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Special cube complexes revisited: a quasi-median generalization

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Abstract. In this article, we generalize Haglund and Wise's theory of special cube complexes to groups acting on quasi-median graphs. More precisely, we define special actions on quasi-median graphs, and we show that a group which acts specially on a quasi-median graph with finitely many orbits of vertices must embed as a virtual retract into a graph product of finite extensions of clique-stabilizers. In the second part of the article, we apply the theory to fundamental groups of some graphs of groups called *right-angled graphs of groups*.

1 Introduction

Haglund and Wise's theory of special cube complexes [40] is one of the major contributions of the study of groups acting on CAT(0) cube complexes. The key point of the theory is that, if a group G can be described as the fundamental group of a nonpositively curved cube complex X, then there exists a simple and natural condition about X which implies that G can be embedded into a right-angled Artin group A. As a consequence, all the properties which are satisfied by right-angled Artin groups and which are stable under taking subgroups are automatically satisfied by our group G, providing valuable information about it. Such properties include:

- two-generated subgroups are either free abelian or free nonabelian [9];
- any subgroup either is free abelian or surjects onto $\mathbb{F}_2$ [6, Corollary 1.6];
- being bi-orderable [25, 26];
- being linear (and, in particular, residually finite) [44];
- being residually torsion-free nilpotent [23, 25, 51].

Even better, as soon as the cube complex X is compact, the theory does not only show that G embeds into A, it shows that it embeds in a very specific way: the image of G in A is a *virtual retract*, i.e., there exists a finite-index subgroup $H \leq A$ containing G and a morphism $r : H \to G$ such that $r_{|G} = \mathrm{Id}_G$. This additional information provides other automatic properties satisfied by our group, including:

- two-generated subgroups are undistorted [16];
- infinite cyclic subgroups are separable [45];
- being conjugacy separable [46].

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One of the most impressive applications of the theory of special cube complexes is Agol's proof of the virtual Haken conjecture [1], showing that any cubulable hyperbolic group must be *cocompact special*. However, the scope of the theory is not restricted to hyperbolic groups and encompasses a large diversity of groups (possibly up to finite index), such as Coxeter groups [41], many 3-manifold groups [38, 50], and graph braid groups [21, 33].

In this article, our goal is to generalize Haglund and Wise's theory by replacing CAT(0) cube complexes with *quasi-median graphs* and right-angled Artin groups with *graph products of groups*.

As shown in [7], quasi-median graphs, a family of graphs generalizing median graphs (or equivalently, one-skeleta of CAT(0) cube complexes) have a long history in metric graph theory. In [28], we introduced them into geometric group theory by showing how they can be exploited in the study of graph products of groups, lamplighter groups, and Thompson-like groups (see also [31]). It turned out that quasi-median graphs provide a particularly relevant point of view in order to study graph products of groups [30, 34, 35].

Recall from [36] that, given a simplicial graph Γ and a family of groups $\mathcal{G} = \{G_u \mid u \in V(\Gamma)\}$ indexed by the vertices of Γ, the *graph product* $\Gamma\mathcal{G}$ is the quotient

$$\left(\mathop{*}_{u \in V(\Gamma)} G_u \right) / \langle\langle [g,h] = 1,\ g \in G_u, h \in G_v, \{u,v\} \in E(\Gamma) \rangle\rangle,$$

where $V(\Gamma)$ and $E(\Gamma)$ denote the vertex and edge sets of Γ. For instance, if the groups in $\mathcal{G}$ are all infinite cyclic, then $\Gamma\mathcal{G}$ coincides with the right-angled Artin group A_Γ; and if all the groups in $\mathcal{G}$ are cyclic of order 2, then $\Gamma\mathcal{G}$ coincides with the right-angled Coxeter group C_Γ. In the same way that the Cayley graphs of A_Γ and C_Γ constructed from the generating set $V(\Gamma)$ are median graphs (or equivalently, that their cube completions are CAT(0) cube complexes), the Cayley graph

$$\mathrm{QM}(\Gamma, \mathcal{G}) := \mathrm{Cayl}\left(\Gamma\mathcal{G}, \bigcup_{u \in V(\Gamma)} G_u \backslash \{1\} \right)$$

of $\Gamma\mathcal{G}$ turns out to be a quasi-median graph.

So, given a group G acting on a quasi-median graph X, we want to identify a simple condition on the action $G \curvearrowright X$ which implies that G naturally embeds into a graph product, possibly as a virtual retract. As shown in Sections 3.1 and 3.2, the following definition includes naturally the groups considered in Haglund and Wise's theory.

Definition 1.1 Let G be a group acting faithfully on a quasi-median graph X. The action is *hyperplane-special* if:

- for every hyperplane J and every element $g \in G$, J and gJ are neither transverse nor tangent;
- for all hyperplanes J_1, J_2 and every element $g \in G$, if J_1 and J_2 are transverse, then J_1 and gJ_2 cannot be tangent.

The action is *special* if, in addition, the action $\mathfrak{S}(J) \curvearrowright \mathscr{S}(J)$ is free for every hyperplane J of X. (Here, $\mathscr{S}(J)$ denotes the collection of all the sectors delimited by J, i.e., the connected components of the graph obtained from X by removing the interiors of all the edges dual to J; and $\mathfrak{S}(J)$ denotes the image of $\mathrm{stab}(J)$ in the permutation group of $\mathscr{S}(J)$.)

The main result of this article is the following embedding theorem. (We refer to Theorems 1.4 and 1.5 for more precise statements.)

Theorem 1.1 *Let G be a group which acts specially on a quasi-median graph with finitely many orbits of vertices. Then G embeds as a virtual retract into a graph product of virtual clique-stabilizers.*

As in Haglund and Wise's theory, knowing that the group we are studying is a subgroup of a graph product provides valuable information about it. For instance:

Corollary 1.2 *Let G be a group which acts specially on a quasi-median graph with finitely many orbits of vertices. Then the following assertions hold.*

- *Assume that clique-stabilizers satisfy the Tits alternative, i.e., every subgroup either contains a nonabelian free subgroup or is virtually solvable. Then G also satisfies the Tits alternative* [6].
- *If clique-stabilizers are linear (resp. residually finite), then so is G* [10, 36].
- *If clique-stabilizers are a-T-menable (resp. weakly amenable), then so is G* [5, 28, 47].

Proof Let $\mathcal{P}$ be one of the group properties under consideration. If clique-stabilizers satisfy $\mathcal{P}$, then G embeds in a graph product of groups satisfying $\mathcal{P}$ according to Theorem 1.1 (and because $\mathcal{P}$ is preserved by commensurability). Since $\mathcal{P}$ is stable under graph products, as a consequence of the references given above, and under taking subgroups, we conclude that G satisfies $\mathcal{P}$. ■

The fact that the image of our embedding is a virtual retract also provides additional information.

Corollary 1.3 *Let G be a group which acts specially on a quasi-median graph with finitely many orbits of vertices. Then the following assertions hold.*

- *For every $n \geq 1$, if clique-stabilizers are of type F_n, then so is G* [3, 4, 19]. *In particular, if clique-stabilizers are finitely generated (resp. finitely presented), then so is G.*
- *If clique-stabilizers are finitely presented, then the coarse inequality*

$$\delta_G \prec \max\left\{n \mapsto n^2, \overline{\delta}_{\mathrm{stab}(C)}\ (C\ clique)\right\}$$

between Dehn functions holds, where

$$\overline{f} : n \mapsto \max\left\{\sum_{i=1}^{k} f(n_i) \mid k \geq 1, \sum_{i=1}^{k} n_i = n\right\}$$

denotes the subnegative closure of the function f [2, 4, 20, 32].
- *If clique-stabilizers are conjugacy separable, then so is G* [27].
- *If clique-stabilizers have their cyclic subgroups separable, then cyclic subgroups of G are separable* [11].
- *If clique-stabilizers are finitely generated and have their infinite cyclic subgroups undistorted, then infinite cyclic subgroups in G are undistorted.*

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Proof The proof of the first four items is similar to the argument used to prove Corollary 1.2, with the stability under taking subgroups replaced with the stability under retraction. This stability is proved in the associated references for the first two items, and for the third and fourth items, it follows from the observation that the image of a retraction in a Hausdorff topological space is necessarily closed and that being separable in group amounts to being closed with respect to the profinite topology.

Finally, let us prove the fifth item. So, we assume that clique-stabilizers are finitely generated and have their infinite cyclic subgroups undistorted. Let $g \in G$ be an infinite-order element. We anticipate on Section 2 and consider the action of $\Gamma\mathcal{G}$ on its quasi-median graph $\mathrm{QM}(\Gamma, \mathcal{G})$, where $\Gamma\mathcal{G}$ is the graph product containing G as given by Theorem 1.1. If g has unbounded orbits in $\mathrm{QM}(\Gamma, \mathcal{G})$, then one can use the translation length and deduce that g is undistorted in G. (For instance, combine [28, Proposition 4.16] and [39, Theorem 1.5].) Otherwise, it follows from [28, Theorem 2.115] that g stabilizes a prism of $\mathrm{QM}(\Gamma, \mathcal{G})$, and hence $g \in h\langle\Lambda\rangle h^{-1}$ for some element $h \in \Gamma\mathcal{G}$ and some complete subgraph $\Lambda \subset \Gamma$ according to Lemma 2.9. For convenience, set $H := h\langle\Lambda\rangle h^{-1}$. Because H is isomorphic to a product of vertex groups, we know that g is undistorted in H. However, H, as a retract of $\Gamma\mathcal{G}$ (a retraction $\Gamma\mathcal{G} \to H$ being given by conjugating the projection $\Gamma\mathcal{G} \to \langle\Lambda\rangle$ that kills the vertex groups indexed by vertices not in Λ), is undistorted in $\Gamma\mathcal{G}$. We also know that G is a retract in $\Gamma\mathcal{G}$, so the metric of G induced by $\Gamma\mathcal{G}$ is coarsely equivalent to the metric of G. We conclude that $\langle g\rangle$ is quasi-isometrically embedded in G. ■

1.1 A word about the proof of the theorem

In Section 3.1, we explain how the fundamental group G of a special cube complex X can be embedded into a right-angled Artin group by looking at the action of G on the universal cover of X, instead of looking for a local isometry of X to the Salvetti complex of a right-angled Artin group. This construction is next generalized to arbitrary quasi-median graphs in Section 3.2 in order to prove the following theorem.

Theorem 1.4 *Let G be a group acting specially on a quasi-median graph X.*

- *Fix representatives $\{J_i \mid i \in I\}$ of hyperplanes of X modulo the action of G.*
- *Let Γ denote the graph whose vertex set is $\{J_i \mid i \in I\}$ and whose edges link two hyperplanes if they have two transverse G-translates.*
- *For every $i \in I$, let G_i denote the group $\mathfrak{S}(J_i) \oplus K_i$, where K_i is an arbitrary group of cardinality the number of orbits of $\mathfrak{S}(J_i) \curvearrowright \mathscr{S}(J_i)$.*

Then there exists an injective morphism $\varphi : G \hookrightarrow \Gamma\mathcal{G}$, where $\mathcal{G} = \{G_i \mid i \in I\}$, and a φ-equivariant embedding $X \hookrightarrow \mathrm{QM}(\Gamma, \mathcal{G})$ whose image is gated.

Notice that, compared to Theorem 1.1, we do not require the action to have only finitely many orbits of vertices. Under this additional assumption, we observe in Corollary 3.19 that each G_i contains a clique-stabilizer as a finite-index subgroup, concluding the first step toward the proof of Theorem 1.1.

The next step is to show that the image of our embedding is a virtual retract. The key point is that the image of $X \hookrightarrow \mathrm{QM}(\Gamma, \mathcal{G})$ in Theorem 1.4 is *gated*, which is a strong convexity condition. Combined with the next statement, the proof of Theorem 1.1 follows.

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Theorem 1.5 *Let Γ be a simplicial graph, and $\mathcal{G}$ a collection of groups indexed by $V(\Gamma)$. A gated-cocompact subgroup $H \leq \Gamma\mathcal{G}$ is a virtual retract.*

Here, a subgroup $H \leq \Gamma\mathcal{G}$ is *gated-cocompact* if there exists a gated subgraph in $\mathrm{QM}(\Gamma, \mathcal{G})$ on which H acts with finitely many orbits of vertices. It is worth noticing that, combined with Theorem 1.4, Theorem 1.5 implies more generally that gated-cocompact subgroups are virtual retracts in arbitrary groups acting specially on quasi-median graphs with finitely many vertices (see Corollary 3.24), generalizing the fact that convex-cocompact subgroups are virtual retracts in cocompact special groups [40].

1.2 Applications

In the second part of the article, we apply the theory of groups acting specially on quasi-median graphs to a specific family of groups originating in [28], namely fundamental groups of *right-angled graphs of groups*. We refer to Section 4.1 for a precise definition, but, roughly speaking, a graph of groups is said right-angled if its vertex groups are graph products and if its edge groups are subgraph products. In Section 4.3, we characterize precisely when the action of the fundamental group of such a graph of groups on the quasi-median graph constructed in [28] is special.

In order to illustrate how special actions on quasi-median graphs can be exploited, let us conclude this introduction by considering an explicit example (detailed in Section 4.4).

Given a group A, define $A^{\rtimes}$ by the relative presentation

$$\langle A, t \mid [a, tat^{-1}] = 1,\ a \in A \rangle.$$

Notice that, if A is infinite cyclic, we recover the group introduced in [15], which was the first example of a fundamental group of a 3-manifold which is not subgroup separable. $A^{\rtimes}$ is an example of a fundamental group of a right-angled graph of groups. It acts on a quasi-median graph, but this action is not special. Such a negative result is not a flaw in the strategy: as a two-generated group which is neither abelian nor free, $\mathbb{Z}^{\rtimes}$ cannot be embedded into a right-angled Artin group. Nevertheless, considering a finite cover of the graph of groups defining $A^{\rtimes}$ naturally leads to a new group, denoted by $A \square A$ and admitting

$$\langle A_1, A_2, t \mid [a_1, a_2] = [a_1, ta_2t^{-1}] = 1,\ a_1 \in A_1, a_2 \in A_2 \rangle,$$

as a relative presentation, where A_1 and A_2 are two copies of A. Then $A \square A$ is a subgroup of $A^{\rtimes}$ of index 2. Now, as the fundamental group of a right-angled graph of groups, $A \square A$ acts specially on a quasi-median graph. By a careful application of Theorem 1.4, we find that $A \square A$ embeds (as a virtual retract) into the graph product

$$G := \mathbb{Z}_2 - A_1 - A_2 - \mathbb{Z}_2,$$

by sending $A_1 \subset A \square A$ to $A_1 \subset G$, $A_2 \subset A \square A$ to $A_2 \subset G$, and $t \in A \square A$ to $xy \in G$, where x and y are generators of the two $\mathbb{Z}_2$.

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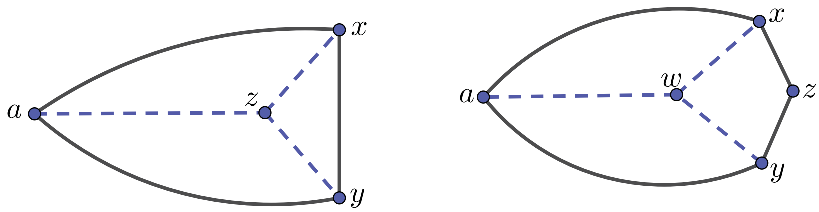

Figure 1: Triangle and quadrangle conditions.

2 Preliminaries

In this section, we give the basic definitions and properties of quasi-median graphs and graph products of groups which will be needed in the rest of the article.

2.1 Quasi-median graphs

There exist several equivalent definitions of quasi-median graphs; see, for instance, [7]. Below is the definition used in [28].

