COUNTING COLOURED GRAPHS

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1. Introduction. A *graph on n labelled nodes* is a set of n objects called "nodes," distinguishable from each other, and a set (possibly empty) of "edges," that is, pairs of nodes. Each edge is said to join its pair of nodes, at most one edge joins any two nodes and no edge joins a node to itself. By a k-colouring of such a graph we mean a mapping of the nodes of the graph onto a set of k distinct colours, such that no two nodes joined by an edge are mapped onto the same colour. We take $k > 1$.

Following Read **(1)**, we write $M_n = M_n(k)$ for the number of such coloured graphs, $F_n = F_n(k)$ for the number of such coloured graphs in which there is at least one node mapped onto each colour, and $f_n = f_n(k)$ for the number of those graphs of the latter set which are connected. We write also $T(\alpha) = 2^\alpha$ and use $\sum$ to denote summation over all i such that $1 \leqslant i \leqslant k$. Read **(1)** showed that

$$M_n(k) = \sum_{(n)} \frac{n!}{s_1! \ldots s_k!} T\left(\frac{1}{2}n^2 - \frac{1}{2}\sum s_i^2\right), \tag{1.1}$$

where $\sum_{(n)}$ denotes summation over all sets of non-negative integers s_i such that

$$\sum s_i = n. \tag{1.2}$$

$F_n(k)$ is the corresponding sum in which every s_i is positive. Read also shows that

$$f_n(k) = F_n(k) - \sum_{r=1}^{n-1} \binom{n-1}{r-1} F_{n-r}(k) f_r(k), \tag{1.3}$$

where $F_1(k) = F_2(k) = \ldots = F_{k-1}(k) = 0$.

If we put

$$\psi = \psi(x) = \sum_{s=1}^{\infty} T(-\tfrac{1}{2}s^2)\frac{x^s}{s!},$$

we have

$$\psi^k = \sum_{n=k}^{\infty} \frac{T(-\tfrac{1}{2}n^2)F_n(k)x^n}{n!} \tag{1.4}$$

and

$$(1 + \psi)^k = 1 + \sum_{n=1}^{\infty} \frac{T(-\tfrac{1}{2}n^2)M_n(k)x^n}{n!}, \tag{1.5}$$

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as remarked by Read. Hence, or directly from (1.1), we have

$$M_n(k) = \sum_{r=1}^{k} \binom{k}{r} F_n(r). \tag{1.6}$$

The series for ψ and the series in (1.4) and (1.5) are convergent for all x and so represent integral functions. On the other hand, Read deduces (1.3) from the formal relationship

$$1 + \sum_{n=k}^{\infty} \frac{F_n(k)x^n}{n!} = \exp\left(\sum_{n=1}^{\infty} \frac{f_n(k)x^n}{n!}\right),$$

in which both series diverge for all non-zero values of x.

Read deduces from (1.4) and (1.5) respectively that

$$F_n(k) = \sum_{r=1}^{n-1} \binom{n}{r} 2^{r(n-r)} F_r(k-1) \tag{1.7}$$

and

$$M_n(k) - M_n(k-1) = \sum_{r=0}^{n-1} \binom{n}{r} 2^{r(n-r)} M_r(k-1) \tag{1.8}$$

and uses these to compute F_n and M_n for small values of k and n. He remarks that M_n, unlike F_n, is a polynomial in k of degree n and this follows from (1.8) by induction. He uses (1.8) to calculate M_n as a polynomial in k for $n = 1, 2, 3, 4$.

