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ON LOGARITHMIC DERIVATIVES OF FUNCTIONS IN A CLASS OF STARLIKE MAPPINGS

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ABSTRACT. The purpose of this paper is to prove some facts about integral means of $(d^2/dz^2)(\log[f(z)/z])$—or equivalently f''/f, for f in a class of starlike mappings of a "singular" nature. In particular it is noted that the Koebe function is not extremal for the Hardy means $M_p(r,f''/f)$ for functions in this class.

1. **Introduction.** Let S denote the class of functions analytic and univalent in the unit disc $\mathbb{D}$ of the complex plane normalized so that $f(0) = f'(0)-1 = 0$, and S^* denote the subclass of S for which $f(\mathbb{D})$ is starlike with respect to the origin; S^* is the class of starlike mappings. If $f \in S^*$, then $\mu(\theta) = \lim_{r\to 1} \arg f(re^{i\theta})$ exists for each θ and is an increasing function with $\mu(\theta) - \theta$ periodic with period 2π, and $\mu(\theta) = \frac{1}{2}[\mu(\theta+0) + \mu(\theta-0)]$ for each θ; see [15], p. 591. Let us call μ the boundary argument function for f.

DEFINITION. A function $f \in S^*$ is said to be in the class S_0^* if there is a closed set $E \subset [0, 2\pi]$ of Lebesgue measure zero such that $[0, 2\pi] - E = \bigcup_k (a_k, b_k)$ with μ constant on each (a_k, b_k).

Members of S_0^* thus have the property that their boundary argument changes only on a closed set of measure zero. The class S_0^* contains, for example, rotations of the Koebe function $K(z) = z(1-z)^{-2}$ and more generally functions of the form $z\Pi_{j=1}^n (1 - ze^{i\theta_j})^{-\alpha_j}$, where $\{\theta_j\}$ are distinct numbers in $[0, 2\pi)$ and $\sum_j \alpha_j = 2, \alpha_j > 0$; these functions map the unit disc onto the plane minus n radial slits making angles $\pi\alpha_j$ with the origin. In fact, since the collection of functions of this form is dense in S^* in the topology of uniform convergence on compact subsets, (see [15], p. 583), it follows that S_0^* is dense in S^*. In Section 3 of this paper it is shown that S_0^* contains some bounded functions, and in Section 4 it is shown that functions in S_0^* are not starlike of order β for any $\beta > 0$.

In this paper we will prove some facts about $\log[f(z)/z]$ for functions in this class, in particular we will deal with the growth classes for $(d^2/dz^2)\big(\log[f(z)/z]\big)$ showing how the Hardy classes H^p to which it belongs is affected by hypotheses on μ, hypotheses which can in some cases be related to the mapping properties of f. (For future reference let us denote $(d^n/dz^n)\big(\log[f(z)/z]\big)$ by $D_l^n f$ for $f \in S$, and let $f^{(n)}$ be the standard n^{th} derivative, $n \geq 1$.) It is noted in particular that the Koebe function is not extremal for integral mean growth of $D_l^2 f$ for $f \in S_0^*$, where the Hardy p-means, $0 < p < \frac{1}{2}$ are used to measure the derivative. We also point out a connection between the smoothness of μ as measured by modulus of continuity and the one dimensional Lebesgue measure of the

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set $\left\{\theta : \frac{\partial}{\partial\theta}\arg f(re^{i\theta}) > \beta\right\}$ for $\beta > 0$, where r is fixed, $0 < r < 1$. The main tools in making these connections are the well-known integral representation formula for star-like mappings, see [14], pp. 209–210, and the work of several authors on growth classes for singular inner functions, [1]–[5].

