text{gp} \bigcup_{-\infty}^{\infty} \gamma^n \theta$ is clearly finitely generated under γ and since the lifting system (\mathbf{R}^r, γ) of (\dot{X}, σ) is hyperbolic, it follows that (\dot{X}, σ) is expansive (see Theorem 1, [2]). We see that \dot{X} is expressed as $\dot{X} = \psi(\mathbf{R}^r) + \{F^- \oplus F^+\}$ by Lemma G (ii). Let $\varepsilon_0 > 0$ be an expansive constant for (\dot{X}, σ) . Then, for $0 < \varepsilon < \varepsilon_0$ we have a coordinate neighborhood $W(\varepsilon) = W^u(\varepsilon) \oplus W^s(\varepsilon)$ of 0 in \dot{X} where $W^u(\varepsilon) \neq \{0\}$ and $W^s(\varepsilon) \neq \{0\}$. It is easily seen that there is $\delta = \delta(\varepsilon) > 0$ such that if $d(x, y) < \delta$ ($x, y \in \dot{X}$) then $\{W^u(\varepsilon) + x\} \cap \{W^s(\varepsilon) + y\}$ consists only of one point. Therefore (\dot{X}, σ) has the P.O.T.P. using (***) (for the proof, see p. 74, R. Bowen [4]). From this fact together with Lemmas 2.3 and 2.4 and Lemma F (ii), we get the conclusion.

§ 3. Proof of Theorem 1

To see the statement (A) \Rightarrow (B), we shall prepare the following three lemmas.

LEMMA 3.1. *If (X, σ) is topologically stable, then $X \setminus \text{Fix}(\sigma)$ is dense in X where $\text{Fix}(\sigma) = \{x \in X: \sigma(x) = x\}$.*

Proof. Notice that $\text{Fix}(\sigma)$ is a subgroup of X . Assume that $X \setminus \text{Fix}(\sigma)$ is not dense in X . Then $\text{Fix}(\sigma)$ is open in X . Since X is connected, we get $X = \text{Fix}(\sigma)$; i.e. $\sigma = \text{id}$. Take $\varepsilon > 0$ with $2\varepsilon < \text{diameter}(X)$ and let $\delta > 0$ be as in the definition of topological stability. Now we can find $a \in X$ with $d(a, 0) < \delta$ such that $\{na: n \in \mathbb{Z}\}$ is dense in X (see [5]). Let $f_a: X \leftarrow$ be a homeomorphism defined by $f_a(x) = x + a(x \in X)$, then $d(f_a(x), x) = d(f_a(x), \sigma(x)) < \delta$. Hence there is a continuous map $h: X \leftarrow$ with $h \circ f_a = \sigma \circ h$ and $d(h(x), x) < \varepsilon (x \in X)$. Since $h(an) = h(f_a^n(0)) = \sigma^n h(0) = h(0)$ for all n , we get $h(x) = h(0)$ for all $x \in X$, and so $\varepsilon > d(h(0), x)$ for $x \in X$. On the other hand, since $\varepsilon < \text{diameter}(X)/2$, there is $y \in X$ with $d(h(0), y) > \varepsilon$, which is a contradiction.

LEMMA 3.2 ([10]). *Let φ be a homeomorphism of a connected metric space Y . Assume that φ is uniformly continuous and $Y \setminus \text{Fix}(\varphi)$ is dense in Y . Take and fix a constant $\delta_1 > 0$ and an integer $k > 0$. Then for every $\{x_i\} \in \text{Orb}^{\delta_1}(\varphi)$ and $\varepsilon_1 > 0$, there is $\{x'_i\} \in \text{Orb}^{3\delta_1}(\varphi)$ such that i) $d(x_i, x'_i) < \varepsilon_1$ for $0 \leq i \leq k$ and ii) $Y_i = \{\varphi(x'_i), x'_{i+1}\} (0 \leq i \leq k - 1)$ are disjoint.*