If we differentiate (1.5) logarithmically with respect to x, rearrange and equate coefficients of x^n, we have

$$M_n(k) = \sum_{r=1}^{n-1} 2^{r(n-r)} M_r(k) \left\{ \binom{n-1}{r} k - \binom{n-1}{r-1} \right\} + k \tag{1.9}$$

and this expresses $M_n(k)$ in terms of $M_r(k)$ for $r < n$ and does not involve $M_n(k-1)$. The polynomial property of $M_n(k)$ follows from (1.9) by induction even more trivially than from (1.8). Again, if we put

$$M_n(k) = \sum_{s=1}^{n} a_{ns} k^s,$$

substitute in (1.9), and equate the coefficients of powers of k, we find the recurrence formula

$$a_{ns} = \sum_{r=s-1}^{n-1} 2^{r(n-r)} \binom{n-1}{r} a_{r,s-1} - \sum_{r=s}^{n-1} 2^{r(n-r)} \binom{n-1}{r-1} a_{rs}$$

for the coefficients in the polynomial $M_n(k)$ when $s > 1$. In particular

$$a_{nn} = 2^{n-1} a_{n-1,n-1} = \ldots = 2^{\frac{1}{2}n(n-1)}.$$

Similarly we can obtain

$$(n-k)F_n(k) = \sum_{s=k}^{n-1} 2^{(s-1)(n-s)} F_s(k) \left\{ \binom{n}{s} k - \binom{n}{s-1} \right\}.$$

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Although $F_n(k)$ is not a polynomial in k, we have

$$F_n(k) = \frac{n!}{(n-k)!} 2^{kn-\frac{1}{2}k^2-\frac{1}{2}k} J_{n-k}(k),$$

where $J_n(k)$ is a polynomial of the nth degree in k such that $J_0 = 1$ and, for $n \geqslant 1$,

$$n(n+1)J_n = \sum_{u=0}^{n-1} 2^{(u-1)(n-u)} \binom{n+1}{u} \{(n-u)k - u\} J_u.$$

2. The main theorems. But these results are fairly trivial. Our purpose here is to find asymptotic formulae for the behaviour of M_n, F_n, and f_n for fixed k as $n \to \infty$. We define a as the least positive residue of n to modulus k. We use A (with or without a suffix) to denote a positive number, not necessarily the same at each occurrence, which depends at most on k and on its suffix, if any. The notation $O(\ )$ refers to the passage of n to infinity and the positive number involved is A_H. We write $K = \frac{1}{2}\{1 - (1/k)\}$.

We shall prove

THEOREM 1. *As $n \to \infty$, we have*

$$M_n = \left(\frac{k}{n \log 2}\right)^{\frac{1}{2}(k-1)} k^n 2^{Kn^2} \left\{ \sum_{h=0}^{H-1} C_h n^{-h} + O(n^{-H}) \right\}, \tag{2.1}$$

where $C_h = C_h(k, a)$ depends on k, h and the residue of n (mod k), but not otherwise on n.

THEOREM 2. *As $n \to \infty$,*

$$F_n \sim f_n \sim M_n.$$

In fact,

$$F_n/M_n = 1 + O(e^{-An^2}), \qquad f_n/M_n = 1 + O(e^{-An})$$

and so (2.1) remains true, with unaltered coefficients $C_0, C_1, \ldots$, if M_n is replaced by F_n or by f_n.

Theorem 2, which we deduce fairly simply from Theorem 1, disposes of F_n and f_n. The coefficients C_h are of interest. Each can be expressed in terms of one or more multiple series. In particular,

$$C_0(k, a) = k^{\frac{1}{2}} (\log 2)^{\frac{1}{2}(k-1)} (2\pi)^{-\frac{1}{2}(k-1)} L(a), \tag{2.2}$$

where

$$L(a) = \sum_{((a))} T\left(-\frac{1}{2} \sum s_i^2 + \frac{a^2}{2k}\right) \tag{2.3}$$

and the sum $\sum_{((a))}$ is over all integral values of the s_i, positive, negative, or zero, subject to the condition

$$\sum s_i = a. \tag{2.4}$$

It can be shown very simply that $L(-a) = L(a)$ and that $L(k+a) = L(a)$. Apart from a trivial factor, $L(a)$ is a generalized theta-function and the transformation theory of such a function may be used to obtain information about the value of $C_0(k, a)$. This involves a good deal of elaborate detail, however, and we postpone it to a sequel. Here we prove more simply that $C_0(k, a)$ differs from 1 by a very small amount.