In the following, the class H^p is the usual Hardy class of functions f analytic in $\mathbb{D}$ with $\sup_{0<r<1}\int_0^{2\pi}|f(re^{i\theta})|^p\frac{d\theta}{2\pi} = M_p^p(r,f) < \infty$, $0 < p < \infty$, and $M_\infty(r,f) = \max_\theta|f(re^{i\theta})|$; N is the Nevanlinna class of functions f for which

$$\sup_{0<r<1}\int_0^{2\pi}\log^+|f(re^{i\theta})|d\theta < \infty,$$

and N^+ is the subset of N for which $\lim_{r\to 1}\int_0^{2\pi}\log^+|f(re^{i\theta})|d\theta = \int_0^{2\pi}\log^+|f(e^{i\theta})|d\theta$. Recall that $H^P \subset N^+ \subset N$ for all p, and $H^p \subset H^q$ if $q < p$. See [6] for details.

2. **Growth classes for $D_L^2 f$—the main theorem.** In this section we prove the main theorem of the paper, which states the relationship between $D_L^2 f$ and μ, for $f \in S_0^*$. We first give some background. There have been many attempts to relate the growth of $f \in S^*$ as measured by integral means of $f^{(n)}$ or $D_L^n f$ to properties of f as a mapping and to μ. For example, if μ has a jump of $\pi\alpha$ at $\theta = \theta_0$, then the image of f contains a maximal "sector" of angle $\pi\alpha$ with vertex at 0, see [15] p. 591; in the case in which this is the greatest jump for μ, then $\lim_{r\to 1}\frac{\log M_\infty(r,f)}{\log(1-r)^{-1}} = \alpha$ ([14] p. 211) and $\lim_{r\to 1}(1-r)M_\infty(r, D_L^1 f) = \alpha$ ([15] p. 211). In [14] are similar limits involving $M_p(r,f^{(n)})$; see p. 605. One also has that if f is bounded, then μ is continuous [14] p. 211. In [8] the authors show that every $f \in S^*$ with finite image area has $M_2^2(r,f'/f) = O\big((1-r)^{-1}\big)$.

For the class S_0^* it is $D_L^2 f$ which can most easily be related to μ. We do this in Theorem 1 below indirectly by making reference to the singular inner function associated with μ; in Section 3 the inner function conditions are replaced by conditions in μ itself in three corollaries. Recall that if σ is a non decreasing function on $[0, 2\pi]$ with $\sigma' = 0$ a.e., then $S_\sigma(z) = \exp\left\{-\int_0^{2\pi}\frac{e^{it}+z}{e^{it}-z}d\sigma(t)\right\}$ is called the *associated singular inner function*, see [6] p. 24 or [11] p. 75. The function $S_\sigma(z)$ satisfies $|S_\sigma(z)| < 1$ in $\mathbb{D}$ with $\lim_{r\to 1}|S_\sigma(re^{i\theta})| = 1$ a.e. Note that if $f \in S_0^*$, then $\mu' = 0$ a.e.

THEOREM 1. *Suppose $f \in S_0^*$ with $\mu(\theta) = \lim_{r\to 1}\arg f(re^{i\theta})$, and $0 < p < 1$. Then $D_L^2 f \in H^p$ iff $S'_\mu \in H^p$. If $S'_\mu \in N$ then $D_L^2 f \in N$, and if $S'_\mu \notin N$, then $D_L^2 f \notin H^p$ for all $p > 0$.*

PROOF. With f and μ as above, we have that

$$f(z) = z\exp\left\{\frac{1}{\pi}\int_0^{2\pi}\log\frac{1}{1-e^{-it}z}\,d\mu(t)\right\},$$

see [14] pp. 209–210; from above, we can form $S_\mu(z) = \exp\left\{-\int_0^{2\pi} \frac{e^{it}+z}{e^{it}-z}\, d\mu(t)\right\}$. Now

$$\begin{aligned}\frac{S'_\mu(z)}{S_\mu(z)} &= -2\int_0^{2\pi} \frac{e^{it}}{(e^{it}-z)^2}\, d\mu(t)\\ &= -2\int_0^{2\pi} \frac{d\mu(t)}{(e^{it}-z)} - 2z\int_0^{2\pi} \frac{d\mu(t)}{(e^{it}-z)^2}\\ &= -2\int_0^{2\pi} \frac{d\mu(t)}{e^{it}-z} - 2z\pi D^2_L f.\end{aligned}$$