Definition 2.1 A connected graph X is *quasi-median* if it does not contain K_4^- and $K_{3,2}$ as induced subgraphs, and it satisfies the following two conditions:

(**Triangle condition**) for all vertices $a, x, y \in X$, if x and y are adjacent and if $d(a,x) = d(a,y)$, then there exists a vertex $z \in X$ which adjacent to both x and y and which satisfies $d(a,z) = d(a,x) - 1$;

(**Quadrangle condition**) for all vertices $a, x, y, z \in X$, if z is adjacent to both x and y and if $d(a,x) = d(a,y) = d(a,z) - 1$, then there exists a vertex $w \in X$ which adjacent to both x and y and which satisfies $d(a,w) = d(a,z) - 2$.

The graph $K_{3,2}$ is the bipartite complete graph, corresponding to two squares glued along two adjacent edges; and K_4^- is the complete graph on four vertices minus an edge, corresponding to two triangles glued along an edge. The triangle and quadrangle conditions are illustrated in Figure 1.

Definition 2.2 Let X be a graph, and let $Y \subset X$ be a subgraph. A vertex $y \in Y$ is a *gate* of another vertex $x \in X$ if, for every $z \in Y$, there exists a geodesic between x and z passing through y. If every vertex of X admits a gate in Y, then Y is *gated*.

It is worth noticing that the gate of x in Y, when it exists, is unique and coincides with the (unique) vertex that minimizes the distance to x in Y. As a consequence, it may be referred to as the *projection* of x onto Y. Gated subgraphs in quasi-median graphs play the role of convex subcomplexes in CAT(0) cube complexes. We record the following useful criterion for future use; a proof can be found in [17] (see also [28, Proposition 2.6]).

Lemma 2.1 *Let X be a quasi-median graph, and let $Y \subset X$ be a connected subgraph. Then Y is gated if and only if it is locally convex (i.e., any four-cycle in X with two adjacent* *edges contained in Y necessarily lies in Y) and if it contains its triangles (i.e., any three-cycle which shares an edge with Y necessarily lies in Y).*

Recall that a *clique* is a maximal complete subgraph, and that cliques in quasi-median graphs are gated [7]. A *prism* is, roughly speaking, a subgraph which is a product of cliques; more precisely, it is an induced subgraph that decomposes as a Cartesian product of complete graphs such that its cliques are also cliques in the whole graph.

Lemma 2.2 *Let X be a quasi-median graph, $x, y \in X$ two vertices, and γ_1, γ_2 two paths between x and y. Then γ_2 can be obtained from γ_1 by flipping squares, shortening triangles, removing backtracks, and inverses of these operations.*

Our lemma requires a few definitions. Given an oriented path γ in our graph X, which we decompose as a concatenation of oriented edges $e_1 \cdots e_n$, one says that γ' is obtained from γ by:

- *flipping a square*, if there exists some $1 \le i \le n-1$ such that

$$\gamma' = e_1 \cdots e_{i-1} \cdot a \cdot b \cdot e_{i+2} \cdots e_n,$$

  where e_i, e_{i+1}, b, a define an unoriented four-cycle in X;
- *shortening a triangle*, if there exists some $1 \le i \le n-1$ such that

$$\gamma' = e_1 \cdots e_{i-1} \cdot a \cdot e_{i+2} \cdots e_n,$$

  where e_i, e_{i+1}, a define an unoriented three-cycle in X;
- *removing a backtrack*, if there exists some $1 \le i \le n-1$ such that

$$\gamma' = e_1 \cdots e_{i-1} \cdot e_{i+2} \cdots e_n,$$

  where e_{i+1} is the inverse of e_i.

Lemma 2.2 follows from the simple connectivity of triangle-square complex obtained from X by filling three- and four-cycles (which is an easy consequence of the triangle and quadrangle conditions; see [13, Lemma 5.5] for details) combined either with a standard argument based on disc diagrams or with observation that flipping squares and shortening triangles provide the relations of the fundamental groupoid of X (see [14, Statement 9.1.6] for details).

2.2 Median graphs

A graph X is a *median graph* if, for all vertices $x, y, z \in X$, there exists a unique vertex $m \in X$ such that

$$\begin{cases} d(x,y) = d(x,m) + d(m,y), \\ d(x,z) = d(x,m) + d(m,z), \\ d(y,z) = d(y,m) + d(m,z). \end{cases}$$

The point m is referred to as the *median point* of the triple x, y, z. Median graphs are known to define the same objects as CAT(0) cube complexes. Indeed, the one-skeleton of a CAT(0) cube complex is a median graph, and the cube-completion of a median graph, namely the cube complex obtained by filling in all the one-skeleta of cubes in the graph with cubes, is a CAT(0) cube complex. We refer to [18] for more information.

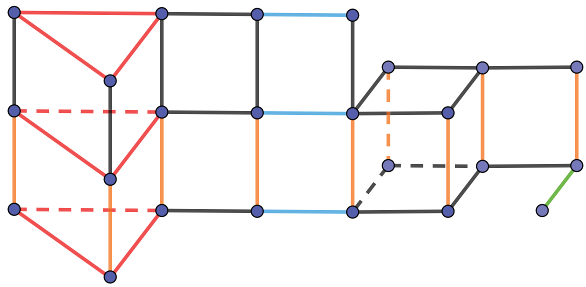

Figure 2: A quasi-median graph and four of its hyperplanes (in orange, red, blue, and green).

2.3 Hyperplanes

Similarly to CAT(0) cube complexes, the notion of *hyperplanes* is fundamental in the study of quasi-median graphs.

Definition 2.3 Let X be a graph. A *hyperplane* J is an equivalence class of edges with respect to the transitive closure of the relation saying that two edges are equivalent whenever they belong to a common triangle or are opposite sides of a square. We denote by $X\backslash\backslash J$ the graph obtained from X by removing the interiors of all the edges of J. A connected component of $X\backslash\backslash J$ is a *sector*. The *carrier* of J, denoted by $N(J)$, is the subgraph induced by all the vertices in the edges of J. Two hyperplanes J_1 and J_2 are *transverse* if there exist two edges $e_1 \subset J_1$ and $e_2 \subset J_2$ spanning a square in X, and they are *tangent* if they are not transverse, but $N(J_1) \cap N(J_2) \neq \emptyset$.

See Figure 2 for examples of hyperplanes in a quasi-median graph. A key observation is that hyperplanes in quasi-median graphs always delimit at least two sectors.

Theorem 2.3 [28, Proposition 2.15]; see also [31, Theorem A.1] *Let X be a quasi-median graph and J a hyperplane. The graph $X\backslash\backslash J$ is disconnected, and the carrier and the sectors of J are gated.*

We refer to [28, Section 2.2] (and more particularly to [28, Proposition 2.30]) for more information about the (fundamental) connection between the geometry of quasi-median graphs and their hyperplanes.

We record the following lemmas for future use.

Lemma 2.4 [28, Lemma 2.25]; see also [31, Lemma A.5] *In a quasi-median graph, two distinct cliques which are dual to the same hyperplane must be disjoint.*

Lemma 2.5 [28, Fact 2.75] *Let X be a quasi-median graph and $e_1, e_2 \subset X$ two edges sharing their initial point. If the hyperplanes dual to e_1 and e_2 are transverse, then e_1 and e_2 span a square.*

Lemma 2.6 *Let X be a quasi-median graph, $x, y \in X$ two vertices, and $[x, y]$ a geodesic from x to y. Let $J_1, \dots, J_n$ denote the hyperplanes crossed by $[x, y]$ in that order. If J_i and J_{i+1} are transverse for some $1 \leq i \leq n-1$, then there exists a geodesic from x to y crossing the hyperplanes $J_1, \dots, J_{i-1}, J_{i+1}, J_i, J_{i+2}, \dots, J_n$ in that order.*

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Proof Decompose the geodesic $[x,y]$ as a concatenation of edges $e_1\cdots e_n$. So, for every $1 \le j \le n$, e_j is dual to the hyperplane J_j. As a consequence of Lemma 2.5, the edges e_i and e_{i+1} span a square. Flipping this square (i.e., replacing e_i and e_{i+1} with their opposite edges in our square) produces a new path between x and y which has the same length as $[x,y]$ (and so is a geodesic) and which crosses the hyperplanes $J_1,\dots,J_{i-1},J_{i+1},J_i,J_{i+2},\dots,J_n$ in that order. ■

2.4 Graph products

We conclude our preliminary section by considering graph products of groups and their quasi-median graphs.

Let Γ be a simplicial graph, and let $\mathcal{G} = \{G_u \mid u \in V(\Gamma)\}$ be a collection of groups indexed by the vertex set $V(\Gamma)$ of Γ. The *graph product* $\Gamma\mathcal{G}$ is defined as the quotient

$$\left(\mathop{*}_{u\in V(\Gamma)} G_u\right) / \langle\langle [g,h]=1 \mid g \in G_u, h \in G_v, \{u,v\} \in E(\Gamma)\rangle\rangle,$$

where $E(\Gamma)$ denotes the edge set of Γ. The groups in $\mathcal{G}$ are referred to as *vertex groups*.

2.5 Convention

In the entire article, we will assume for convenience that the groups in $\mathcal{G}$ are nontrivial. Notice that it is not a restrictive assumption, since a graph product with some trivial factors can be described as a graph product over a smaller graph all of whose factors are nontrivial.

A *word* in $\Gamma\mathcal{G}$ is a product $g_1\cdots g_n$ where $n \ge 0$ and where, for every $1 \le i \le n$, $g_i \in G$ for some $G \in \mathcal{G}$; the g_i are the *syllables* of the word, and n is the *length* of the word. Clearly, the following operations on a word does not modify the element of $\Gamma\mathcal{G}$ it represents:

Cancellation: delete the syllable $g_i = 1$;
Amalgamation: if $g_i, g_{i+1} \in G$ for some $G \in \mathcal{G}$, replace the two syllables g_i and g_{i+1} by the single syllable $g_i g_{i+1} \in G$;
Shuffling: if g_i and g_{i+1} belong to two adjacent vertex groups, switch them.

A word is *graphically reduced* if its length cannot be shortened by applying these elementary moves. Every element of $\Gamma\mathcal{G}$ can be represented by a graphically reduced word, and this word is unique up to the shuffling operation. This allows us to define the *length* of an element $g \in \Gamma\mathcal{G}$, denoted by $|g|$, as the length of any graphically reduced word representing g. For more information on graphically reduced words, we refer to [36] (see also [32, 43]).

We record the following definition for future use.

Definition 2.4 The *tail* of a graphically reduced word $g = g_1\cdots g_n$ is the set of syllables that can be shuffled to the end of the word. More precisely, the syllable g_i belongs to the tail of g if the vertex group containing g_i is adjacent to the vertex group containing g_j for every $j > i$. Observe that applying a shuffling to a graphically reduced word does not modify its tail, so, because an element of $\Gamma\mathcal{G}$ is represented by a unique graphically

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reduced word up to the shuffling operation, one can define the *tail* of an element of $\Gamma\mathcal{G}$ as the tail of any graphically reduced word representing it.

The connection between graph products and quasi-median graphs is made explicit by the following statement [28, Proposition 8.2].

Theorem 2.7 *Let Γ be a simplicial graph and $\mathcal{G}$ a collection of groups indexed by $V(\Gamma)$. The Cayley graph* $\mathrm{QM}(\Gamma, \mathcal{G}) := \mathrm{Cay}\left(\Gamma\mathcal{G}, \bigcup_{G\in\mathcal{G}} G\backslash\{1\}\right)$ *is a quasi-median graph.*

Notice that the graph product $\Gamma\mathcal{G}$ naturally acts by isometries on $\mathrm{QM}(\Gamma, \mathcal{G})$ by left multiplication. We refer to [28, Section 8.1] (and [35, Section 2.2]) for more information about the geometry of $\mathrm{QM}(\Gamma, \mathcal{G})$. Here, we only mention the following two statements, which describe the cliques and the prisms of $\mathrm{QM}(\Gamma, \mathcal{G})$.

Lemma 2.8 [28, Lemma 8.6]; see also [35, Lemma 2.4] *Let Γ be a simplicial graph and $\mathcal{G}$ a collection of groups indexed by $V(\Gamma)$. The cliques of* $\mathrm{QM}(\Gamma, \mathcal{G})$ *are the subgraphs induced by the vertices in the cosets of the form gG_u, where $g \in \Gamma\mathcal{G}$ and $u \in V(\Gamma)$.*

Lemma 2.9 [28, Corollary 8.7]; see also [35, Lemma 2.6] *Let Γ be a simplicial graph and $\mathcal{G}$ a collection of groups indexed by $V(\Gamma)$. The prisms of* $\mathrm{QM}(\Gamma, \mathcal{G})$ *are the subgraphs induced by the vertices in the cosets of the form $g\langle\Lambda\rangle$, where $g \in \Gamma\mathcal{G}$, where Λ is a complete subgraph of Γ, and where $\langle\Lambda\rangle$ denotes the subgroup of $\Gamma\mathcal{G}$ generated by the vertex groups labeling the vertices of Λ.*

3 Special actions on quasi-median graphs

3.1 Warm up: special cube complexes revisited

As introduced in [40], special cube complexes are nonpositively curved cube complexes which do not contain configurations of hyperplanes referred to as *pathological*. Then, the key observation is that, given such a cube complex X, there exists a graph Γ (namely, the crossing graph of the hyperplanes in X) and a local isometry $X \hookrightarrow X_\Gamma$, where X_Γ is a nonpositively curved cube complex with the right-angled Artin group A_Γ as fundamental group (namely, a *Salvetti complex*). As local isometries between nonpositively curved cube complexes are π_1-injective, it follows that $\pi_1(X)$ is a subgroup of A_Γ. Similar arguments can be conducted when A_Γ is replaced with the right-angled Coxeter group C_Γ, but considering either A_Γ or C_Γ is essentially equivalent because a right-angled Artin groups always embeds as a finite-index subgroup into a right-angled Coxeter group [22].

In this section, we sketch an alternative approach which illustrates the more general arguments from the next section. So, we fix a group G which acts *specially* on a CAT(0) cube complex X, i.e.,

- for every hyperplane J and every element $g \in G$, J and gJ are neither transverse nor tangent;
- for all hyperplanes J_1, J_2 and every element $g \in G$, if J_1 and J_2 are transverse, then J_1 and gJ_2 cannot be tangent.

Observe that this definition is compatible with the terminology introduced by Definition 1.1 from the introduction: because a hyperplane in a CAT(0) cube complex

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always delimits exactly two subspaces, and the action $\mathfrak{S}(J) \curvearrowright \mathscr{S}(J)$ is automatically free for every hyperplane J, so special and hyperplane-special actions coincide.

Let Γ denote the graph whose vertices are the G-orbits of hyperplanes and whose edges link two orbits if they contain at least two transverse hyperplanes. Naturally, each hyperplane of X is labeled by a vertex of Γ, namely the G-orbits it belongs to. Define the *label* of an oriented path γ in X as the word $\ell(\gamma)$ given by the sequence of the G-orbits of hyperplanes it crosses. Fixing a basepoint $x_0 \in X$, we consider

$$\Phi : \begin{cases} X & \to & X(\Gamma), \\ x & \mapsto & \ell(\text{path from } x_0 \text{ to } x), \end{cases}$$

where $X(\Gamma)$ denotes the usual CAT(0) cube complex on which the right-angled Coxeter group C_Γ acts, namely the cube completion of the Cayley graph $\mathrm{Cay}(C_\Gamma, V(\Gamma))$. Notice that Φ naturally induces

$$\varphi : \begin{cases} G & \to & C_\Gamma, \\ g & \mapsto & \Phi(g \cdot x_0). \end{cases}$$

It turns out that φ is an injective morphism, that Φ is a φ-equivariant embedding, and that the image of Φ is a convex subcomplex of C_Γ. These observations are based on the following three claims, for which we sketch justifications.