We shall describe here a proof given in [10] for completeness. We can assume $\varepsilon_1 < \delta_1$. For this ε_1 , there is $\varepsilon'_1 > 0$ with $\varepsilon_1 > \varepsilon'_1$ such that $d(x, y) < \varepsilon'_1$ implies $d(\varphi(x), \varphi(y)) < \varepsilon_1$. First we can find $x'_i \in Y (0 \leq i \leq k)$ such that $x'_i \neq x'_j (i \neq j)$ and $d(x_i, x'_i) < \varepsilon'_1 (0 \leq i \leq k)$. Next we shall show by induction that Y_0, \dots, Y_{k-1} are disjoint by taking x'_i suitably. Assume that $Y_i = \{\varphi(x'_i), x'_{i+1}\} (0 \leq i \leq k - 2)$ are disjoint. We shall show that, by changing x'_{k-1} and x'_k if necessary, $Y_i \cap Y_j = \emptyset (0 \leq i \neq j \leq k - 1)$. Consider the point $\varphi(x'_{k-1})$ and assume $\varphi(x'_{k-1}) \in \bigcup_{i=0}^{k-2} Y_i$. Then there is a unique $i \leq k - 1$ such that $\varphi(x'_{k-1}) = x'_i$ since $x'_{k-1} \neq x'_j (j \leq k - 2)$ implies $\varphi(x'_{k-1}) \neq \varphi(x'_j)$. If $i \leq k - 2$, we can find x''_{k-1} near x'_{k-1} such that $\varphi(x''_{k-1}) \neq x'_i$. If $i = k - 1$ ($\varphi(x'_{k-1}) = x'_{k-1}$), then we can find x''_{k-1} near x'_{k-1} such that $\varphi(x''_{k-1}) \neq x'_{k-1}$, since $Y \setminus \text{Fix}(\varphi)$ is dense and open in Y . We denote x''_{k-1} by x'_{k-1} again. Then we can assume that $x'_k \notin \bigcup_{i=0}^{k-1} Y_i$, since $\bigcup_{i=0}^{k-2} Y_i$ is a finite set. Thus we have proved that Y_0, Y_1, \dots, Y_{k-1} are disjoint.

For $i < 0$ (resp. $i > k$) we define $x'_i = \varphi^{-i}(x'_0)$ (resp. $x'_i = \varphi^{i-k}(x'_k)$). Then we see that $\{x'_i\} \in \text{Orb}^{3\delta_1}(\varphi)$ since

$$\begin{aligned} d(\varphi(x'_i), x'_{i+1}) &\leq d(\varphi(x'_i), \varphi(x_i)) + d(\varphi(x_i), x_{i+1}) + d(x_{i+1}, x'_{i+1}) \\ &< \varepsilon_1 + \delta_1 + \varepsilon'_1 < 3\delta_1 \end{aligned}$$

for $0 \leq i \leq k - 1$. This completes the proof of Lemma 3.2.

LEMMA 3.3 ([8]). *Let M be a differentiable manifold of $\dim(M) \geq 2$ with a metric d . Let $M_i = \{p_i, q_i\}$ ($1 \leq i \leq k$) be a subset of M consisting of at most two points p_i and q_i with $d(p_i, q_i) < \delta$. Assume that $M_i \cap M_j = \emptyset$ ($i \neq j$). Then there is an onto homeomorphism $\eta: M \leftarrow$ such that $d(\eta(x), x) < 2\pi\delta$ for $x \in M$, and that $\eta(p_i) = q_i$ ($1 \leq i \leq k$).*

Proof of the statement (A) \Rightarrow (B). The proof will be done along the following two cases.

Case (1). For the case $\dim(X) = 1$, we get that (R^1, γ) is hyperbolic by applying Lemma 3.1. Therefore (X, σ) has the P.O.T.P. by Theorem 2.

Case (2). We can use Lemma 3.3 when $\dim(X) \geq 2$. For every $\varepsilon > 0$, choose $\delta > 0$ ($\delta < \varepsilon$) as in the definition of topological stability. Since $\overline{\psi(R^r)} = X$ by Lemma B, for every $\{x_i\} \in \text{Orb}^{\delta/12\pi}(\sigma)$ there is $\{x'_i\} \subset \psi(R^r)$ such that $d(x_i, x'_i) < \delta/24\pi$, $d(\sigma(x_i), \sigma(x'_i)) < \delta/24\pi$ and $\sigma(x'_i) - x'_{i+1} \in W(\delta/6\pi)$ ($i \in \mathbb{Z}$) where $W(\delta/6\pi)$ is a closed neighborhood with the radius $\delta/6\pi$ of 0 in X . Since (X, σ) is topologically stable and $\psi(R^r)$ is connected, we have that $\psi(R^r) \setminus (\psi(R^r) \cap \text{Fix}(\sigma))$ is dense in $\psi(R^r)$ (by using Lemma 3.1).