The first term of this last expression is in H^p for all p, $0 < p < 1$, see [6] p. 39. It follows that $D^2_L f \in H^p$ iff $S'_\mu / S_\mu \in H^p$. However, by [3] Theorem 4 p. 118, $S'_\mu / S_\mu \in H^p$ iff $S'_\mu \in H^p$, thus the H^p part of the theorem follows. If $S'_\mu \in N$, then again by [3] Corollary 4, p. 118, $S'_\mu / S_\mu \in N^+$, hence in N. Finally, if $S'_\mu \notin N$, then by the same corollary, $S'_\mu / S_\mu \notin N^+$, hence $S'_\mu / S_\mu \notin H^p$ for all $p > 0$. The proof is complete. ■

For purposes of orientation let us note the following: first, for all $f \in S^*$, $\log f(z)/z \in \bigcap_{p<\infty} H^p$. Now if $f \in S^*$, $D^1_L f = \frac{1}{\pi}\int_0^{2\pi} \frac{d\mu(t)}{(e^{it}-z)}$ where μ is the boundary argument function for f, where μ need not be singular [14], pp. 209–210; thus $D^1_L f \in \bigcap_{p<1} H^p$. Recall that by a theorem of Hardy and Littlewood ([6] p. 88) $g' \in H^p$, $0 < p < 1$ implies $g \in H^{\frac{p}{1-p}}$, so $C_1 = \{f : D^1_L f \in \bigcap_{p<1} H^p\} \subset \bigcap_{p<\infty} H^p$, and thus any $f \in S^*$ is in the subclass C_1 of $\bigcap_{p<\infty} H^p$. It is then natural to ask whether $S^* \subset C_2 = \{f : D^2_L f \in \bigcap_{p<1/2} H^p\} \subset C_1 \subset \bigcap_{p<\infty} H^p$. Theorem 1 and the corollaries in Section 3 will show that the first inclusion does not always hold; counterexamples will be members of S^*_0. Of course, for arbitrary starlike functions one can easily have $D^n_L f \in \bigcap_{p<\infty} H^p$ for any n; take $f(z) = z$, for example. See also [8] for other subclasses of $\bigcap_{p<\infty} H^p$ to which S or S^* may belong.

We close with some final remarks on the quantity $D^2_L f$ for $f \in S^*$. Note that $D^1_L f(z) = f'(z)/f(z) = 1/z$ if $z \neq 0$, and since $D^1_L f \in \bigcap_{p<1} H^p$, it follows that $M_p(r, f'/f) < \infty$ as $r \longrightarrow 1$, for all $p < 1$. The identity $D^2_L f(z) = f''(z)/f(z) - [f'(z)/f(z)]^2 + 1/z^2$ for $z \neq 0$ thus gives that $D^2_L f \in H^p$ iff $M_p(r, f''/f) < \infty$ as $r \longrightarrow 1$, if $0 < p < 1/2$. So we may use $D^2_L f$ and f''/f interchangeably in any statements of our theorems. Secondly, we have that $1 + zD^1_L f = zf'(z)/f(z)$, so $\mathrm{Re}[1 + zD^1_L f] = \frac{\partial}{\partial\theta} \arg f(re^{i\theta})$ where $z = re^{i\theta}$; this shows that $D^1_L f$ is related to $\frac{\partial}{\partial\theta} \arg f(re^{i\theta})$. For $D^2_L f$ we have:

PROPOSITION. *Let $f \in S^*$. If $D^2_L f \in H^p$ for some $0 < p < 1$, then $M_p(r, g) < \infty$ as $r \longrightarrow 1$, where $g(re^{i\theta}) = \frac{\partial^2}{\partial^2\theta} \arg f(re^{i\theta})$.*

PROOF. Differentiating both sides of the identity $1 + zD^1_L f(z) = zf'(z)/f(z)$ with respect to θ and taking real parts gives

$$\mathrm{Re}[iz^2 D^2_L f(z) + izD^1_L f(z)] = \mathrm{Re}\left[\frac{\partial}{\partial\theta}\frac{zf'(z)}{f(z)}\right] = \frac{\partial}{\partial\theta}\mathrm{Re}\left[\frac{zf'(z)}{f(z)}\right] = \frac{\partial^2}{\partial\theta^2}\arg f(re^{i\theta}).$$

The result follows since $D^1_L f \in \bigcap_{p<1} H^p$. ■

Thus a mean growth condition of this sort on $D^2_L f$ implies one of the same sort on $\frac{\partial^2}{\partial\theta^2} \arg f(re^{i\theta})$.