Claim 3.1 The map Φ is well defined, i.e., for every vertex $x \in X$, the vertex of $X(\Gamma)$ represented by the label of a path from x_0 to x does not depend on the path we choose.

First, consider an oriented path of the form ee^{-1}, namely a backtrack. Then $\ell(ee^{-1}) = \ell(e)^2$ equals 1 in C_Γ. Next, consider an oriented path of the form $e \cdot f$ where e and f are consecutive edges in a square. Because the hyperplanes dual to e and f are transverse, the generators $\ell(e)$ and $\ell(f)$ commute in C_Γ, so

$$\ell(e \cdot f) = \ell(e)\ell(f) = \ell(f)\ell(e) = \ell(e' \cdot f'),$$

where $e' \cdot f'$ denotes the image of $e \cdot f$ under the diagonal reflection in the square which contains $e \cdot f$. Therefore, the label of a path remains the same if we add or remove a backtrack or if we *flip a square*. In a CAT(0) cube complex, any two paths with the same endpoints can be obtained from one to another thanks to such elementary operations, so the desired conclusion follows. ■

Claim 3.2 The map $\varphi : G \to C_\Gamma$ is a morphism.

Fix two elements $g, h \in G$. We have

$$\begin{aligned} \varphi(gh) &= \ell([x_0, ghx_0]) = \ell([x_0, gx_0] \cdot g[x_0, hx_0]) \\ &= \ell([x_0, gx_0])\ell([x_0, hx_0]) = \varphi(g)\varphi(h), \end{aligned}$$

where the second equality is justified by Claim 3.1, and the third one by the fact that the labeling map ℓ is G-invariant. (Here, $[\cdot, \cdot]$ refers to some arbitrary choice of a geodesic between the two vertices under consideration.) ■

So far, the specialness of the action has not been used, and the morphism $\varphi : G \to C_\Gamma$ is well defined for every action of G on a CAT(0) cube complex. However, this

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assumption is crucial in the proof of the injectivity of Φ (and φ), which follows from the next assertion.

Claim 3.3 For every vertex $x \in X$ and every geodesic $[x_0, x]$ in X, the word $\ell([x_0, x])$ is graphically reduced in C_Γ.

Assume that there exists a vertex $x \in X$ and a geodesic $[x_0, x]$ such that the word $\ell([x_0, x])$ is not graphically reduced. So, if we write $[x_0, x]$ as a concatenation of oriented edges $e_1 \cdots e_n$, then there exist two indices $1 \leq i < j \leq n$ such that $\ell(e_i) = \ell(e_j)$ and such that $\ell(e_k)$ commutes with $\ell(e_i)$ for every $i < k < j$. Assume that $j - i \geq 2$. Because $\ell(e_i)$ and $\ell(e_{i+1})$ commute, the hyperplane J_i dual to e_i has a G-translate which is transverse to the hyperplane J_{i+1} dual to e_{i+1}. Because the action is special, the hyperplanes J_i and J_{i+1} cannot be tangent, so they are transverse. As a consequence, the edges e_i and e_{i+1} span a square, and by flipping this square, we can replace our geodesic $[x_0, x]$ with a new geodesic so that $j - i$ decreases. By iterating the process, we end up with a geodesic $[x_0, x]$ such that $j = i + 1$. In other words, $[x_0, x]$ contains two successive edges with the same label; or equivalently, if J and H denote the hyperplanes dual to these two edges, J and H belong to the same G-orbit. However, J and H are either tangent or transverse, which contradicts the specialness of the action. ■

3.2 Embeddings into graph products

In this section, we define special actions on quasi-median graphs and we show, given a group admitting such an action, how to embed it into a graph product. We begin by introducing the following notation.

Notation 3.4 Let G be a group acting on a quasi-median graph X. For every hyperplane J of X, let $\mathscr{S}(J)$ denote the collection of sectors delimited by J, and $\mathfrak{S}(J)$ the image of $\mathrm{stab}_G(J)$ in the permutation group of $\mathscr{S}(J)$.

Special actions on quasi-median graphs are defined as follows.

Definition 3.1 Let G be a group acting faithfully on a quasi-median graph X. The action is *hyperplane-special* if:

- for every hyperplane J and every element $g \in G$, J and gJ are neither transverse nor tangent;
- for all hyperplanes J_1, J_2 and every element $g \in G$, if J_1 and J_2 are transverse, then J_1 and gJ_2 cannot be tangent.

The action is *special* if, in addition, the action $\mathfrak{S}(J) \curvearrowright \mathscr{S}(J)$ is free for every hyperplane J of X.

It is worth noticing that our definition agrees with the definition of special actions on median graphs we used in the previous section. In other words, an action on a median graph is special if and only if it is hyperplane-special. Indeed, hyperplanes in median graphs delimit exactly two sectors, and a faithful action on a set of cardinality two is automatically free.

As a preliminary observation, note that:

Lemma 3.5 *If G acts specially on a quasi-median graph X, then vertex stabilizers are trivial.*

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Proof Assume that $g \in G$ fixes a vertex $x \in X$.

Let $y \in X$ be a neighbor of x. Let C denote the clique which contains the edge connecting x and y, and J the hyperplane containing C. Because J and gJ are neither tangent nor transverse, necessarily $gJ = J$, so that $gC = C$ as a consequence of Lemma 2.4. Because the action $\mathfrak{S}(J) \curvearrowright \mathscr{S}(J)$ is free, necessarily g stabilizes all the sectors delimited by J, which implies that g fixes C pointwise, and in particular $gy = y$.

Thus, we have proved that g fixes x and all its neighbors. By reproducing the argument to the neighbors of y, and so on, we deduce that g fixes X pointwise. As the action of G on X is faithful, we conclude that g must be trivial. ∎

The rest of the section is almost entirely dedicated to the proof of the following embedding theorem.

Theorem 3.6 *Let G be a group acting specially on a quasi-median graph X.*

- *Fix representatives $\{J_i \mid i \in I\}$ of hyperplanes of X modulo the action of G.*
- *Let Γ denote the graph whose vertex set is $\{J_i \mid i \in I\}$ and whose edges link two hyperplanes if they have two transverse G-translates.*
- *For every $i \in I$, let G_i be an arbitrary group containing $\mathfrak{S}(J_i)$ as a subgroup of index at least the number of orbits of $\mathfrak{S}(J_i) \curvearrowright \mathscr{S}(J_i)$.*

Then there exists an injective morphism $\varphi : G \hookrightarrow \Gamma\mathcal{G}$, where $\mathcal{G} = \{G_i \mid i \in I\}$, and a φ-equivariant embedding $\Phi : X \hookrightarrow \mathrm{QM}(\Gamma, \mathcal{G})$. Moreover, if the index of $\mathfrak{S}(J_i)$ in G_i is exactly the number of orbits of $\mathfrak{S}(J_i) \curvearrowright \mathscr{S}(J_i)$ for every $i \in I$, then the image of Φ is a gated subgraph.

Proof We fix a basepoint $x_0 \in X$, and, for every hyperplane J, we denote by $S(J)$ the sector delimited by J containing x_0. Given an $i \in I$, we want to fix an injective map $\lambda_i : \mathscr{S}(J_i) \hookrightarrow G_i$ and a coloring of $\mathscr{S}(J_i)$ such that:

- λ_i is $\mathfrak{S}(J_i)$-equivariant and $\lambda_i(S(J_i)) = 1$;
- if the index of $\mathfrak{S}(J_i)$ in G_i coincides with the number of orbits of $\mathfrak{S}(J_i) \curvearrowright \mathscr{S}(J_i)$, then λ_i is a bijection;
- no two points of $\mathscr{S}(J_i)$ in the same $\mathfrak{S}(J_i)$-orbit have the same color;
- for every point $x \in \mathscr{S}(J_i)$ and every color c, there exists a unique $g \in \mathfrak{S}(J_i)$ such that gx has color c.

Let us prove that such objects exist. Fix representatives S_α, $\alpha \in A$, in $\mathscr{S}(J_i)$ under the action of $\mathfrak{S}(J_i)$; and representatives g_β, $\beta \in B$, in G_i under the action by left-multiplication of $\mathfrak{S}(J_i)$. We know by assumption that the cardinality of B is at least the cardinality of A, so we can assume that B contains A, with equality if they both have the same cardinality. Because $\mathfrak{S}(J_i)$ acts freely on $\mathscr{S}(J_i)$, every point in $\mathscr{S}(J_i)$ can be uniquely written as gS_α where $g \in \mathfrak{S}(J_i)$ and $\alpha \in A$. Therefore, setting $\lambda_i(gS_\alpha) := gg_\alpha$ for all $g \in \mathfrak{S}(J_i)$ and $\alpha \in A$ defines an $\mathfrak{S}(J_i)$-equivariant injection $\mathscr{S}(S_j) \to G_i$, which is also surjective if $A = B$. Of course, we can assume that $S(J_i)$ is one of our representatives, say S_γ for some $\gamma \in A$, and that $g_\gamma = 1$. Then $\lambda_i(S(J_i)) = 1$. Thus, our λ_i satisfies the desired properties. Next, declare that two sectors $R, S \in \mathscr{S}(J_i)$ have the same color if there exist $g \in \mathfrak{S}(J_i)$ and $\alpha, \beta \in A$ such that $R = gS_\alpha$ and $S = gS_\beta$. The third item is satisfied because $\mathfrak{S}(J_i)$ acts freely on $\mathscr{S}(J_i)$, and the fourth item is satisfied by construction (the uniqueness being also a consequence of the freeness of the action of $\mathfrak{S}(J_i)$).

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From now on, we fix, for every $i \in I$, such a map λ_i and such a coloring of $\mathscr{S}(J_i)$. The specific construction does not matter; only the properties recorded by the four items will be used in the sequel. Observe that, in particular, the sectors delimited by J_i are labeled by elements in G_i under λ_i. We want to extend such a labeling equivariantly to all the hyperplanes of X.

Claim 3.7 For every hyperplane J, there exist $i \in I$ and $g \in G$ such that $gJ = J_i$ and such that $gS(J)$ and $S(J_i)$ have the same color. ■

Of course, there exist $i \in I$ and $h \in G$ such that $hJ = J_i$. By definition of our colouing of $\mathscr{S}(J_i)$, there exists some $k \in \text{stab}(J_i)$ such that $k \cdot hS(J)$ and $S(J_i)$ have the same color. Setting $g := kh$ proves the claim.

3.3 Labeling the sectors

If J is an arbitrary hyperplane of X, let $i \in I$ and $g \in G$ be as given by Claim 3.7. A sector S delimited by J is labeled by $\ell(S) := \lambda_i(gS) \in G_i$.

Notice that the label of S does not depend on the choice of g. Indeed, let $h \in G$ be another element satisfying Claim 3.7. Then, gh^{-1} stabilizes J_i, and the sectors $gS(J)$, $S(J_i)$, and $hS(J)$ all have the same color. In other words, gh^{-1} defines an element of $\mathfrak{S}(J_i)$ that sends some element of $\mathscr{S}(J_i)$ to an element of the same color, which implies that gh^{-1} represents the trivial element of $\mathfrak{S}(J_i)$. We conclude that $gS = gh^{-1} \cdot hS = hS$.

3.4 Labeling the oriented paths

If $e \subset X$ is an oriented edge, let S_1 (resp. S_2) denote the sector delimited by the hyperplane dual to e which contains the initial endpoint of e (resp. the terminal endpoint of e). The label of e is defined as $\ell(e) := \ell(S_1)^{-1}\ell(S_2)$. More generally, if $\gamma = e_1 \cdots e_n$ is an oriented path, then its label is defined as the word $\ell(\gamma) := \ell(e_1)\cdots\ell(e_n)$, most of the time thought of as an element of $\Gamma\mathcal{G}$.

Because we may consider the label of an oriented path either as a word or as an element of $\Gamma\mathcal{G}$, we will use the following notation in order to avoid any ambiguity. Given two labels a and b, we denote by $a = b$ the equality in the group $\Gamma\mathcal{G}$, and $a \equiv b$ the equality as words.

We record below two fundamental facts about the labeling we have constructed: it is G-invariant, and it sends geodesics to graphically reduced words.

Claim 3.8 Let $e \subset X$ be an oriented edge and $g \in G$ an element. Then $\ell(g \cdot e) = \ell(e)$.

Let J denote the hyperplane dual to e. According to Claim 3.7, there exist $i \in I$ and $h, k \in G$ such that $hJ = J_i = k \cdot gJ$ and such that $S(J_i)$, $hS(J)$, and $kS(gJ)$ all have the same color. As a consequence, kgh^{-1} stabilizes J_i, so it defines an element σ of $\mathfrak{S}(J_i)$. Notice that, if S is an arbitrary sector delimited by J, then σ sends hS to $k \cdot gS$ (as elements of $\mathscr{S}(J_i)$). Hence,

$$\ell(gS) = \lambda_i(kgS) = \lambda_i(\sigma hS) = \sigma\lambda_i(hS) = \sigma\ell(S),$$

in G_i. The key observation is that σ does not depend on S. Therefore, if S_1 (resp. S_2) denotes the sector delimited by J which contains the initial endpoint of e (resp. the

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terminal endpoint of e), then

$$\ell(ge) = \ell(gS_1)^{-1}\ell(gS_2) = \ell(S_1)^{-1}\sigma^{-1}\sigma\ell(S_2) = \ell(S_1)^{-1}\ell(S_2) = \ell(e),$$

concluding the proof of our claim.

Claim 3.9 For all vertices $x, y \in X$ and every geodesic $[x, y]$ from x to y, the word $\ell([x, y])$ is graphically reduced in $\Gamma\mathcal{G}$.

Assume for contradiction that there exist vertices $x, y \in X$ and a geodesic $[x, y]$ from x to y, which we decompose as a concatenation of edges $e_1 \cdots e_r$, such that $\ell([x, y])$ is not graphically reduced in $\Gamma\mathcal{G}$. So, there exist two indices $1 \leq i < j \leq r$ such that $\ell(e_i)$ and $\ell(e_j)$ belong to the same vertex group of $\Gamma\mathcal{G}$ and such that $\ell(e_k)$ belongs to an adjacent vertex group for every $i < k < j$. In other words, if J_k denotes the hyperplane dual to e_k for every $1 \leq k \leq r$, then J_i and J_j belong to the same G-orbit and, for every $i < k < j$, a G-translate of J_k is transverse to J_j. Because G acts specially on X, notice that, if $j \geq i + 2$, then the hyperplane J_{j-1} cannot be tangent to J_j, so J_{j-1} and J_j are transverse. As a consequence of Lemma 2.6, there exists a geodesic from x to y which crosses the hyperplanes $J_1, \dots, J_{j-2}, J_j, J_{j-1}, J_{j+1}, \dots, J_r$ in that order. By iterating the argument, it follows that we can choose carefully our geodesic $[x, y]$ so that $j = i + 1$. In other words, J_i and J_j are tangent or transverse. However, we know that J_i and J_j belong to the same G-orbit, contradicting the specialness of the action. The proof of our claim is complete.

3.5 The embedding

Fix a second basepoint $x_1 \in X$, possibly different from x_0. In order to prove our theorem, we want to show that

$$\Phi : \begin{cases} X & \to & X(\Gamma, \mathcal{G}) \\ x & \mapsto & \ell\,(\text{path from } x_1 \text{ to } x) \end{cases}$$

defines an embedding whose image, that

$$\varphi : \begin{cases} G & \to & \Gamma\mathcal{G} \\ g & \to & \Phi(g \cdot x_1) \end{cases}$$

is an injective morphism, and that Φ is φ-equivariant.