Take and fix an integer $k > 0$. Notice that σ and σ^{-1} are uniformly continuous on $\psi(R^r)$. By Lemma 3.2 there is a sequence $\{x''_i\} \in \text{Orb}^{\delta/2\pi}(\sigma)$ such that $d(x'_i, x''_i) < \delta/2\pi$ ($0 \leq i \leq k$) and $\{\sigma(x''_i), x''_{i+1}\}$ ($0 \leq i \leq k - 1$) are mutually disjoint. Choose two closed balls B' and B of 0 in $V_1 \oplus V_2$ such that $\{x''_i\}_1^k \cup \{\sigma x''_i\}_1^{k-1} \subset \psi^*(B') \subseteq \psi^*(B)$. Since $\psi^*(B)$ is a differentiable manifold, as in Lemma 3.3 there is an onto homeomorphism $\eta_1: \psi^*(B) \leftarrow$ such that $d(\eta_1(x), x) < \delta$ ($x \in \psi^*(B)$), $\eta_1(x) = x$ for $x \in \psi^*(B) \setminus \psi^*(B')$ and $\eta_1 \sigma(x''_i) = x''_{i+1}$ ($1 \leq i \leq k - 1$). Choose a small open subgroup F' of F such that $\psi^*(B + B) \cap F' = \{0\}$ and $\psi^*(B) + F'$ is a closed neighborhood of X . Define a map $\eta'_1: \psi^*(B) + F' \leftarrow$ by $\eta'_1(x + y) = \eta_1(x) + y$ ($x \in \psi^*(B)$ and $y \in F'$). Then it is clear that η'_1 is 1-1 and onto. Let η be a map from X onto itself defined by

$$\eta(x) = \begin{cases} \eta'_1(x) & \text{if } x \in \psi^*(B) + F' \\ x & \text{if } x \notin \psi^*(B) + F' . \end{cases}$$

It is easily checked that $\eta: X \leftarrow$ is an onto homeomorphism having the properties; $d(\eta(x), x) < \delta$ for $x \in X$, $\eta(x) = x$ on $\psi^*(B') + F$ and $\eta\sigma(x'_i) = x''_{i+1}$ ($1 \leq i \leq k - 1$). Put $\varphi = \eta \circ \sigma$. Since $d(\varphi(x), \sigma(x)) < \delta$ for $x \in X$, we have $\{\varphi^i(x)\} \in \text{Orb}^\delta(\sigma)$. By the property for $\delta > 0$, there is a continuous map $h: X \leftarrow$ such that $h \circ \varphi = \sigma \circ h$ and $d(h(x), x) < \varepsilon$ for $x \in X$. If $x = h(x''_i)$, then for $0 \leq i \leq k$

$$\begin{aligned} d(\sigma^i(x), x_i) &= d(\sigma^i(h(x''_i)), x_i) = d(h(\varphi^i(x''_i)), x_i) \\ &\leq d(h(x''_i), x''_i) + d(x''_i, x'_i) + d(x'_i, x_i) < 2\varepsilon, \end{aligned}$$

which shows $x \in \text{Tr}^{2\varepsilon}(\{x_i\}_0^k, \sigma)$. Since k is arbitrary, (X, σ) has the P.O.T.P.

It remains to prove the statement (B) \Rightarrow (A). First we shall prepare the following two lemmas.