3. **Corollaries.** We now use the results in [1]–[5] to relate growth classes for S'_μ to conditions on μ itself, and thus obtain corollaries relating $D_L^2 f$ to μ directly.

We first assume that μ is continuous. An example of $f \in S_0^*$ with μ continuous is given in [12], Section 5, where μ is the Lebesgue function of the standard Cantor set on $[0, 2\pi]$ normalized so that $\mu(2\pi) - \mu(0) = 2\pi$; similar examples will occur in Corollary 1. Note that this function has an image which contains no angular sectors with vertex at the origin and positive angular spread ([15], p. 591). Also, this function is shown to be bounded in [12].

For arbitrary μ continuous we have the standard modulus of continuity $\omega_\mu(t) = \sup_{|x-y|\leq t} |\mu(x) - \mu(y)|$; since μ is continuous, $\omega_\mu(t) \longrightarrow 0$ as $t \longrightarrow 0$, and since μ is singular, $\omega_\mu(t)/t \longrightarrow \infty$ as $t \longrightarrow 0$ ([1], p. 315). Our first corollary involves a condition on $\omega_\mu(t)$:

COROLLARY 1. *Suppose that $f \in S_0^*$, μ is continuous and $\omega_\mu(t) = O(t^\alpha)$, for some α, $0 < \alpha < 1$. Then $D_L^2 f \notin H^{\frac{1-\alpha}{2-\alpha}}$. Furthermore, this is best possible in the sense that for each α, $0 < \alpha < 1$, there is an $f_\alpha \in S_0^*$ with argument function μ_α such that $\omega_{\mu_\alpha}(t) = O(t^\alpha)$ and $f'_\alpha \in H^p$ for all $p < \frac{1-\alpha}{2-\alpha}$.*

PROOF. Suppose $f \in S_0^*$ with μ continuous and $\omega_\mu(t) = O(t^\alpha)$, for some α, $0 < \alpha < 1$. Then if S_μ is the associated singular inner function, we have by [1] p. 341 that $S'_\mu \notin H^{\frac{1-\alpha}{2-\alpha}}$, so by our Theorem 1 the first statement follows. For the second part, let $0 < \alpha < 1$ be fixed, and define $\omega_\alpha(t) = t^\alpha/(2\pi)^\alpha$. Then Ahern in [1] pp. 323-326 constructs a Cantor set "of constant ratio $2^{-\frac{1}{\alpha}}$" whose Lebesgue function λ_α has a modulus of continuity $\omega_{\lambda_\alpha}(t)$ satisfying $\frac{1}{2}t^\alpha/(2\pi)^\alpha \leq w_{\lambda\alpha}(t) \leq 4t^\alpha/(2\pi)^\alpha$, and thus the associated singular inner function S_{λ_α} has $S'_{\lambda_\alpha} \in H^p$ for all $p < \frac{1-\alpha}{2-\alpha}$ by [1], p. 346. By replacing λ_α by $\mu_\alpha = 2\pi\lambda_\alpha$ we obtain a new singular function S_{μ_α} with modulus of continuity $O(t^\alpha)$ where $S'_{\mu_\alpha} \in H^p$ for all $p < \frac{1-\alpha}{2-\alpha}$ ([3], Theorem 4 can be used to see this last statement). Now define $f_\alpha(z) = z\exp\left\{\frac{1}{\pi}\int_0^{2\pi} \log \frac{1}{1-e^{-it}z}\, d\mu_\alpha(t)\right\}$; then $f \in S_0^*$ with $f'_\alpha \in H^p$ for all $p < \frac{1-\alpha}{2-\alpha}$ by Theorem 1; it is easy to see that μ_α is the boundary argument function for f_α. We are done. ∎

COMMENTS. 1) The corollary says that in some sense the more smoothly the set of arguments of f is distributed, the worse the behavior of $D_L^2 f$.