First of all, we claim that Φ is well defined, i.e., the label of a path from x_1 to x (as an element of $\Gamma\mathcal{G}$) does not depend on the path we choose. As a consequence of Lemma 2.2, it suffices to show that flipping a square, shortening a triangle, and removing a backtrack do not modify the label of a path.

We begin by noticing that, if $e \cdot f$ is an oriented path between two opposite vertices of a square and if $e' \cdot f'$ denotes the image of $e \cdot f$ under the reflection along the diagonal of our square, then $e \cdot f$ and $e' \cdot f'$ have the same label. Indeed, observe that the endpoints of e and f' belong to the same sectors delimited by the hyperplane dual to e and f', and similarly for f and e', so $\ell(e \cdot f) \equiv \ell(e)\ell(f)$ and $\ell(e' \cdot f') \equiv \ell(f)\ell(e)$. However, $\ell(e)$ and $\ell(f)$ belong to two vertex groups of $\Gamma\mathcal{G}$ which are adjacent since the two hyperplanes dual to e and f are transverse. Therefore,

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$$\ell(e \cdot f) = \ell(e)\ell(f) = \ell(f)\ell(e) = \ell(e' \cdot f'),$$

so that flipping a square in a path does not modify its label (in $\Gamma\mathcal{G}$). Next, if $e \cdot f$ is a backtrack, then

$$\ell(e \cdot f) = \ell(e)\ell(f) = \ell(e)\ell(e)^{-1} = 1,$$

so that removing a backtrack to a path does not modify its label (in $\Gamma\mathcal{G}$) either. Finally, let $e \cdot f$ be the concatenation of two successive edges in a triangle, and let e' denote the edge of this triangle with the same endpoints as $e \cdot f$. Let J denote the hyperplane containing our triangle, S_1 the sector delimited by J which contains the initial point of e, S_2 the sector delimited by J which contains the terminal endpoint of e, and S_3 the sector delimited by J which contains the terminal point of f. Then

$$\ell(e \cdot f) = \ell(e)\ell(f) = \ell(S_1)^{-1}\ell(S_2) \cdot \ell(S_2)^{-1}\ell(S_3) = \ell(S_1)^{-1}\ell(S_3) = \ell(e'),$$

so that shortening a triangle does not modify the label of a path. Thus, we have proved that Φ is well defined.

It is worth noticing that our map Φ essentially does not depend on the basepoint x_1 we choose. When we allow the basepoint x_1 to vary, we denote by Φ_z the map obtained from Φ by replacing x_1 with another vertex $z \in X$. Then:

Claim 3.10 For all vertices $p, q \in X$, we have the commutative diagram

$$\begin{array}{ccc} X & \xrightarrow{\Phi_q} & \mathrm{QM}(\Gamma, \mathcal{G}) \\ \downarrow{\scriptstyle \Phi_p} & \swarrow{\scriptstyle m_g} & \\ \mathrm{QM}(\Gamma, \mathcal{G}) & & \end{array}$$

where the isometry m_g denotes the left-multiplication by $g := \ell([p, q])$.

Indeed,

$$\Phi_p(x) = \ell([p, x]) = \ell([p, q] \cdot [q, x]) = \ell([p, q]) \cdot \Phi_q(x),$$

for every vertex $x \in X$.

We are now ready to show that φ is an injective morphism and that Φ is a φ-equivariant embedding.

Claim 3.11 The map Φ is an isometric embedding. In particular, it is injective.

Let $x, y \in X$ be two vertices. Fix a geodesic $[x, y]$ between x and y in X. As a consequence of Claim 3.10,

$$d(\Phi(x), \Phi(y)) = d(\Phi_x(x), \Phi_x(y)) = d(1, \ell([x, y]).$$

However, $\ell([x, y]$ is a graphically reduced word according to Claim 3.9, so $d(1, \ell([x, y]))$ coincides with the length of $\ell([x, y])$, or equivalently with the number of edges of $[x, y]$. We conclude that $d(\Phi(x), \Phi(y)) = d(x, y)$.

Claim 3.12 For all $x \in X$ and $g \in G$, $\Phi(gx) = \varphi(g)\Phi(x)$.

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By fixing arbitrary paths $[x_1, gx]$, $[x_1, gx_1]$, and $[x_1, x]$ in X, we have

$$\Phi(gx) = \ell([x_1, gx]) = \ell([x_1, gx_1] \cdot g[x_1, x]) = \ell([x_1, gx_1])\ell([x_1, x]) = \varphi(g)\Phi(x),$$

where the penultimate equality is justified by Claim 3.8. Our claim is proved.

Notice that Claim 3.12 implies that φ is a morphism. Indeed, for every $g, h \in G$, we have

$$\varphi(gh) = \Phi(gh \cdot x_1) = \varphi(g)\Phi(h \cdot x_1) = \varphi(g)\varphi(h).$$

Moreover, the injectivity of φ follows from the injectivity of Φ provided by Claim 3.11, combined with Lemma 3.5, and Claim 3.12 precisely means that Φ is φ-equivariant.

From now on, we assume that, for every $i \in I$, the index of $\mathfrak{S}(J_i)$ in G_i coincides with the number of orbits of $\mathfrak{S}(J_i) \curvearrowright \mathscr{S}(J_i)$; by construction, this implies that $\lambda_i : \mathscr{S}(J_i) \to G_i$ is a bijection. We want to show that, under this assumption, the image of Φ is a gated subgraph of $\mathrm{QM}(\Gamma, \mathcal{G})$.

Claim 3.13 Let $x \in X$ be a vertex and $i \in I$ an index. If there exists some $a \in G_i$ such that x is the initial vertex of an edge of X labeled by a, then, for every $b \in G_i$, x is the initial vertex of an edge labeled by b; moreover, this edge belongs to the same clique as the edge labeled by a.

Fix an element $b \in G_i$. Let C denote the clique of X containing our edge labeled by a, and let J denote the hyperplane of X which contains it. By construction, the sectors delimited by J are labeled by elements of G_i, and conversely every element of G_i labels a sector delimited by J. Let e be the edge of C which connects x to the sector delimited by J which is labeled by cb, where $c \in G_i$ is the label of the sector containing x. Then $\ell(e) = c^{-1} \cdot cb = b$, so e is the edge we are looking for.

Claim 3.14 The image under Φ of a clique of X is a clique of $\mathrm{QM}(\Gamma, \mathcal{G})$. As a consequence, the image of Φ contains its triangles.

Let C be a clique of X. Fix an arbitrary vertex $x \in C$. The edges of C are all labeled by the same group G_i, $i \in I$. It follows from Claim 3.13 that $\Phi(C) \supset \Phi(x)G_i$. On the other hand, $\Phi(x)G_i$ is a clique in $\mathrm{QM}(\Gamma, \mathcal{G})$ according to Lemma 2.8, so $\Phi(C) \subset \Phi(x)G_i$. This proves the first assertion of our claim. Now, let T be a triangle with at least one edge e in the image of Φ. Let A be a clique of X such that $\Phi(A)$ contains e, and let B be the clique of $\mathrm{QM}(\Gamma, \mathcal{G})$ that contains T. Then $\Phi(A)$ and B are two cliques of $\mathrm{QM}(\Gamma, \mathcal{G})$ containing e. However, in a quasi-median graph, the intersecting between two distinct cliques contains at most one vertex (which is an immediate consequence of the fact that there is no induced K_4^-); hence, $T \subset B = \Phi(A)$, proving that the image of Φ contains its triangle, as desired.

Claim 3.15 The image of Φ is locally convex.

Let $e_1, e_2 \subset X$ be two edges which share their initial point and such that $\Phi(e_1)$ and $\Phi(e_2)$ span a square S. Necessarily, $\ell(e_1)$ and $\ell(e_2)$ belong to adjacent vertex groups, which means that the hyperplane dual to e_1 has a G-translate which is transverse to the hyperplane dual to e_2. Because G acts specially on X, it follows that the hyperplanes dual to e_1 and e_2 are transverse, so that e_1 and e_2 span a square in X according to

Lemma 2.5. The image of this square under Φ must be S as $\mathrm{QM}(\Gamma, \mathcal{G})$ does not contain $K_{3,2}$ as an induced subgraph, concluding the proof of our claim.

By combining Lemma 2.1 with Claims 3.14 and 3.15, we conclude that the image of Φ is a gated subgraph. The proof of our theorem is complete.

Remark 3.16 Observe that, if X is a median graph in Theorem 3.6, then the graph product we obtain is a right-angled Coxeter group. Indeed, for every $i \in I$, $\mathscr{S}(J_i)$ has cardinality 2, so either $\mathfrak{S}(J_i)$ has order 2 and $G_i = \mathfrak{S}(J_i)$; or $\mathfrak{S}(J_i)$ is trivial and G_i has order 2. Consequently, G_i is cyclic of order 2 for every $i \in I$, and $\Gamma\mathcal{G}$ is a right-angled Coxeter group. So, we recover that groups acting specially on CAT(0) cube complexes embed into right-angled Coxeter groups.

When applying Theorem 3.6, it may be difficult to understand the groups $\mathfrak{S}(J)$. Our next statement shows that, when the group acts with finitely many orbits of vertices, these groups are essentially clique-stabilizers (which are much easier to understand).

Proposition 3.17 *Let G be a group acting specially on a quasi-median graph X with finitely many orbits of vertices. For every hyperplane J and every clique $C \subset J$,* $\mathrm{stab}(C)$ *embeds as a finite-index subgroup in $\mathfrak{S}(J)$.*

Proof Fix a hyperplane J of X and a clique $C \subset J$. The fact that the image of $\mathrm{stab}(C)$ in $\mathfrak{S}(J)$ is faithful is a direct consequence of Lemma 3.5. Because $\mathfrak{S}(J)$ acts freely on $\mathscr{S}(J)$, it suffices to show that (the image of) $\mathrm{stab}(C)$ acts on $\mathscr{S}(J)$ with finitely many orbits in order to deduce that (the image of) $\mathrm{stab}(C)$ has finite index in $\mathfrak{S}(J)$. In fact, we claim that $\mathrm{stab}(C)$ acts on C with finitely many orbits of vertices, which is sufficient.

Notice that, if two vertices x and y of C are in the same G-orbit, then they are in the same $\mathrm{stab}(C)$-orbit. Indeed, let $g \in G$ be such that $gx = y$. Then the cliques C and gC either are identical or they intersect along a single vertex. However, the latter case cannot happen because the action is special (indeed, otherwise g would send the hyperplane containing C to a hyperplane that is transverse or tangent to it), so $g \in \mathrm{stab}(C)$. As G acts on X with finitely many orbits of vertices, it follows that:

Fact 3.18 C contains only finitely many $\mathrm{stab}(C)$-orbits of vertices. ∎

This last observation concludes the proof of our proposition.

As a consequence of Proposition 3.17, we better understand the vertex groups of the graph product into which we embed our group in Theorem 3.6, under the additional assumption that the action on the quasi-median graph has only finitely many orbits of vertices.

Corollary 3.19 *Let G be a group which acts specially on a quasi-median graph X with finitely many orbits of vertices. Following the notation in Theorem 3.6, for every $i \in I$, $\mathfrak{S}(J_i)$ has finite index in G_i and contains a clique-stabilizer as a finite-index subgroup; in particular, G_i is virtually a clique-stabilizer.*

Proof Recall that, for every $i \in I$, the index of $\mathfrak{S}(J_i)$ in G_i coincides with the number of $\mathfrak{S}(J_i)$-orbits in $\mathcal{S}(J_i)$. As a consequence of Proposition 3.17, our corollary follows from the observation that, for every hyperplane J of X, $\mathfrak{S}(J)$ acts on $\mathscr{S}(J)$ with finitely many orbits. This is a direct consequence of Fact 3.18. ∎

By combining Theorem 3.6 with Corollary 3.19, one immediately gets the following corollary.

Corollary 3.20 *Let G be a group which acts specially on a quasi-median graph X with finitely many orbits of vertices. Then G embeds into a graph product of virtual clique-stabilizers.*

3.6 Gated-cocompact subgroups are virtual retracts

We saw in the previous section that a group acting specially on a quasi-median graph can be embedded into a graph product. In the present section, our goal is to show, under the additional assumption that the group acts with only finitely many orbits of vertices, that the image of this embedding is a virtual retract. Our proof is based on the following concept.

Definition 3.2 Let G be a group acting on a quasi-median graph X. A subgroup $H \leq G$ is *gated-cocompact* if there exists a gated subgraph $Y \subset X$ on which H acts with finitely many orbits of vertices.

Unless stated otherwise, a gated-cocompact subgroup of a graph product $\Gamma\mathcal{G}$ always refers to the action of $\Gamma\mathcal{G}$ on $\mathrm{QM}(\Gamma, \mathcal{G})$. The main result of this section is that such subgroups are virtual retracts.

Theorem 3.21 *Let Γ be a simplicial graph and $\mathcal{G}$ a collection of groups indexed by $V(\Gamma)$. A gated-cocompact subgroup $H \leq \Gamma\mathcal{G}$ is a virtual retract.*

Before turning to the proof of our theorem, we need to introduce a few definitions. So, let X be a quasi-median graph and G a group acting on it.

- The *rotative-stabilizer* of a hyperplane J is $\mathrm{stab}_{\circlearrowleft}(J) := \bigcap\{\mathrm{stab}(C) \mid C \subset J \text{ clique}\}$.
- Given a G-invariant collection of hyperplanes $\mathcal{J}$, the action $G \curvearrowright X$ is *$\mathcal{J}$-rotative* if, for every $J \in \mathcal{J}$, the action $\mathrm{stab}_{\circlearrowleft}(J) \curvearrowright \mathscr{S}(J)$ is transitive and free.
- Given a vertex $x \in X$, a collection of hyperplanes $\mathcal{J}$ is *x-peripheral* if there do not exist $J_1, J_2 \in \mathcal{J}$ such that J_1 separates x from J_2.

For instance, the action of $\Gamma\mathcal{G}$ on $\mathrm{QM}(\Gamma, \mathcal{G})$ is *fully rotative* [35, Proposition 2.21], i.e., it is $\mathcal{J}$-rotative where $\mathcal{J}$ denotes the collection of all the hyperplanes of $\mathrm{QM}(\Gamma, \mathcal{G})$.

Lemma 3.22 *Let G be a group acting on a quasi-median graph X with trivial vertex-stabilizers. Fix a basepoint $x_0 \in X$, and let $\mathcal{J}$ be an x_0-peripheral collection of hyperplanes. Assume that the action of G on X is $\mathcal{J}$-rotative. Then*

$$Y := \bigcap_{J \in \mathcal{J}} \text{sector delimited by } J \text{ containing } x_0$$

is a fundamental domain for the action of $R := \langle \mathrm{stab}_{\circlearrowleft}(J) \mid J \in \mathcal{J} \rangle$ on X.

Proof Let $x \in X$ be an arbitrary vertex. Assume that $x \notin, Y$ and let $y \in Y$ denote its projection onto Y (which exists since, as an intersection of gated subgraphs, Y is gated). The last edge of a geodesic $[x, y]$ must be dual to a hyperplane J in $\mathcal{J}$. Because

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the action is $\mathcal{J}$-rotative, there exists some $g \in \mathrm{stab}_{\circlearrowleft}(J)$ which sends x in the sector delimited by J which contains Y. Notice that g sends $[x, y]$ minus its last edge to a path between gx and y, so

$$d(gx, Y) \leq d(gx, y) \leq d(x, y) - 1.$$

By iterating the argument, we conclude that there exists $r \in R$ such that $rx \in Y$.