LEMMA 3.4. *Let α_0 be as in (*). For every $f \in \mathcal{H}(X)$ with $\max\{d(\sigma, f), d(\sigma^{-1}, f^{-1})\} < \alpha_0/2$, there exist $y_0 \in F$ and $f_0 \in \mathcal{H}(X)$ with $f_0(\psi(\mathbf{R}^r)) = \psi(\mathbf{R}^r)$ such that $f(x) = y_0 + f_0(x)$ for $x \in X$.*

Proof. Put $g(x) = f(x) - y'_0$ ($x \in X$) where $y'_0 = f(0)$. Since $d(f(0), \sigma(0)) = d(y'_0, 0) < \alpha_0/2$, we get $d(g(x), f(x)) < \alpha_0/2$ ($x \in X$). Notice that $\{x \in X: d(x, 0) < \alpha_0\} \subset \psi B(\alpha_0) \oplus F$ (see (**)). Put $\kappa(x) = g(x) - \sigma(x)$ ($x \in X$). Since $d(\kappa(x), 0) \leq d(g(x), f(x)) + d(f(x), \sigma(x)) < \alpha_0$ for all $x \in X$, we have $\kappa(x) \in \psi B(\alpha_0) \oplus F$ for $x \in X$ and so $\kappa(\psi \mathbf{R}^r) \subset \psi B(\alpha_0) \oplus F$. Since $\psi(\mathbf{R}^r)$ contains the identity 0 and $\kappa(\psi \mathbf{R}^r)$ is connected, we have $\kappa(\psi \mathbf{R}^r) \subset \psi B(\alpha_0)$, from which $g(\psi \mathbf{R}^r) \subset \psi(\mathbf{R}^r)$. In the same way, it follows that $g^{-1}(\psi \mathbf{R}^r) \subset \psi(\mathbf{R}^r)$, so that $g(\psi \mathbf{R}^r) = \psi(\mathbf{R}^r)$. Since $y'_0 \in X = \psi(\mathbf{R}^r) + F$, y'_0 splits into the sum $y'_0 = \psi(v) + y_0$ with $\psi(v) \in \psi(\mathbf{R}^r)$ and $y_0 \in F$. Put $f_0(x) = \psi(v) + g(x)$ ($x \in X$). Then $f_0(x)$ satisfies all the conditions of the lemma.

LEMMA 3.5 ([10]). *Let Y be a metric space such that every bounded set is relatively compact. Let $f: Y \leftarrow$ be a homeomorphism with the P.O.T.P.. If (Y, f) is expansive, then (Y, f) is topologically stable.*

We shall give here a proof due to A. Morimoto [10]. The proof will be used in proving the statement (B) \Rightarrow (A). For every $\varepsilon > 0$, there is $\delta > 0$ such that $\{x_i\} \in \text{Orb}^\delta(f)$ implies $\text{Tr}^\varepsilon(\{x_i\}, f) = \phi$. We can assume $\varepsilon < \varepsilon_0/4$, where ε_0 is an expansive constant of f . Take a $g \in \mathcal{H}(Y)$ with $d(g, f) < \delta$. We shall prove that there exists a continuous map $h: Y \leftarrow$ having the property in the definition of topological stability. Take a point $x \in Y$. It is easy to see that $\{g^i(x)\} \in \text{Orb}^\delta(f)$. Hence there is $y \in$

$\text{Tr}^e(\{g^i(x)\}, f)$; i.e. $d(f^i(y), g^i(x)) \leq \varepsilon$ for $i \in \mathbb{Z}$. If $y' \in \text{Tr}^e(\{g^i(x)\}, f)$, then we have

$$d(f^i(y), f^i(y')) \leq d(f^i(y), g^i(x)) + d(g^i(x), f^i(y')) \leq 2\varepsilon < \varepsilon_0,$$

which implies $y = y'$. Thus by putting $h(x) = y$, we get a well-defined map $h: Y \leftarrow$ with the property

$$(1) \quad d(f^i(h(x)), g^i(x)) \leq \varepsilon \quad \text{for } i \in \mathbb{Z}.$$