2) This shows the existence of $f \in S^*$ for which $f \notin C_2 = \{f : f'' \in \bigcap_{p<1/2} H^p\} \in \bigcap_{p<\infty} H^p$.

3) Given any subclass of S, a problem of considerable interest has been to find extremal functions for integral means for functions f in the class, as well as for $f^{(n)}$ and $D_L^n f$. Baernstein's theorem ([7] p. 215) says that the Koebe function $K(z)$ is extremal for $M_p(r,f)$, $f \in S$, $0 < p < \infty$, but for $f^{(n)}$, $n \geq 1$ the Koebe function does not necessarily play this role. If $p > 2/5$, then $M_p(r,f^{(n)}) = O\big(M_p(r, K^{(n)})\big)$, $r \longrightarrow 1$ for all n [10]. It is also known that $M_p(r,f^{(n)}) \leq M_p(r, K^{(n)})$, $0 < p < \infty$, $0 < r < 1, n \geq 1$ for all f in the close to convex class [13]. For starlike functions the bound $M_2^2(r, D_L^1 f) = O\big(M_2^2(r, D_L^1 K)\big)$ was obtained in [8] and in [9] it was shown that $M_p\big(r, zf'(z)/f(z)\big) \leq M_p\big(r, zK'(z)/K(z)\big)$

for all f starlike, $0 < p < \infty$. Now $D_L^2 K \in H^p$ for all $p < 1/2$, but clearly K cannot be extremal for $M_p(r, D_L^2 f), f \in S^*$ for any $p < 1/2$ by this corollary. It is interesting that this happens even though $\log f(z)/z$ is subordinate to $\log K(z)/z$ for all $f \in S^*$ ([7], p. 213).

We next remove the requirement that μ be continuous and focus on restrictions on the set $[0, 2\pi] - E = \bigcup_k (a_k, b_k)$.

COROLLARY 2. *Suppose $f \in S_0^*$ and $d\mu$ is supported on E, where $[0, 2\pi] - E = \bigcup_k (a_k, b_k)$, and $\mu(\theta) = \lim_{r \to 1} \arg f(re^{i\theta})$. Let $0 < \gamma < 1$. Then*

a) If $\sum_k |b_k - a_k|^\gamma < \infty$, then $D_L^2 f \in H^{\frac{1-\gamma}{2}}$.

b) If $\sum_k |b_k - a_k| \log \frac{1}{|b_k - a_k|} < \infty$, then $D_L^2 f \in N$.

PROOF. Both of these results follow immediately from [5] Theorem 1, p. 284 and our Theorem 1. ∎

The corollary may be viewed as saying that the faster the lengths of intervals of constant boundary argument for f go to zero, the better the behavior of $D_L^2 f$.

Finally, we turn to boundary argument functions μ which are essentially step functions: assume that $0 < a_1 < b_1 = a_2 < b_2 = a_3 < b_3 = \cdots < 2\pi$ and that μ has a jump of $\pi\lambda_k$ at a_k, where $\lambda_k > 0$. Thus the measure $d\mu$ is purely atomic with weights $\pi\lambda_k$ at a_k; let us call such an argument function purely atomic also. Recall that if $f \in S_0^*$ has purely atomic boundary argument function, then the image of f contains a maximal sector of angle $\pi\lambda_k$ for each k [actually, the existence of such a sector of argument $\pi\lambda_k$ is an equivalent condition for having a jump $\pi\lambda_k$ in μ] (see [15], p. 591). We also must have $\sum_k \lambda_k = 2$.