Now, fix an arbitrary vertex $x \in Y$. For every $J \in \mathcal{J}$, let X_J denote the union of all the sectors delimited by J which are disjoint from Y. Notice that:

- If $J_1, J_2 \in \mathcal{J}$ are transverse, then g_1 and g_2 commute for all $g_1 \in \mathrm{stab}_{\circlearrowleft}(J_1)$ and $g_2 \in \mathrm{stab}_{\circlearrowleft}(J_2)$ [28, Lemma 8.46] (see also [35, Fact 2.22]).
- If $J_1, J_2 \in \mathcal{J}$ are transverse, then $g \cdot X_{J_2} \subset X_{J_2}$ for every $g \in \mathrm{stab}_{\circlearrowleft}(J_1)$ [28, Lemma 8.47] (see also [35, Proposition 2.21]).
- If $J_1, J_2 \in \mathcal{J}$ are distinct and not transverse, then $g \cdot X_{J_2} \subset X_{J_1}$ for every $g \in \mathrm{stab}_{\circlearrowleft}(J_1)\backslash\{1\}$.
- For every $J \in \mathcal{J}$ and every $g \in \mathrm{stab}_{\circlearrowleft}(J)\backslash\{1\}$, we have $g \cdot x \in X_J$.

Therefore, [28, Proposition 8.44] (see also [31, Proposition 3.26]) applies, and we deduce from [28, Fact 8.45] (see also [31, Fact 3.27]) that $g \cdot x \in \bigcup_{J \in \mathcal{J}} X_J$ for every nontrivial $g \in R$; in particular, $g \cdot x \notin Y$. Thus, we have proved that Y is a fundamental domain for $R \curvearrowright X$. ∎

Proof Let $Y \subset \mathrm{QM}(\Gamma, \mathcal{G})$ be a gated subgraph on which H acts with finitely many orbits of vertices. Let $\mathcal{J}$ denote the collection of the hyperplanes of $\mathrm{QM}(\Gamma, \mathcal{G})$ which are *tangent* to Y (i.e., with no edges in Y but whose carriers intersect Y). We set $R := \langle \mathrm{stab}_{\circlearrowleft}(J) \mid J \in \mathcal{J} \rangle$ and $H^+ := \langle R, H \rangle$. Notice that $\mathcal{J}$ is H-invariant, so R is a normal subgroup of H^+. Moreover, Y coincides with

$$\bigcap_{J \in \mathcal{J}} \text{sector delimited by } J \text{ containing } Y,$$

which is a fundamental domain of R according to Lemma 3.22. Therefore, $H \cap R = \{1\}$. It follows that $H^+ = R \rtimes H$, so that H is a retract in H^+. Moreover, since Y is a fundamental domain of R and because H acts on Y with finitely many orbits of vertices, necessarily H^+ acts on $\mathrm{QM}(\Gamma, \mathcal{G})$ with finitely many orbits of vertices, which means that H^+ is a finite-index subgroup of $\Gamma\mathcal{G}$. Thus, we have proved that H is a virtual retract in $\Gamma\mathcal{G}$. ∎

According to Theorem 3.6, if a group G acts specially on a quasi-median graph X, then there exists an embedding $\varphi : G \hookrightarrow \Gamma\mathcal{G}$ such that X embeds φ-equivariantly into $\mathrm{QM}(\Gamma, \mathcal{G})$ as a gated subgraph. As a consequence, if G acts on X with finitely many vertices, then the image of φ is a gated-cocompact subgroup of $\Gamma\mathcal{G}$, so Theorem 3.21 directly implies that:

Corollary 3.23 *Let G be a group which acts specially on a quasi-median graph X with finitely many orbits of vertices. The image of the embedding $G \hookrightarrow \Gamma\mathcal{G}$ provided by Theorem 3.6 is a virtual retract in $\Gamma\mathcal{G}$.*

It also follows from Theorem 3.21 that gated-cocompact subgroups of our group G are virtual retracts in G itself.

Corollary 3.24 *Let G be a group which acts specially on a quasi-median graph X. Gated-cocompact subgroups of G are virtual retracts in G.*

Proof According to Theorem 3.6, there exist a graph product $\Gamma\mathcal{G}$, an injective morphism $\varphi : G \hookrightarrow \Gamma\mathcal{G}$, and a φ-equivariant embedding $X \hookrightarrow \mathrm{QM}(\Gamma, \mathcal{G})$ whose image is gated. As a consequence, any gated-cocompact subgroup H of G (with respect to its action on X) is a gated-cocompact subgroup of $\Gamma\mathcal{G}$ (with respect to its action on $\mathrm{QM}(\Gamma, \mathcal{G})$). Therefore, H is a virtual retract in $\Gamma\mathcal{G}$ according to Theorem 3.21, which implies that H is a virtual retract in G. ■

As subgraphs in median graphs are gated if and only if they are convex, we recover from Corollary 3.24 that convex-cocompact subgroups in cocompact special groups are virtual retracts [40].

4 Right-angled graphs of groups

4.1 Graphs of groups

We begin this section by fixing the basic definitions and notations related to graphs of groups; essentially, we follow [49]. So far, our graphs were always one-dimensional simplicial complexes, but we need a different definition in order to define graphs of groups. In order to avoid ambiguity, we will refer to the latter as *abstract graphs*.

Definition 4.1 An *abstract graph* is the data of a set of vertices V, a set of arrows E, a fixed-point-free involution $e \mapsto \bar{e}$ on E, and two maps $s, t : E \to V$ satisfying $t(e) = s(\bar{e})$ for every $e \in E$.

Notice that the elements of E are referred to as arrows and not as edges. This terminology will allow us to avoid confusion between arrows of abstract graphs and edges of quasi-median graphs. Below, we define graphs of groups and their associated fundamental groupoids as introduced in [42].

Definition 4.2 A *graph of groups* $\mathfrak{G}$ is the data of an abstract graph $(V, E, \bar{\cdot}, s, t)$, a collection of groups indexed by $V \sqcup E$ such that $G_e = G_{\bar{e}}$ for every $e \in E$, and a monomorphism $\iota_e : G_e \hookrightarrow G_{s(e)}$ for every $e \in E$. The *fundamental groupoid* $\mathfrak{F} = \mathfrak{F}(\mathfrak{G})$ of $\mathfrak{G}$ is the groupoid which has vertex set V, which is generated by the arrows of E together with $\bigsqcup_{v \in V} G_v$ (an element of G_v being thought of as a loop based at v), and which satisfies the relations:

- for all $v \in V$ and $g, h, k \in G_v$, $gh = k$ if the equality holds in G_v;
- for all $e \in E$ and $g \in G_e$, $\iota_e(g) \cdot e = e \cdot \iota_{\bar{e}}(g)$.

Notice in particular that, for every $e \in E$, $\bar{e}$ is an inverse of e in $\mathfrak{F}$. Fixing some vertex $v \in V$, the *fundamental group* of $\mathfrak{G}$ (based at v) is the vertex group $\mathfrak{F}_v$ of $\mathfrak{F}$, i.e., the loops of $\mathfrak{F}$ based at v.

We record the following definition for future use.

Definition 4.3 The *terminus* of an element g of $\mathfrak{F}$ is the vertex of V which corresponds to the terminal point of g when thought of as an arrow of $\mathfrak{F}$.

The following normal form, proved in [42], is central in the quasi-median geometry of right-angled graphs of groups.

Proposition 4.1 *Let $\mathfrak{G}$ be a graph of groups. For every $e \in E$, fix a set T_e of left-coset representatives of $\iota_e(G_e)$ in $G_{s(e)}$ containing $1_{s(e)}$. Any element of $\mathfrak{F}$ can be written uniquely as a word $g_1 \cdot e_1 \cdots g_n \cdot e_n \cdot g_{n+1}$, where:*

- *$(e_1, \ldots, e_n)$ is an oriented path in the underlying abstract graph;*
- *$g_i \in T_{e_i}$ for every $1 \leq i \leq n$, and g_{n+1} is an arbitrary element of $G_{t(e_n)}$;*
- *if $e_{i+1} = \bar{e}_i$ for some $1 \leq i \leq n-1$ then $g_{i+1} \neq 1$.*

Such a word will be referred to as a *normal word*.

Roughly speaking, we will be interested in graphs of groups gluing graph products. In order to get something interesting for our purpose, we need to control the gluings.

Definition 4.4 Given two graph products $\Gamma\mathcal{G}$ and $\Lambda\mathcal{H}$, a morphism $\Phi : \Gamma\mathcal{G} \to \Lambda\mathcal{H}$ is a *graphical embedding* is there exist an embedding $f : \Gamma \to \Lambda$ and isomorphisms $\varphi_v : G_v \to H_{f(v)}$, $v \in V(\Gamma)$, such that $f(\Gamma)$ is an induced subgraph of Λ and $\Phi(g) = \varphi_v(g)$ for every $v \in V(\Gamma)$ and every $g \in G_v$.

Typically, we glue graph products along "subgraph products" in a canonical way. We refer to Section 4.4 for examples.

Definition 4.5 A *right-angled graph of groups* is a graph of groups such that each (vertex and edge)group has a fixed decomposition as a graph product and such that each monomorphism of an edge group into a vertex group is a graphical embedding (with respect to the structures of graph products we fixed).

In the following, a *factor* will refer to a vertex group of one of these graph products that label the vertices in our graph of groups. In order to avoid possible confusion, in the sequel vertex groups will only refer to the groups labeling the vertices of the underlying abstract graph of our graph of groups.

Given a right-angled graph of groups $\mathfrak{G}$ and an arrow $e \in E$, there exists a natural set T_e of left-coset representatives of $\iota_e(G_e)$ in $G_{s(e)}$: the set of elements of $G_{s(e)}$ represented by graphically reduced words whose tails (see Definition 2.4) do not contain any element of the vertex groups in $\iota_e(G_e)$. From now on, we fix this choice, and any normal word will refer to this convention.

4.2 Quasi-median geometry

Fix a right-angled graph of groups $\mathfrak{G}$, and a vertex $\omega \in V$ of its underlying abstract graph. Let $\mathfrak{S} \subset \mathfrak{F}$ denote the union of the arrows of E together with the factors (minus the identity) of the graph products G_v, $v \in V$. By definition, $\mathfrak{S}$ is a generating set of the fundamental groupoid $\mathfrak{F}$ of $\mathfrak{G}$.

Definition 4.6 The graph $\mathfrak{X} = \mathfrak{X}(\mathfrak{G}, \omega)$ is the connected component of the Cayley graph $\mathfrak{X}(\mathfrak{G})$ of the groupoid $\mathfrak{F}$, constructed from the generating set $\mathfrak{S}$, which contains the neutral element 1_ω based at ω. In other words, $\mathfrak{X}$ is the graph whose vertices are the arrows of $\mathfrak{F}$ starting from ω and whose edges link two elements $g, h \in \mathfrak{F}$ if $g = h \cdot s$ for some $s \in \mathfrak{S}$.

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It is worth noticing that an edge of $\mathfrak{X}$ is naturally labeled either by an arrow of E or by a factor.

Proposition 4.2 [28, Proposition 11.8] *The graph $\mathfrak{X}$ is quasi-median.*

Notice that the fundamental group $\mathfrak{F}_\omega$ of $\mathfrak{G}$ based at ω naturally acts by isometries on $\mathfrak{X}$ by left multiplication. Moreover:

Lemma 4.3 *Two vertices of $\mathfrak{X}$ belong to the same $\mathfrak{F}_\omega$-orbit if and only if they have the same terminus.*

Proof If $g \in \mathfrak{F}_\omega$ and $h \in \mathfrak{X}$, it is clear that h and gh have the same terminus. Conversely, if $h, k \in \mathfrak{X}$ have the same terminus, then the product kh^{-1} is well defined, and it belongs to $\mathfrak{F}_\omega$. Since $kh^{-1} \cdot h = k$, it follows that h and k belong to the same $\mathfrak{F}_\omega$-orbit. ■

We record the following definition for future use.

Definition 4.7 A *leaf* of $\mathfrak{X}$ is the subgraph induced by the set of vertices gG_v, where G_v is a vertex group of $\mathfrak{G}$ and where $g \in \mathfrak{F}$ is some arrow starting from ω and ending at $v \in V$.

Notice that, by construction, a leaf is isometric to the Cayley graph of a graph product as given by Theorem 2.7. (See [28, Lemma 11.11] for more details.)

4.2.1 Path morphisms

Let $\mathfrak{G}$ be a right-angled graph of groups, and let $(V, E, \bar{\cdot}, s, t)$ denote its underlying abstract graph. Given an arrow $e \in E$, we denote by $\varphi_e : \iota_e(G_e) \to \iota_{\bar{e}}(G_e)$ the isomorphism $\iota_{\bar{e}} \circ \iota_e^{-1}$. A priori, φ_e is not defined on $G_{s(e)}$ entirely, but for every subset $S \subset G_{s(e)}$, we can define $\varphi_e(S)$ as $\varphi_e\left(S \cap \iota_e(G_e)\right)$. By extension, if an oriented path γ decomposes as a concatenation of arrows $e_1 \cdots e_n$, we denote by φ_γ the composition $\varphi_{e_n} \circ \cdots \circ \varphi_{e_1}$.

Notice that, if G is a factor contained in a vertex group G_u of $\mathfrak{G}$ and if γ is a path in the graph of $\mathfrak{G}$ starting from u, then $\varphi_\gamma(G)$ is either trivial (i.e., reduced to $\{1\}$) or a factor (different from G in general). Moreover, in the latter case, the equality

$$a \cdot e_1 \cdots e_n = e_1 \cdots e_n \cdot \varphi_\gamma(a)$$

holds for every $a \in G$, where $e_1 \cdots e_n$ is a decomposition of γ as a concatenation of arrows.

Given a right-angled graph of groups, a subgroup of automorphisms is naturally associated to each factor.

Definition 4.8 For every factor G contained in a vertex-group G_u of $\mathfrak{G}$,

$$\Phi(G) := \{\varphi_c \mid c \text{ closed path based at } u \text{ such that } \varphi_c(G) = G\} \leq \mathrm{Aut}(G).$$

These groups of automorphisms are crucial in the study of right-angled graphs of groups. Indeed, as noticed by [28, Example 11.36], cyclic extensions of an arbitrary group are fundamental groups of right-angled graphs of groups, but we cannot expect to find a geometry common to all the cyclic extensions, so we need additional

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restrictions on the graphs of groups we look at. As suggested by [28, Proposition 11.26] and Proposition 4.9, typically we require the $\Phi(G)$ to be trivial, or at least finite.

4.2.2 Cliques and prisms

Let $\mathfrak{G}$ be a right-angled graph of groups. The description of the cliques in our quasi-median graph $\mathfrak{X}$ is given by the following lemma.

Lemma 4.4 [28, Lemma 11.15] *A clique of $\mathfrak{X}$ is either an edge labeled by an arrow or a complete subgraph gG where G is a factor in some vertex-group G_u and $g \in \mathfrak{F}$ an element with u as its terminus.*

About the prisms of $\mathfrak{X}$, notice that we already understand the prisms which lie in leaves, as a consequence of Lemma 2.9. The other prisms are described by our next lemma.

Lemma 4.5 [28, Lemma 11.18] *For every prism Q of $\mathfrak{X}$ which is not included in a leaf, there exist some $e \in E$ and some prism P which is included into a leaf, such that Q is induced by the set of vertices $\{g, \ ge \mid g \in P\}$.*

4.2.3 Hyperplanes

Let $\mathfrak{G}$ be a right-angled graph of groups. The rest of the section is dedicated to the description of the hyperplanes of $\mathfrak{X}$. It is worth noticing that, as a consequence of [28, Fact 11.14 and Lemma 11.16], a hyperplane has all its edges labeled either an arrow of $\mathfrak{G}$ or by factors (not a single one in general). In the former case, the hyperplane is *of arrow-type*; and in the latter case, the hyperplane is *of factor-type*. Notice that, as a consequence of [28, Fact 11.14], two hyperplanes of arrow-type cannot be transverse.