Putting $i = 0$ in (1) we get

$$(2) \quad d(h(x), x) \leq \varepsilon \quad \text{for } x \in Y.$$

Next we have, again by (1) for x and $g(x)$,

$$\begin{aligned} d(f^i(f(h(x))), f^i(h(g(x)))) &\leq d(f^i(f(h(x))), g^i(g(x))) \\ &\quad + d(f^i(h(g(x))), g^i(g(x))) \leq 2\varepsilon < \varepsilon_0 \end{aligned}$$

for every $i \in \mathbb{Z}$, which implies $f(h(x)) = h(g(x))$. Finally we shall prove the continuity of h . Assume that h is not continuous at $x_0 \in Y$. Then, there is a sequence $x_\nu \rightarrow x_0$ ($\nu \rightarrow \infty$) such that $y_\nu = h(x_\nu)$ does not tend $y_0 = h(x_0)$ as $\nu \rightarrow \infty$. Since $\{x_\nu\}$ is bounded and $d(h(x_\nu), x_\nu) \leq \varepsilon$ for $\nu > 0$, the set $\{h(x_\nu)\}$ is also bounded. Hence we can assume, by taking a subsequence if necessary, that $y_\nu \rightarrow y'_0 \neq y_0$ ($\nu \rightarrow \infty$). Since f is expansive, there is $k \in \mathbb{Z}$ such that $d(f^k(y'_0), f^k(y_0)) \geq \varepsilon_0$. Fixing k , we can find $\nu_0 > 0$ such that for $\nu > \nu_0$

$$(3) \quad d(f^k(y_\nu), f^k(y'_0)) < \varepsilon_0/4,$$

since f^k is continuous and $y_\nu \rightarrow y'_0$ ($\nu \rightarrow \infty$). We can assume

$$(4) \quad d(g^k(x_\nu), g^k(x_0)) < \varepsilon \quad (\nu \geq \nu_0),$$

since g^k is continuous and $x_\nu \rightarrow x_0$ ($\nu \rightarrow \infty$). Now we have, using (2) and (4),

$$\begin{aligned} d(f^k(y_\nu), f^k(y_0)) &= d(f^k(h(x_\nu)), h(g^k(x_0))) \\ &= d(h(g^k(x_\nu)), h(g^k(x_0))) \\ &\leq d(h(g^k(x_\nu)), g^k(x_\nu)) + d(g^k(x_\nu), g^k(x_0)) \\ &\quad + d(g^k(x_0), h(g^k(x_0))) < 3\varepsilon \end{aligned}$$

and hence by (3) we obtain

$$\begin{aligned} \varepsilon_0 &\leq d(f^k(y'_0), f^k(y_0)) \leq d(f^k(y'_0), f^k(y_\nu)) + d(f^k(y_\nu), f^k(y_0)) \\ &\leq \varepsilon_0/4 + 3\varepsilon < \varepsilon_0, \end{aligned}$$

which is a contradiction. This completes the proof of Lemma 3.5.

Proof of the statement (B) ⇒ (A). Remark that $\psi^*(V_1 \oplus V_2) = \psi(\mathbf{R}^r)$ (by Lemma D (i)). By Theorem 2, (\mathbf{R}^r, γ) is hyperbolic. Hence (\mathbf{R}^r, γ) , and so $(V_1 \oplus V_2, \gamma)$, is expansive and has the P.O.T.P. (see Theorem A). Using Lemma 3.5, we get that $(V_1 \oplus V_2, \gamma)$ is topologically stable; i.e. take $\varepsilon \in (0, \alpha_0/2)$ such that $\gamma B(3\varepsilon) \subset B(\alpha_0)$ and $\gamma^{-1}B(3\varepsilon) \subset B(\alpha_0)$ and let $\delta > 0$ ($\delta < \varepsilon$) be the number with the property of topological stability.

Take $f \in \mathcal{H}(X)$ with $d(f, \sigma) < \delta$ (Then we may assume that the number δ is chosen such that $d(f^{-1}, \sigma^{-1}) < \alpha_0/2$). By Lemma 3.4 there are $y_0 \in F$ and $f_0 \in \mathcal{H}(X)$ such that $f_0(\psi(\mathbf{R}^r)) = \psi(\mathbf{R}^r)$ and $f(x) = f_0(x) + y_0$ for $x \in X$. Since $\overline{\psi(\mathbf{Z}^r)} = F$ by Lemma D (iii), we can choose in $\psi(\mathbf{Z}^r)$ a sequence $\{y_n\}_{n \geq 1}$ such that $y_n \rightarrow y_0$ as $n \rightarrow \infty$. Put $f_n(x) = y_n + f_0(x)$. Then $d(f_n, \sigma) < \delta$ for n large enough. Fix such an integer n and define $\hat{f}_n(v) = \psi^{*-1} f_n \psi^*(v)$ ($v \in V_1 \oplus V_2$). Then we claim that $\hat{f}_n: V_1 \oplus V_2 \leftarrow$ is uniformly continuous. Indeed, we denote by $F(\varepsilon)$ a closed neighborhood with the radius ε of 0 in F . Since $f_n: X \leftarrow$ is uniformly continuous, for every $\lambda \in (0, \alpha_0]$ there is $\alpha > 0$ with $\alpha < \lambda$ such that for every $v \in V_1 \oplus V_2$