COROLLARY 3. *Suppose $f \in S_0^*$ and $\mu(\theta) = \lim_{r \to 1} \arg f(re^{i\theta})$ is purely atomic with jumps $\pi\lambda_k$ at a_k. Let $0 < \gamma < 1/2$. Then if $\sum_k \lambda_k^\gamma < \infty$, we have $D_L^2 f \in H^p$, for all $p < 1/2$.*

PROOF. This follows from [4], see also [1] p. 346. ∎

This corollary may be viewed as saying that if the wedge arguments in the image of f go to zero "faster than $1/k^2$", then the behavior of $D_L^2 f$ is the best possible over the class S_0^*. It is perhaps interesting in this regard that ([14] p. 211) if α is the largest wedge argument, then $M_\infty(r, f) \geq C(1 - r)^{-\alpha}$ for $0 < r < 1$, so the smaller the wedge the tamer the maximum modulus is allowed to be.

We conclude this section by noting that in [1] are other results stating conditions on E such that $S'_\mu \in H^p$ for some $p < 1/2$, these relate to the "type" of E defined by $p(\epsilon) =$ Lebesgue measure of $\{\theta : |\theta - E| < \epsilon\}$ and to functions related to $\omega_\mu(t)$, see pp. 344–345. These conditions can then be related to $f \in S_0^*$ as we have done in the corollaries.

4. **The rate of change of the argument function.** In this final section we move from considerations of the quantity $D_L^2 f$ to $\frac{\partial}{\partial\theta} \arg f(re^{i\theta})$. For any starlike function f with

boundary argument function μ the relation

$$\frac{\partial}{\partial\theta}\arg f(re^{i\theta}) = \mathrm{Re}\left[\int_0^{2\pi}\frac{e^{it}+re^{i\theta}}{e^{it}-re^{i\theta}}\,d\mu(t)\right] = \int_0^{2\pi}\frac{1-r^2}{1-2r\cos(\theta-t)+r^2}\,d\mu(t) = P_\mu(r,\theta)$$

holds, where the next to last expression is the familiar Poisson integral of μ. If $f \in S_0^*$, then μ is singular, so, for example, we have $\lim_{r\to 1} P_\mu(r,\theta) = 0$ a.e., with respect to Lebesgue measure and $\lim_{r\to 1} P_\mu(r,\theta) = \infty$ a.e. $[d\mu]$, see [2] or [11], p. 77. For any $\beta > 0$, $0 < r < 1$, we may define $E(r,\beta) = \{\theta : P_\mu(r,\theta) > \beta\}$; if μ has compact support of measure zero (as it will for $f \in S_0^*$) then $|E(r,\beta)| \to 0$ as $r \to 1$, where $|E(r,\beta)|$ denotes the Lebesgue measure of $E(r,\beta)$, see [2], p. 1. In [2] are found bounds from above and below on the rate at which $|E(r,\beta)| \to 0$ as $r \to 1$; thus we have bounds on the rate at which $\left|\left\{\theta : \frac{\partial}{\partial\theta}\arg f(re^{i\theta}) > \beta\right\}\right| \to 0$ as $r \to 1$. Note that the fact that $|E(r,\beta)| \to 0$ for any β as $r \to 1$ says that $f \in S_0^*$ is never starlike of order β for any $\beta > 0$.

In Theorem 2 below we state some bounds from below on the rate of decay to zero of $|\{\theta : \partial/\partial\theta \arg f(re^{i\theta}) > \beta\}|$. These bounds are not stated in the most general form, greater generality for bounds from above or below may be obtained by referring to Theorems 2 and 4 in [2].

THEOREM 2. *Suppose $f \in S_0^*$. Then for any $\beta > 0$ there is a constant $C = C(\beta,f)$ such that $\left|\left\{\theta : \frac{\partial}{\partial\theta}\arg f(re^{i\theta}) > \beta\right\}\right| \geq C\sqrt{1-r}$ as $r \to 1$. If in addition f has continuous boundary argument μ with modulus of continuity $\omega_\mu(t) = O(t^\alpha)$ for some α, $0 < \alpha < 1$, then for any β there is a constant $C(f,\beta)$ such that*

$$\left|\left\{\theta : \frac{\partial}{\partial\theta}\arg f(re^{i\theta}) > \beta\right\}\right| \geq C(1-r)^{\frac{1-\alpha}{2-\alpha}}, \text{ as } r \to 1.$$

PROOF. This is [2] Theorems 4 and 5; see also [1] for further details on the calculation of the quantity $\delta(r)$. ∎

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