Roughly speaking, the carrier of the hyperplane dual to a clique labeled by some factor G is induced by the vertices corresponding to elements of $\mathfrak{F}$ which "commute" with all the elements of G. Because commutation is not well-defined in groupoids, we need to define carefully this idea, which is done by the following definition.

Definition 4.9 Let G be a factor contained in a vertex group G_v of $\mathfrak{G}$. An element $h \in \mathfrak{F}$ with initial vertex v belongs to the *link* of G, denoted by $\mathrm{link}(G)$, if it can be written as a normal word $h_1 e_1 \cdots h_n e_n h_{n+1}$ such that:

- $\varphi_{e_1 \cdots e_n}(G)$ is nontrivial;
- h_1 belongs to the subgroup of G_v generated by the factors adjacent to G;
- for every $1 \leq i \leq n$, h_{i+1} belongs to the subgroup of $G_{t(e_i)}$ generated by the factors adjacent to $\varphi_{e_1 \cdots e_i}(G)$.

We are now ready to describe the hyperplanes of factor-type of $\mathfrak{X}$ and their stabilizers.

Proposition 4.6 [28, Proposition 11.21] *Let $C = gG$ be a clique where G is a factor and where $g \in \mathfrak{X}$. Let J denote the hyperplane dual to C. An edge $e \subset \mathfrak{X}$ is dual to J if and only if $e = g(h_1\ell, h_2\ell)$ for some $h_1, h_2 \in G$ distinct and $\ell \in \mathrm{link}(G)$. As a consequence, $N(J) = gG \cdot \mathrm{link}(G)$ and the fibers of J are the $gh \cdot \mathrm{link}(G)$ where $h \in G$.*

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Corollary 4.7 [28, Corollary 11.22] *Let $C = gG$ be a clique where G is a factor and where $g \in \mathfrak{X}$. Let J denote the hyperplane dual to C. Then*

$$\text{stab}(J) = g\{kh \mid k \in G,\ h \in \text{link}(G), \varphi_h(G) = G\}g^{-1}.$$

In this statement, φ_h is defined as follows. Writing h as a normal word $h_1 e_1 \cdots h_n e_n h_{n+1}$ as in Definition 4.9 (this representation being unique according to Proposition 4.1), we refer to $e_1 \cdots e_n$ as the *path associated to h*. Then, $\varphi_h := \varphi_{e_1 \cdots e_n}$. Notice that, by definition of link(G), φ_h always sends G to another factor, or, in other words, $\varphi_h(G)$ cannot be trivial.

About the hyperplanes of arrow type of $\mathfrak{X}$, a complete description is not required here. The following statement will be sufficient.

Lemma 4.8 [28, Lemma 11.24] *Let J be a hyperplane of arrow type in $\mathfrak{X}$. Then J has exactly two fibers, and they are both stabilized by* stab(J).

4.3 When is the action special?

In this section, we want to understand when the action of the fundamental group of a right-angled graph of groups on the quasi-median graph constructed in Section 4.2 is special. Our main result in this direction is the following statement.

Proposition 4.9 *Let $\mathfrak{G}$ be a right-angled graph of groups. The action of the fundamental group $\mathfrak{F}_\omega$ of $\mathfrak{G}$ on $\mathfrak{X}(\mathfrak{G}, \omega)$ is special if and only if the following conditions are satisfied:*

(i) *for every factor G and every cycle c in the abstract graph of $\mathfrak{G}$ based at the vertex group containing G, $\varphi_c(G) = \{1\}$ or G;*
(ii) *there do not exist two vertices u, v in the graph of $\mathfrak{G}$, two paths α, β from u to v, two commuting factors $A_1, A_2 \subset G_u$, and two noncommuting factors $B_1, B_2 \subset G_v$ such that $\varphi_\alpha(A_1) = B_1$ and $\varphi_\beta(A_2) = B_2$;*
(iii) *every arrow in the abstract graph of $\mathfrak{G}$ has distinct endpoints;*
(iv) *for every factor G, the equality $\Phi(G) = \{\text{Id}\}$ holds.*

Observe that, if (iii) does not hold, then we can modify our graph of groups by subdividing the loops and by indexing the new vertices and edges by the edge groups of the corresponding loops. This operation does not modify the fundamental group of the graph of groups, it does not interfere with the conditions (i), (ii), and (iv), and the new graph of groups satisfies (iii).

We begin by proving the following preliminary lemma.

Lemma 4.10 *Let $\mathfrak{G}$ be a right-angled graph of groups. Let G be a factor of $\mathfrak{G}$, and let C denote a clique labeled by G, say $C = gG$. Furthermore, let J denote the hyperplane containing C. The action $\mathfrak{S}(J) \curvearrowright \mathscr{S}(J)$ is transitive, and it is free if and only if $\Phi(G) = \{\text{Id}\}$. Moreover, if this is the case, then the image of* stab$(C) = gGg^{-1}$ *in $\mathfrak{S}(J)$ is faithful and surjective.*

Proof In order to shorten the notation, we assume that g is trivial. As a consequence of Proposition 4.6, it is clear that stab$(C) = G$ acts faithfully, freely, and transitively on $\mathscr{S}(J)$. Therefore, the action $\mathfrak{S}(J) \curvearrowright \mathscr{S}(J)$ is transitive, and it is free if and only if the image of stab$(C) = G$ in $\mathfrak{S}(J)$ is surjective. However, we know from Proposition 4.6

and Corollary 4.7 that $\mathfrak{S}(J)$ is the set of permutations of $\mathscr{S}(J) = \{h \cdot \text{link}(G) \mid h \in G\}$ induced by

$$\text{stab}(J) = \{km \mid k \in G, m \in \text{link}(G), \varphi_m(G) = G\}.$$

Observe that, for all $km \in \text{stab}(J)$ and $h \cdot \text{link}(G) \in \mathscr{S}(J)$, we have

$$km \cdot (h \cdot \text{link}(G)) = k\varphi_m^{-1}(h)m \cdot \text{link}(G) = k\varphi_m^{-1}(h) \cdot \text{link}(G),$$

where $\varphi_m \in \Phi(G)$. If $\Phi(G)$ is trivial, then it is clear that the action of km on $\mathscr{S}(J)$ coincides with the action of $k \in G = \text{stab}(C)$. Conversely, if the action of km on $\mathscr{S}(J)$ coincides with the action of some $f \in G$, then $k\varphi_m^{-1}(h) \cdot \text{link}(G) = fh \cdot \text{link}(G)$ for every $h \in G$. As a consequence of Proposition 4.1 and of the definition of $\text{link}(G)$, this amounts to saying that $k\varphi_m^{-1}(h) = fh$ for every $h \in G$. Hence, $k = f$ and $\varphi_m = \text{Id}$. Because the conclusion holds for every $km \in \text{stab}(J)$, necessarily $\Phi(G) = \{\text{Id}\}$. ∎

The next observation will be fundamental in our proof.

Lemma 4.11 *Let $\mathfrak{G}$ be a right-angled graph of groups, and $e, f \subset \mathfrak{X}$ two edges. Let A, B denote the two factors labeling e, f, respectively, and let u, v denote the vertices of the graph of $\mathfrak{G}$ such that A and B are factors of G_u and G_v, respectively. If e and f are dual to the same hyperplane, then there exists a path γ in the graph of $\mathfrak{G}$ from u to v such that $\varphi_\gamma(A) = B$.*

Proof Write $e = (p, pa)$ and $f = (q, qb)$ where $a \in A$ and $b \in B$. As a consequence of Proposition 4.6, $f = p(a_1\ell, a_2\ell)$ for some distinct $a_1, a_2 \in A$ and some $\ell \in \text{link}(A)$. We have

$$b = q^{-1} \cdot qb = \ell^{-1}a_1^{-1}p^{-1} \cdot pa_2\ell = \varphi_\ell\left(a_1^{-1}a_2\right) = \varphi_\gamma\left(a_1^{-1}a_2\right),$$

where γ is the path in the graph of $\mathfrak{G}$ associated to ℓ. Because φ_γ sends a factor to $\{1\}$ or to another factor, we conclude that γ is a path from u to v and that $\varphi_\gamma(A) = B$, as desired. ∎

Now, we are ready to determine when the action of the fundamental group of a right-angled graph of groups on its quasi-median graph is hyperplane-special.

Lemma 4.12 *Let $\mathfrak{G}$ be a right-angled graph of groups. The action of the fundamental group $\mathfrak{F}_\omega$ of $\mathfrak{G}$ on $\mathfrak{X}(\mathfrak{G}, \omega)$ is hyperplane-special if and only if the following conditions are satisfied:*

- (i) *For every factor G and every cycle c in the abstract graph of $\mathfrak{G}$ based at the vertex group containing G, $\varphi_c(G) = \{1\}$ or G.*
- (ii) *There do not exist two vertices u, v in the graph of $\mathfrak{G}$, two paths α, β from u to v, two commuting factors $A_1, A_2 \subset G_u$, and two noncommuting factors $B_1, B_2 \subset G_v$ such that $\varphi_\alpha(A_1) = B_1$ and $\varphi_\beta(A_2) = B_2$.*
- (iii) *Every arrow in the abstract graph of $\mathfrak{G}$ has distinct endpoints.*

Proof First, assume that the action of the fundamental group of $\mathfrak{G}$ on $\mathfrak{X}$ is not hyperplane-special. There are several cases to consider.

Case 1: There exist a hyperplane J of $\mathfrak{X}$ and an element $g \in \mathfrak{F}_\omega$ such that gJ and J are transverse or tangent.

It is clear that, if there exist two distinct intersecting edges of $\mathfrak{X}$ which are labeled by the same arrow of $\mathfrak{G}$, then the endpoints of this arrow coincide, so that (iii) does not hold. So, from now on, we assume that J is of factor type. Fix two distinct edges $e_1 \subset J$ and $e_2 \subset gJ$ which share their initial point, and let A, B denote the distinct factors which label them. Notice that, because e_1 and e_2 intersect, our factors A and B belong to the same vertex group of $\mathfrak{G}$, say G_u. Because ge_1 is labeled by the factor A and is dual to the same hyperplane as e_2, namely gJ, it follows from Lemma 4.11 that there exists in the abstract graph of $\mathfrak{G}$ a closed path c based at u such that $\varphi_c(A) = B$. In particular, $\varphi_c(A)$ is neither trivial nor A, contradicting (i).

Case 2: There exist two tangent hyperplanes J_1, J_2 of $\mathfrak{X}$ and an element $g \in \mathfrak{F}_\omega$ such that J_1 and gJ_2 are transverse.

We distinguish three cases, depending on whether J_1 and J_2 are of arrow type or of factor type.

Case 2.1: J_2 is of arrow type.

Fix a geodesic $\gamma \subset N(J_1)$ whose initial point belongs to $N(J_1) \cap N(J_2)$ and whose last edge is dual to gJ_2. Crossing J_2 corresponds to right-multiplying by the arrow e (or its inverse) which labels J_2. However, such a multiplication is allowed only if the element of the groupoid under consideration has as terminus an endpoint of e (the initial or terminal point of e depending on whether we are multiplying by e or e^{-1}). Consequently, the initial point of γ and one of the last two points of γ have the same terminus. According to Lemma 4.3, these two points belong to the same $\mathfrak{F}_\omega$-orbit. So, there exists some $h \in \mathfrak{F}_\omega$ such that the initial point of γ belongs to $N(hgJ_2) \cap N(hJ_1)$. We already know from Case 1 that, if J_1 and hJ_1 are tangent or transverse, then (i) cannot hold, so (since their carriers intersect) we suppose that they coincide. Similarly, we suppose that $J_2 = hgJ_2$. As gJ_2 and J_1 are transverse, it follows that hgJ_2 and hJ_1 must be transverse as well; but J_2 and J_1 are tangent, a contradiction.

Case 2.2: J_1 and J_2 are both of factor type.

Fix two edges $e_1 \subset J_1$ and $e_2 \subset J_2$ which share their initial point, and let A_1 and A_2 denote the factors which label them, respectively. Notice that, because e_1 and e_2 intersect, A_1 and A_2 belong to the same vertex group of $\mathfrak{G}$, say G_u. Moreover, because e_1 and e_2 do not span a square, A_1 and A_2 do not commute in the graph product G_u. Next, fix two edges $f_1 \subset J_1$ and $f_2 \subset gJ_2$ which share their initial endpoint and which span a square, and let B_1 and B_2 denote the factors which label them, respectively. Notice that, because f_1 and f_2 intersect, B_1 and B_2 belong to the same vertex group of $\mathfrak{G}$, say G_v. Moreover, because f_1 and f_2 span a square, B_1 and B_2 commute in the graph product G_v. As e_1 and f_1 are dual to the same hyperplane, namely J_1, it follows from Lemma 4.11 that there exists a path α in the graph of $\mathfrak{G}$ from u to v such that $\varphi_\alpha(A_1) = B_1$. Similarly, because f_2 and ge_2 are dual to gJ_2, there exists a path β from u to v such that $\varphi_\beta(A_2) = B_2$. We conclude that (ii) does not hold.

Case 2.3: J_1 is of arrow type and J_2 of factor type.

Observe that gJ_2 and gJ_1 are tangent, that $g^{-1} \cdot gJ_1 = J_1$ is transverse to gJ_2, and that gJ_2 is of factor type. Therefore, we conclude thanks to Case 2.1.

Thus, we have proved that, if the conditions (i)–(iii) of our proposition hold, then the action of $\mathfrak{F}_\omega$ on $\mathfrak{X}$ is hyperplane-special. Conversely, assume that one of the conditions (i)–(iii) does not hold.

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If (i) does not hold, then there exist a loop c in the graph of $\mathfrak{G}$ based at some vertex u and a factor G in the graph product G_u such that $\varphi_c(G)$ is a factor of G_u distinct from G. Fix an arbitrary vertex $h \in \mathfrak{X}$ whose terminus is u (for instance, a concatenation of arrows from ω to u). Furthermore, fix a nontrivial element $g \in G$ and write c as a concatenation of arrows $e_1 \cdots e_n$. Notice that, for every $0 \leq i \leq n$, we have

$$hge_1 \cdots e_i = he_1 \cdots e_i \varphi_{e_1 \cdots e_i}(g) \text{ where } \varphi_{e_1 \cdots e_i}(g) \neq 1,$$

so $he_1 \cdots e_i$ and $hge_1 \cdots e_i$ are adjacent vertices of $\mathfrak{X}$. As a consequence, for every $0 \leq i \leq n-1$, the four vertices $he_1 \cdots e_i$, $hge_1 \cdots e_i$, $he_1 \cdots e_{i+1}$, and $hge_1 \cdots e_{i+1}$ span a square in $\mathfrak{X}$, so the two edges (h, hg) and $(he_1 \cdots e_n, hge_1 \cdots e_n)$ are dual to the same hyperplane, say J. However, $he_1 \cdots e_n h^{-1} \in \mathfrak{F}_\omega$ sends the edge (h, hg) to the edge $(he_1 \cdots e_n, he_1 \cdots e_n g)$, and the two edges $(he_1 \cdots e_n, hge_1 \cdots e_n)$ and $(he_1 \cdots e_n, he_1 \cdots e_n g)$ are distinct because

$$hge_1 \cdots e_n = he_1 \cdots e_n \varphi_c(g) \text{ where } \varphi_c(g) \notin G.$$

Therefore, the hyperplanes $he_1 \cdots e_n h^{-1} J$ and J are either tangent or transverse (depending on whether G and $\varphi_c(G)$ commute in G_u). So, the action of $\mathfrak{F}_\omega$ on $\mathfrak{X}$ is not hyperplane-special.