$$f_n(\psi^*B(\alpha) \oplus F(\alpha) + \psi^*(v)) \subset \psi^*B(\lambda) \oplus F(\lambda) + f_n\psi^*(v),$$

from which

$$f_n(\psi^*B(\alpha) + \psi^*(v)) \subset \psi^*B(\lambda) + f_n\psi^*(v).$$

Hence, $\hat{f}_n(B(\alpha) + v) \subset B(\lambda) + \hat{f}_n(v)$; i.e. our requirement was obtained.

Let d_0 be a metric of $V_1 \oplus V_2$ defined as in Section 1. Since $\gamma = \psi^{-1} \circ \sigma \circ \psi$, we have

$$\delta > d(\sigma\psi^*(v), f_n\psi^*(v)) = d_0(\psi^{*-1}\sigma\psi^*(v), \psi^{*-1}f_n\psi^*(v)) = d_0(\gamma(v), \hat{f}_n(v)),$$

so that $\{\hat{f}_n(v)\} \in \text{Orb}^\delta(\gamma)$ for $v \in V_1 \oplus V_2$. Hence there is $w \in V_1 \oplus V_2$ with $d_0(\hat{f}_n^j(v), \gamma^j(w)) < \varepsilon$ ($j \in \mathbf{Z}$) since $(V_1 \oplus V_2, \gamma)$ has the P.O.T.P.. Put $w = \hat{h}_n(v)$. Since $(V_1 \oplus V_2, \gamma)$ is expansive, from the proof of Lemma 3.5 it follows that $\hat{h}_n: V_1 \oplus V_2 \leftarrow$ is a continuous map such that $\hat{h}_n \circ \hat{f}_n = \alpha \circ \hat{h}_n$ and $d_0(\hat{h}_n, \text{id}) < \varepsilon$. Put $h_n = \psi^* \circ \hat{h}_n \circ \psi^{*-1}$. Obviously, $h_n \circ f_n = \sigma \circ h_n$ on $\psi(\mathbf{R}^r)$ and $d(h_n(x), x) < \varepsilon$ for $x \in \psi(\mathbf{R}^r)$.

We now prove that h_n is uniformly continuous. Since $d(\hat{f}_n^j(v), \gamma^j\hat{h}_n(v)) < \varepsilon$ for $j \in \mathbf{Z}$ and $v \in V_1 \oplus V_2$, we have

$$\psi^{*-1}f_n^j\psi^*(v) - \gamma^j\hat{h}_n(v) = \hat{f}_n^j(v) - \gamma^j\hat{h}_n(v) \in B(\varepsilon),$$

so that for all $v \in V_1 \oplus V_2$ and all $j \in Z$

$$(5) \quad f_n^j \psi^*(v) - \sigma^j h_n \psi^*(v) \in \psi^* B(\epsilon).$$

Since (R^r, γ) is expansive, it is easily checked that for every $\lambda > 0$ ($\lambda < \epsilon$) there is $N \geq 1$ such that $d(h_n(x), h_n(y)) < \lambda$ when $\sigma^j h_n(x) - \sigma^j h_n(y) \in \psi^* B(3\epsilon)$ for j with $|j| \leq N$. Take $\alpha > 0$ such that if $d(x, y) < \alpha$ for $x, y \in \psi(R^r)$ then $\max \{d(f_n^j(x), f_n^j(y)) : -N \leq j \leq N\} < \lambda$. Then for j with $|j| \leq N$,

$$\begin{aligned} d(\sigma^j h_n(x), \sigma^j h_n(y)) &\leq d(\sigma^j h_n(x), f_n^j(x)) + d(f_n^j(x), f_n^j(y)) \\ &\quad + d(f_n^j(y), \sigma^j h_n(y)) < 2\epsilon + \lambda < 3\epsilon, \end{aligned}$$

which shows that $d(h_n(x), h_n(y)) < \lambda$. Indeed, fix $y \in \psi(R^r)$ and put $\kappa_y(x) = f_n^j(x) - f_n^j(y)$. Then $\kappa_y(W(\alpha) + y) \subset \psi^* B(\lambda) \oplus F(\lambda)$ where $W(\alpha) = \psi^* B(\alpha) \oplus F(\alpha)$. Since $\kappa_y(\psi(R^r)) = \psi(R^r)$ and $\psi(R^r) = \cup \kappa_y(W(\alpha) + y)$, we have $\kappa_y(x) \in \kappa_y(W(\alpha) + y) \subset \psi^* B(\lambda)$ and hence $\sigma^j h_n(x) - \sigma^j h_n(y) \in \psi^* B(3\epsilon)$.

Therefore h_n is uniformly extended to a continuous map from X into itself. We shall denote it by the same symbol. By using (5) we have for m and n large enough and for all $x \in X$

$$\sigma^j \{h_n(x) - h_m(x)\} + \{f_m^j(x) - f_n^j(x)\} \in \psi^* B(2\epsilon).$$

Since for fixed j

$$\lim_{n, m \rightarrow \infty} d(f_n^j, f_m^j) = 0,$$

there is $N(j) > 0$ such that $f_n^j(x) - f_m^j(x) \in \psi^* B(\epsilon)$ for $n, m \geq N(j)$ and $x \in X$. Hence for $n, m \geq N(j)$ and $x \in X$

$$(6) \quad h_n(x) - h_m(x) \in \sigma^{-j} \psi^* B(3\epsilon).$$

As before we have $B(3\epsilon) = B(3\epsilon)^u \oplus B(3\epsilon)^s$ where $B(3\epsilon)^u = B(3\epsilon) \cap E^u$ and $B(3\epsilon)^s = B(3\epsilon) \cap E^s$. Hence $\psi^* B(3\epsilon) = \psi^* B(3\epsilon)^u \oplus \psi^* B(3\epsilon)^s$. Since $\gamma B(3\epsilon) \subset B(\alpha_0)$ and $\gamma^{-1} B(3\epsilon) \subset B(\alpha_0)$, obviously $\sigma \psi^* B(3\epsilon) \subset \psi^* B(\alpha_0)$ and $\sigma^{-1} \psi^* B(3\epsilon) \subset \psi^* B(\alpha_0)$. It follows easily that $\sigma \psi^* B(3\epsilon)^u \subset \psi^* B(\alpha_0)^u$ and $\sigma^{-1} \psi^* B(3\epsilon)^s \subset \psi^* B(\alpha_0)^s$. Hence $\bigcap_{-\infty}^{\infty} \sigma^j \psi^* B(3\epsilon) = \bigcap_{-\infty}^{\infty} \sigma^j \psi^* B(3\epsilon)^u \oplus \bigcap_{-\infty}^{\infty} \sigma^j \psi^* B(3\epsilon)^s = \{0\}$. From (6) we have for $n, m \geq \max \{N(j) : -i \leq j \leq i\}$ and $x \in X$

$$h_n(x) - h_m(x) \in \bigcap_{j=-i}^i \sigma^j \psi^* B(3\epsilon).$$

For any open neighborhood U of 0 there is $i > 0$ such that $\bigcap_{j=-i}^i \sigma^j \psi^* B(3\epsilon) \subset U$. This implies that $\lim_{n, m \rightarrow \infty} d(h_n, h_m) = 0$; i.e. $\{h_n\}$ converges uni-

formly to some continuous map h of X . Since $h_n \circ f_n = \sigma \circ h_n$ on X and $d(h_n, \text{id}) < \varepsilon$ for an arbitrary large n , it follows that $h \circ f = \sigma \circ h$ on X and $d(h, \text{id}) \leq \varepsilon$. Therefore (X, σ) is topologically stable. The proof of Theorem 1 is completed.

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Tokyo Metropolitan University
Department of Mathematics
Tokyo, Japan