If (ii) does not hold, then there exist two vertices u, v in the graph of $\mathfrak{G}$, a path α from u to v, a path β from v to u, two commuting factors $A_1, A_2 \subset G_u$, and two noncommuting factors $B_1, B_2 \subset G_v$ such that $\varphi_\alpha(A_1) = B_1$ and $\varphi_\beta(B_2) = A_2$. Fix an arbitrary vertex h of $\mathfrak{X}$ whose terminus is v (for instance, a concatenation of arrows from ω to v) and nontrivial elements $p \in B_1$, $b \in B_2$, and $a \in A_1$. Moreover, write α as the concatenation of arrows $a_1 \cdots a_s$ and β as $b_1 \cdots b_r$. Notice that, for every $0 \leq i \leq r$, the vertices $hb_1 \cdots b_i$ and $hbb_1 \cdots b_i$ are adjacent as

$$hbb_1 \cdots b_i = hb_1 \cdots b_i \varphi_{b_1 \cdots b_i}(b) \text{ where } \varphi_{b_1 \cdots b_i}(b) \neq 1.$$

Consequently, for every $0 \leq i \leq r-1$, the vertices $hb_1 \cdots b_i$, $hbb_1 \cdots b_i$, $hb_1 \cdots b_{i+1}$, and $hbb_1 \cdots b_{i+1}$ span a square. See Figure 3. Similarly, for every $0 \leq i \leq s$, the vertices $h\beta a_1 \cdots a_i$ and $h\beta a a_1 \cdots a_i$ are adjacent because

$$h\beta a a_1 \cdots a_i = h\beta a_1 \cdots a_i \varphi_{a_1 \cdots a_i}(a) \text{ where } \varphi_{a_1 \cdots a_i}(a) \neq 1;$$

so, for every $0 \leq i \leq s-1$, the vertices $h\beta a_1 \cdots a_i$, $h\beta a_1 \cdots a_{i+1}$, $h\beta a a_1 \cdots a_i$, and $h\beta a a_1 \cdots a_{i+1}$ span a square. Notice that the edges (h, hb) and (h, hp) do not span a square because B_2 and B_1 do not commute, so the hyperplane J_1 dual to (h, hb) is tangent to the hyperplane J_2 dual to (h, hp). Next, because A_1 and A_2 commute, we have

$$hb\beta a = h\beta \varphi_\beta(b) a = h\beta a \varphi_\beta(b) \text{ where } \varphi_\beta(b) \in A_2 \backslash \{1\},$$

so the vertices $h\beta$, $hb\beta$, $h\beta a$, and $hb\beta a$ span a square. As a consequence, the hyperplane J_3 dual to the edge $(h\beta, h\beta a)$ is transverse to J_1. Finally, observe that $\beta\alpha$ is a loop based at v in the graph of $\mathfrak{G}$, so $g := h\beta\alpha h^{-1}$ represents an element of $\mathfrak{F}_\omega$. Moreover, $g(h, hp) = (h\beta\alpha, h\beta\alpha p)$ belongs to the same clique as the edge $(h\beta\alpha, h\beta a\alpha)$ because

$$h\beta a\alpha = h\beta\alpha\varphi_\alpha(a) \text{ and } p, \varphi_\alpha(a) \in B_1,$$

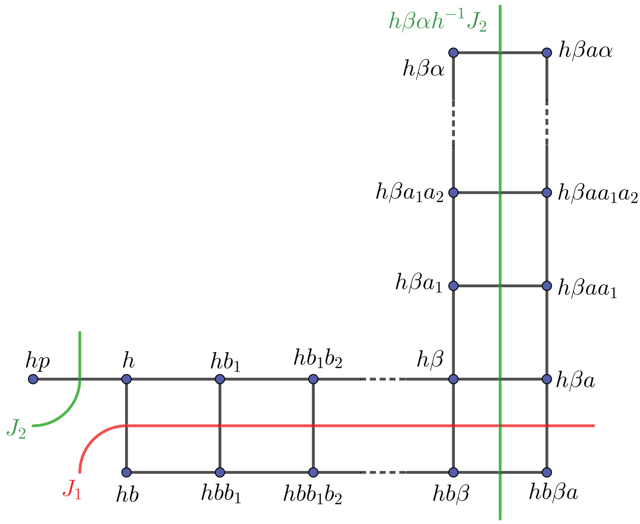

Figure 3: Configuration of vertices when (ii) does not hold.

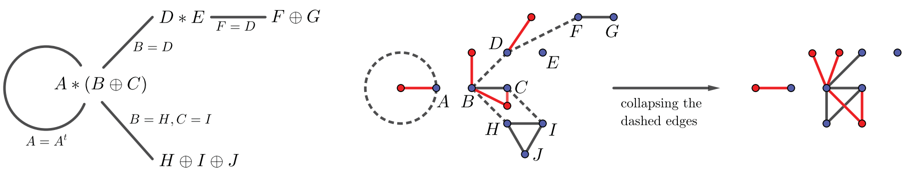

Figure 4: On the left, a graph of groups; and, on the right, the graph Ψ that Proposition 4.13 associates to it (with factors in blue and arrows in red).

and hence $J_3 = gJ_2$. Thus, we have proved that J_1 and J_2 are tangent, but J_1 and gJ_2 are transverse, showing that the action of $\mathfrak{F}_\omega$ on $\mathfrak{X}$ is not hyperplane-special. ■

Finally, if (iii) does not hold, then there exists an arrow e which is a loop based at some vertex u of the graph of $\mathfrak{G}$. Fix an arbitrary vertex h of $\mathfrak{X}$ whose terminus is u (for instance, a concatenation of arrows from ω to u). Then $h^{-1}eh$ defines an element of $\mathfrak{F}_\omega$ which acts on the bi-infinite line $\{he^n \mid n \in \mathbb{Z}\} \subset \mathfrak{X}$ as a translation of length 1. Consequently, if J is any hyperplane crossing this line, then J and $h^{-1}ehJ$ are tangent, proving that the action of $\mathfrak{F}_\omega$ on $\mathfrak{X}$ is not hyperplane-special.

Proof Our proposition is an immediate consequence of Lemmas 4.8, 4.10, and 4.12. ■

As a consequence of Proposition 4.9, one obtains a sufficient condition which implies that the fundamental group of a right-angled graph of groups embeds into a graph product. Our next proposition describes such a graph product; we refer to Figure 4 for an illustration of the graph constructed in its statement.

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Proposition 4.13 *Let $\mathfrak{G}$ be a right-angled graph of groups satisfying the assumptions of Proposition 4.9. Let Γ denote the underlying abstract graph of $\mathfrak{G}$, and, for every vertex $u \in V(\Gamma)$, let Γ_u denote the graph corresponding to our decomposition of the vertex group G_u as a graph product. On the disjoint union of the Γ_u, we define an equivalence relation $\sim$ by declaring that two vertices $a \in \Gamma_u$ and $b \in \Gamma_v$ are $\sim$-equivalent if u and v are linked by an arrow e in Γ and if φ_e sends the factor corresponding to a to the factor corresponding to b. Let Ψ denote the graph obtained from $\Psi_0 := \left(\bigcup_{u \in V(\Gamma)} \Gamma_u \right) / \sim$ by adding a vertex for each element in $\{\{e, \bar{e}\} \mid e \in E(\Gamma)\}$ and by linking $\{e, \bar{e}\}$ to each vertex of $\Gamma_{s(e)}, \Gamma_{s(\bar{e})}$ corresponding to a factor in the image of $\iota_e, \iota_{\bar{e}}$. Finally, let $\mathcal{G}$ denote the collection of groups indexed by $V(\Psi)$ such that a group indexed by (the image in Ψ_0 of) a vertex of Γ_u is the corresponding factor and such that the groups indexed by arrows are cyclic of order 2. Then $\mathfrak{F}_\omega$ embeds into $\Psi\mathcal{G}$.*

Proof As a consequence of Lemma 4.4, $\mathfrak{F}_\omega$-orbits of cliques in $\mathfrak{X}$ are bijectively indexed by the vertices of $\{\{e, \bar{e}\} \mid e \in E(\Gamma)\} \cup \bigcup_{u \in V(\Gamma)} \Gamma_u$. Notice that two hyperplanes labeled by distinct arrows lie in distinct $\mathfrak{F}_\omega$-orbits of hyperplanes and that:

Claim 4.14 Let A and B be two factors, respectively, in the vertex groups G_u and G_v. Two hyperplanes J and H dual to cliques C and D, respectively, labeled by A and B belong to the same $\mathfrak{F}_\omega$-orbit if and only if there exists a path γ from u to v such that $\varphi_\gamma(A) = B$. ■

If J and H belong to the same $\mathfrak{F}_\omega$-orbit, then there exists some $g \in \mathfrak{F}_\omega$ such that gC and D are dual to the same hyperplane, namely H. The desired conclusion follows from Lemma 4.11.

Conversely, assume that there exists a path γ from u to v such that $\varphi_\gamma(A) = B$. Write $C = gA$ and $D = hB$ for some $g, h \in \mathfrak{X}$, and write γ as a concatenation of arrows $e_1 \cdots e_k$. In addition, fix a nontrivial element $a \in A$. Notice that, for every $0 \leq i \leq k$, the vertices $ge_1 \cdots e_i$ and $gae_1 \cdots e_i$ are adjacent because

$$gae_1 \cdots e_i = ge_1 \cdots e_i \varphi_{e_1 \cdots e_i}(a) \text{ where } \varphi_{e_1 \cdots e_i}(a) \neq 1.$$

Consequently, for every $0 \leq i \leq k-1$, the vertices $ge_1 \cdots e_i$, $gae_1 \cdots e_i$, $ge_1 \cdots e_{i+1}$, and $gae_1 \cdots e_{i+1}$ span a square. It follows that the edges $(g, ga) \subset C$ and $(ge_1 \cdots e_k, gae_1 \cdots e_k)$ are dual to the same hyperplane, namely J. By noticing that

$$gae_1 \dots e_k = ge_1 \cdots e_k \varphi_{e_1 \cdots e_k}(a) \text{ where } \varphi_{e_1 \cdots e_k}(a) = \varphi_\gamma(a) \in B,$$

we deduce that the edge $(ge_1 \cdots e_k, gae_1 \cdots e_k)$ is a translate of an edge of the clique D. As a consequence, J and H belong to the same $\mathfrak{F}_\omega$-orbit, concluding the proof of our claim.

So far, we have proved that the $\mathfrak{F}_\omega$-orbits of hyperplanes in $\mathfrak{X}$ are bijectively indexed by the vertices of Ψ. Next, we need to verify that two $\mathfrak{F}_\omega$-orbits of hyperplanes contain transverse representatives if and only if they correspond to two adjacent vertices in Ψ. This observation is an immediate consequence of the description of prisms in $\mathfrak{X}$ given by Lemma 4.5 (the prisms in leaves being described by Lemma 2.9) and of the description of $\mathfrak{F}_\omega$-orbits of hyperplanes given by Claim 4.14.

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Finally, notice that, if J is a hyperplane containing a clique labeled by a factor G, then $\mathfrak{S}(J)$ is isomorphic to G and it acts transitively on $\mathscr{S}(J)$ according to Lemma 4.10. Moreover, if J is a hyperplane labeled by an arrow, then, according to Lemma 4.8, $\mathfrak{S}(J)$ is trivial and $\mathscr{S}(J)$ has cardinality 2. Therefore, the embedding described by our proposition follows from Theorem 3.6.

4.4 Examples

In practice, Proposition 4.9 most of the time does not apply, but its assumptions are just too strong. However, it turns out that the conditions (i)–(iii) are often satisfied *up to a finite cover*, so that the condition (iv) seems to be the central condition of our criterion.

Let $\mathfrak{G}$ be an arbitrary graph of groups, and let Γ denote its underlying graph. If $\pi : \Gamma' \to \Gamma$ is a cover, then we naturally define a graph of groups $\mathfrak{G}'$ which has Γ' as its underlying graph by defining, for every vertex $u \in V(\Gamma')$ and every edge $e \in E(\Gamma')$, the vertex group G_u as $G_{\pi(u)}$, the edge-group G_e as $G_{\pi(e)}$, and the monomorphism $\iota_e : G_e \hookrightarrow G_{s(e)}$ as $\iota_{\pi(e)} : G_{\pi(e)} \hookrightarrow G_{s(\pi(e))}$. One obtains a covering of graphs of groups $\mathfrak{G}' \to \mathfrak{G}$ as defined in [8], so that the fundamental group of $\mathfrak{G}'$ embeds into the fundamental group of $\mathfrak{G}$; moreover, if $\Gamma' \to \Gamma$ is a finite cover, then the image of this embedding has finite index. (More topologically, one can say that the (finite sheeted) cover $\Gamma' \to \Gamma$ induces a (finite sheeted) cover from the graph of spaces defining $\mathfrak{G}'$ to the graph of spaces defining $\mathfrak{G}$; see [48] for more details on graphs of spaces and their connection with graphs of groups.)

Although taking a well-chosen finite cover of graphs of groups often allows us to apply Proposition 4.9, we were not able to prove that this strategy always works, and leave the following question open (for which we expect a positive answer).

Question 4.15 Let $\mathfrak{G}$ be a right-angled graph of groups. Assume that the graph of $\mathfrak{G}$ is finite, that its vertex groups are graph products over finite graphs, and that $\Phi(G)$ is finite for every factor G. Does there exists a finite cover $\mathfrak{G}' \to \mathfrak{G}$ such that $\mathfrak{G}'$ satisfies the assumptions of Proposition 4.9?

In the rest of the section, we explain how to exploit Proposition 4.9 in specific examples. The examples of right-angled graphs of groups given below are taken from [28]. We emphasize that, in the embeddings given below, the $\mathbb{Z}_2$ can be replaced with arbitrary nontrivial groups.

Example 4.16 Given a group A, consider the graph of groups with a single vertex, labeled by $A \times A$, and a single edge, labeled by A, such that the edge group A is sent into $A \times A$ first as the left factor and next as the right factor. Let $A^{\rtimes}$ denote the fundamental group of this graph of groups. The group $A^{\rtimes}$ admits

$$\langle A, t \mid [a, tat^{-1}] = 1,\ a \in A \rangle,$$

as a (relative) presentation. Notice that, if A is infinite cyclic, we recover the group introduced in [15], which was the first example of a fundamental group of a 3-manifold which is not subgroup separable.

By construction, $A^{\rtimes}$ is the fundamental group of a right-angled graph of groups, so it acts on a quasi-median graph. However, the conditions (i) and (iii) in Proposition

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4.9 are not satisfied, so this action is not special. Nevertheless, it is sufficient to consider a new graph of groups, which is a two-sheeted cover of the previous one.

More generally, fix another group B, and consider the graph of groups which has two vertices, both labeled by $A \times B$, and two edges between these vertices, labeled by A and B, such that the edge group A is sent into the vertex groups as the left factor A and such that the edge group B is sent into the vertex groups as the right factor B. The fundamental group of this graph of groups is denoted by $A \square B$, and has

$$\langle A, B, t \mid [a, b] = [a, tbt^{-1}] = 1,\ a \in A, b \in B \rangle,$$

as a (relative) presentation. Observe that $A \square A$ is naturally a subgroup of $A^{\rtimes}$ of index 2, and that the right-angled graph of groups defining $A \square B$ satisfies the assumptions of Proposition 4.9. Let Γ denote the graph which is a path of length 3 $a - b - c - d$, and let $\mathcal{G}_{A,B} = \{G_a = \mathbb{Z}_2, G_b = A, G_c = B, G_d = \mathbb{Z}_2\}$. By applying Proposition 4.13, it follows that $A \square B$ embeds into $\Gamma\mathcal{G}_{A,B}$. Such an embedding is given by sending $A \subset A \square B$ to $A \subset \Gamma\mathcal{G}_{A,B}$, $B \subset A \square B$ to $B \subset \Gamma\mathcal{G}_{A,B}$, and $t \in A \square B$ to $xy \in \Gamma\mathcal{G}_{A,B}$ where $x \in G_a, y \in G_b$ are nontrivial.

Thus, we have found a subgroup $A \square A$ of index 2 in $A^{\rtimes}$, and we have constructed an embedding $A \square A \hookrightarrow \Gamma\mathcal{G}_{A,A}$ whose image is a virtual retract.

Notice that, if we replace the $\mathbb{Z}_2$ with infinite cyclic groups, then it follows that the group $\mathbb{Z}^{\rtimes}$ from [15] virtually embeds into the right-angled Artin group defined by a path of length 3. Here, we see that taking a finite-index subgroup is necessary as $\mathbb{Z}^{\rtimes}$ does not embed directly into a right-angled Artin group. Indeed, $\mathbb{Z}^{\rtimes}$ is two-generated, but it is neither abelian nor free [9].

Example 4.17 The previous example can be generalized in the following way. Consider a graph product $\Gamma\mathcal{G}$, and fix two vertices $u, v \in V(\Gamma)$ such that there exists an isomorphism $\varphi : G_u \to G_v$. The HNN extension $G := \Gamma\mathcal{G}*_\varphi$ is a simple example of a fundamental group of right-angled graph of groups. Notice that G contains a subgroup of index 2 H which decomposes as a graph of groups with two vertices, both labeled by $\Gamma\mathcal{G}$; with two edges between these vertices, both labeled by G_u; such that one edge group is sent into the first $\Gamma\mathcal{G}$ as G_u and into the second $\Gamma\mathcal{G}$ as G_v (through φ); and such that the second edge group is sent into the first $\Gamma\mathcal{G}$ as G_v (through φ) and into the second $\Gamma\mathcal{G}$ as G_u. Now, Proposition 4.9 applies to H. Let Ψ denote the graph obtained from two copies of Γ by identifying u, v in the first copy of Γ, respectively, with v, u in the second copy of Γ; and by adding a new neighbor to each of the two vertices in the intersection of the two copies of Γ. Moreover, let $\mathcal{H}$ denote the collection of groups indexed by $V(\Psi)$ such that a vertex w of a copy of Γ is labeled by $G_w \in \mathcal{G}$ and such that the two new vertices are labeled by $\mathbb{Z}_2$. According to Proposition 4.13, our group H embeds into $\Psi\mathcal{H}$.

Thus, $\Gamma\mathcal{G}*_\varphi$ has a subgroup of index 2 which embeds (as a virtual retract if Γ is finite) into the graph product $\Psi\mathcal{H}$.

For instance, the HNN extension

$$G_{p,q} = \langle t, x_i\ (0 \leq i \leq p-1) \mid tx_0t^{-1} = x_2, x_i^q = [x_i, x_{i+1}] = 1\ (i \bmod p) \rangle$$

of the Bourdon group $\Gamma_{p,q}$ [12] has a subgroup of index 2 $\bar{G}_{p,q}$ which embeds as a gated-cocompact subgroup into the graph product $\Pi(p,q)$, illustrated by Figure 5 for

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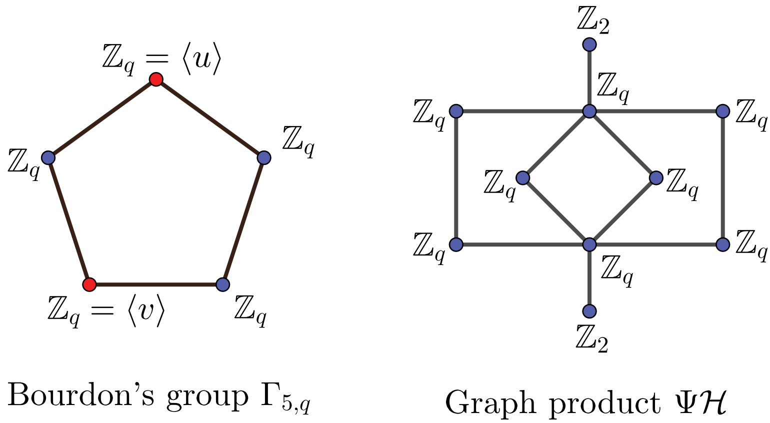

Figure 5: The HNN extension $\Gamma_{5,q} *_{u^t=v}$ virtually embeds into $\Psi\mathcal{H}$.

$p = 5$. As an application, let us deduce that $G_{p,q}$ is relatively hyperbolic for $p \geq 5$ and $q \geq 2$. Indeed, it follows from [28, Theorem 8.35] that the graph product $\Pi(p,q)$ is hyperbolic relative to the free product $\mathbb{Z}_q^2 * \mathbb{Z}_q^2$ (given by the central four-cycle of Ψ in Figure 5). Because $\Pi(p,q)$ is a graph product of finite groups, it acts properly and cocompactly on its quasi-median graph, so $\bar{G}_{p,q}$, as a gated-cocompact subgroup, is undistorted. Then, it follows from [24, Theorem 1.8] that $G_{p,q}$ is hyperbolic relative to products of free groups. (As a particular case, $G_{p,2}$ is toral relatively hyperbolic.)

Example 4.18 In our last example, we consider the group operation

$$G \bullet H = \langle G, H, t \mid [g, t^n h t^{-n}] = 1,\ g \in G, h \in H, n \geq 0 \rangle,$$

introduced in [37]. As observed in [29], $\mathbb{Z} \bullet \mathbb{Z} = \langle a, b, t \mid [a, t^n b t^{-n}] = 1, n \geq 0 \rangle$ is a simple example of finitely generated but not finitely presented subgroup of $\mathbb{F}_2 \times \mathbb{F}_2$. We would like to generalize such an embedding for arbitrary factors.

The product $G \bullet H$ can be decomposed as a right-angled graph of groups, since, given infinitely many copies G_n, H_m of G, H, respectively ($n, m \in \mathbb{Z}$), it admits

$$\left\langle t, H_n, G_m, n, m \in \mathbb{Z} \;\middle|\; \begin{array}{l} [g_{(n)}, h_{(m)}] = 1, m \geq n \\ t g_{(n)} t^{-1} = g_{(n+1)}, t h_{(m)} t^{-1} = h_{(m+1)}, n, m \in \mathbb{Z} \end{array}, g \in G, h \in H \right\rangle,$$

as an alternative (relative) presentation, where $g_{(n)}$ (resp. $h_{(m)}$) denotes the element $g \in G$ in the copy G_n (resp. the element $h \in H$ in the copy H_m). However, such a graph of groups (and each of its finite covers) does not satisfy Proposition 4.9. So, here, we have an example of a fundamental group of a right-angled graph of groups for which the methods developed in the article do not work, even though nice embeddings exist (as sketched below).

In order to embed $G \bullet H$ into a graph product, an alternative approach is to consider $G \bullet H$ as a diagram product [37] and to look at its action on the quasi-median graph constructed in [28]. We do not give details here, but the action turns out to be special, and an application of Proposition 4.13 shows that $G \bullet H$ embeds into

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$(G * \mathbb{Z}_2) \times (H * \mathbb{Z}_2)$ by sending G to G, H to H, and t to yx where x (resp. y) is a nontrivial element of the left (resp. the right) $\mathbb{Z}_2$. (By replacing the $\mathbb{Z}_2$ with infinite cyclic groups, we recover the embedding $\mathbb{Z} \bullet \mathbb{Z} \hookrightarrow \mathbb{F}_2 \times \mathbb{F}_2$ found in [29].)

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References

[1] I. Agol, *The virtual Haken conjecture*. Doc. Math. **18**(2013), 1045–1087, with an appendix by I. Agol, D. Groves, and J. Manning.
[2] J. Alonso, *Inégalités isopérimétriques et quasi-isométries*. C. R. Acad. Sci. Paris Sér. I Math. **311**(1990), 761–764.
[3] J. Alonso, *Finiteness conditions on groups and quasi-isometries*. J. Pure Appl. Algebra **95**(1994), 121–129.
[4] J. Alonso, *Dehn functions and finiteness properties of graph products*. J. Pure Appl. Algebra **107**(1996), 9–17.
[5] Y. Antolín and D. Dreesen, *The Haagerup property is stable under graph products*. Preprint, 2014. arXiv:1305.6748
[6] Y. Antolín and A. Minasyan, *Tits alternatives for graph products*. J. Reine Angew. Math. **704**(2015), 55–83.
[7] H.-J. Bandelt, H. M. Mulder, and E. Wilkeit, *Quasi-median graphs and algebras*. J. Graph Theory **18**(1994), no. 7, 681–703.
[8] H. Bass, *Covering theory for graphs of groups*. J. Pure Appl. Algebra **89**(1993), nos. 1–2, 3–47.
[9] A. Baudisch, *Subgroups of semifree groups*. Acta Math. Acad. Sci. Hungar. **38**(1981), nos. 1–4, 19–28.
[10] F. Berlai and J. de la Nuez González, *Linearity of graph products*. Preprint, 2019. arXiv:1906.11958
[11] F. Berlai and M. Ferov, *Separating cyclic subgroups in graph products of groups*. J. Algebra **531**(2019), 19–56.
[12] M. Bourdon, *Immeubles hyperboliques, dimension conforme et rigidité de Mostow*. Geom. Funct. Anal. **7**(1997), no. 2, 245–268.
[13] B. Brešar, J. Chalopin, V. Chepoi, T. Gologranc, and D. Osajda, *Bucolic complexes*. Adv. Math. **243**(2013), 127–167.
[14] R. Brown, *Topology and groupoids*, BookSurge, LLC, Charleston, SC, 2006, Third edition of it elements of modern topology (McGraw-Hill, New York, 1968), with 1 CD-ROM (Windows, Macintosh, and UNIX).
[15] R. Burns, A. Karrass, and D. Solitar, *A note on groups with separable finitely generated subgroups*. Bull. Aust. Math. Soc. **36**(1987), no. 1, 153–160.
[16] M. Carr, *Two-generator subgroups of right-angled Artin groups are quasi-isometrically embedded*. Preprint, 2015. arXiv:1412.0642
[17] V. Chepoi, *Classification of graphs by means of metric triangles*. Metody Diskret. Analiz. **49**(1989), no. 96, 75–93.
[18] V. Chepoi, *Graphs of some CAT(0) complexes*. Adv. in Appl. Math. **24**(2000), no. 2, 125–179.
[19] D. Cohen, *Projective resolutions for graph products*. Proc. Edinb. Math. Soc. (2) **38**(1995), no. 1, 185–188.
[20] D. Cohen, *Isoperimetric functions for graph products*. J. Pure Appl. Algebra **101**(1995), no. 3, 305–311.
[21] J. Crisp and B. Wiest, *Embeddings of graph braid and surface groups in right-angled Artin groups and braid groups*. Algebr. Geom. Topol. **4**(2004), no. 1, 439–472.
[22] M. Davis and T. Januszkiewicz, *Right-angled Artin groups are commensurable with right-angled Coxeter groups*. J. Pure Appl. Algebra **153**(2000), no. 3, 229–235.
[23] C. Droms, *Graph groups*. Ph.D. thesis, Syracuse University, 1983.
[24] C. Druţu and M. Sapir, *Tree-graded spaces and asymptotic cones of groups*. Topology **44**(2005), no. 5, 959–1058, with an appendix by D. Osin and M. Sapir.
[25] G. Duchamp and D. Krob, *The lower central series of the free partially commutative group*. Semigroup Forum **45**(1992), no. 3, 385–394.

[26] G. Duchamp and J.-Y. Thibon, *Simple orderings for free partially commutative groups*. Internat. J. Algebra Comput. **2**(1992), no. 3, 351–355.
[27] M. Ferov, *On conjugacy separability of graph products of groups*. J. Algebra **447**(2016), 135–182.
[28] A. Genevois, *Cubical-like geometry of quasi-median graphs and applications in geometric group theory*. Ph.D. thesis, 2017. Preprint, arXiv:1712.01618
[29] A. Genevois, *Hyperplanes of Squier's cube complexes*. Algebr. Geom. Topol. **18**(2018), no. 6, 3205–3256.
[30] A. Genevois, *Quasi-isometrically rigid subgroups in right-angled Coxeter groups*. Algebraic & Geometric Topology. Preprint, 2019. arXiv:1909.04318
[31] A. Genevois, *Embeddings into Thompson's groups from quasi-median geometry*. Groups Geom. Dyn. **13**(2019), no. 4, 1457–1510.
[32] A. Genevois, *On the geometry of van Kampen diagrams of graph products of groups*. Preprint, 2019. arXiv:1901.04538
[33] A. Genevois, *Negative curvature in graph braid groups*. Internat. J. Algebra Comput. **31**(2021), no. 1, 81–116.
[34] A. Genevois, *Negative curvature of automorphism groups of graph products with applications to right-angled Artin groups*. Automorphisms of graph products of groups and acylindrical hyperbolicity. Preprint, 2022. arXiv:1807.00622
[35] A. Genevois and A. Martin, *Automorphisms of graph products of groups from a geometric perspective*. Proc. Lond. Math. Soc. (3) **119**(2019), no. 6, 1745–1779.
[36] E. Green, *Graph products of groups*. Ph.D. thesis, University of Leeds, 1990.
[37] V. Guba and M. Sapir, *On subgroups of the R. Thompson group F and other diagram groups. Mat. Sb.* **190**(1999), no. 8, 3–60.
[38] M. Hagen and P. Przytycki, *Cocompactly cubulated graph manifolds*. Israel J. Math. **207**(2015), no. 1, 377–394.
[39] F. Haglund, *Isometries of CAT(0) cube complexes are semi-simple*. Preprint, 2007. arXiv:0705.3386
[40] F. Haglund and D. Wise, *Special cube complexes*. Geom. Funct. Anal. **17**(2008), no. 5, 1551–1620.
[41] F. Haglund and D. Wise, *Coxeter groups are virtually special*. Adv. Math. **224**(2010), no. 5, 1890–1903.
[42] P. Higgins, *The fundamental groupoid of a graph of groups*. J. Lond. Math. Soc. (2) **s2–13**(1976), no. 1, 145–149.
[43] T. Hsu and D. Wise, *On linear and residual properties of graph products*. Michigan Math. J. **46**(1999), 251–259.
[44] S. Humphries, *On representations of Artin groups and the Tits conjecture*. J. Algebra **169**(1994), no. 3, 847–862.
[45] V. Metaftsis and E. Raptis, *On the profinite topology of right-angled Artin groups*. J. Algebra **320**(2008), no. 3, 1174–1181.
[46] A. Minasyan, *Hereditary conjugacy separability of right-angled Artin groups and its applications*. Groups Geom. Dyn. **6**(2012), no. 2, 335–388.
[47] E. Reckwerdt, *Weak amenability is stable under graph products*. J. Lond. Math. Soc. (2) **96**(2017), no. 1, 133–155.
[48] P. Scott and T. Wall, *Topological methods in group theory, Homological group theory (Proc. Sympos., Durham)*. London Math. Soc. Lecture Note Ser. **36**(1977), no. 1979, 137–203.
[49] J.-P. Serre, *Trees*, Springer Monographs in Mathematics, Springer, Berlin, 2003, translated from the French original by J. Stillwell, corrected 2nd printing of the 1980 English translation.
[50] J. Tidmore, *Cocompact cubulations of mixed 3-manifolds*. Groups Geom. Dyn. **12**(2018), no. 4, 1429–1460.
[51] R. Wade, *The lower central series of a right-angled Artin group*. Enseign. Math. **61**(2015), nos. 3–4, 343–371.

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