THEOREM 3. *If* $\epsilon = 1.33 \times 10^{-6}$, *then*

$$1 - \epsilon < C_0(k, a) < 1 + \epsilon$$

for $k \leqslant 1000$. *For* $k > 1000$, *we have*

$$(1 - \epsilon)(1 - 10^{-12})^k < C_0(k, a) < (1 + \epsilon)(1 + 10^{-12})^k.$$

But $C_0(k, a)$ is not independent of a. In fact, we shall show that

$$C_0(2, 0) \neq C_0(2, 1). \tag{2.5}$$

To put it roughly, $C_0(2, a)$ does depend on a, though only very little.

It is not surprising that M_n (and F_n and f_n), like other enumerative functions, should depend on the residue of $n \pmod k$ as well as on the size of n. For example, the number of partitions of n into k parts can be expressed as a sum of powers of n up to n^{k-1}, the coefficients of which depend on the residue of $n \pmod{k!}$. But the coefficients of the larger powers, in particular n^{k-1}, do not depend on this residue. Again, the well-known "singular series" in Waring's Problem depends on the arithmetical properties of n. But these enumerative functions of n are fairly small. The asymptotic expansions of the larger enumerative functions (for example, $p(n)$, the number of partitions of n into any number of parts, for which $p(n) \sim B_0 n^{-1} \exp(B_1 \sqrt{n})$) do not have the coefficients of their dominant terms dependent on the congruence properties of n. Thus it is a somewhat unusual phenomenon that M_n, which is very large indeed, has C_0 depending on the residue of $n \pmod k$ but, according to Theorem 3, only a little. The distinction between the size of n and its arithmetical properties, and indeed the whole of the remarks of this paragraph, are deliberately vague. But the point involved seems in some ways the most interesting part of the results.

3. Proof of Theorem 2. If we write $u_i = ks_i - n$ and

$$s_o = 1, \quad S_m = \sum u_i^m = \sum (ks_i - n)^m \quad (m > 0)$$

and suppose (1.2) to be satisfied we have

$$S_1 = \sum (ks_i - n) = k(\sum s_i - n) = 0, \tag{3.1}$$

$$S_2 = k^2 \sum s_i^2 - 2kn \sum s_i + kn^2 = k^2 \sum s_i^2 - kn^2 \tag{3.2}$$

and so

$$k^2(n^2 - \sum s_i^2) = k(k-1)n^2 - S_2 = 2k^2 K n^2 - S_2. \tag{3.3}$$

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Since $S_2 \geqslant 0$, we have

$$T(\tfrac{1}{2}n^2 - \tfrac{1}{2}\sum s_i^2) = T(Kn^2 - \tfrac{1}{2}k^{-2}S_2) \leqslant T(Kn^2).$$

and, by (1.1),

$$M_n(k) \leqslant T(Kn^2)\sum_{(n)} \frac{n!}{s_1! \ldots s_k!} = k^n T(Kn^2).$$

This is true for all k and all n.

It follows from the definitions that

$$0 \leqslant f_n(k) < F_n(k) \leqslant M_n(k)$$

for all $n > k$. Again, by (1.6),

$$\begin{aligned} M_n(k) - F_n(k) &= \sum_{r=1}^{k-1} \binom{k}{r} F_n(r) \leqslant \sum_{r=1}^{k-1} \binom{k}{r} M_n(r) \\ &\leqslant (k-1)^n T\left\{\frac{1}{2}n^2\left(1 - \frac{1}{k-1}\right)\right\} \sum_{r=1}^{k-1} \binom{k}{r} \\ &\leqslant 2^k (k-1)^n T\{Kn^2 - \tfrac{1}{2}n^2/(k^2-k)\}. \end{aligned}$$

If we now assume Theorem 1 to be true, we have

$$(3.4) \qquad \begin{aligned} (M_n - F_n)/M_n &< An^{\frac{1}{2}(k-1)}\{1 - (1/k)\}^n T\{-\tfrac{1}{2}n^2/(k^2-k)\} \\ &< Ae^{-An^2}. \end{aligned}$$

Next, by (1.3),

$$\begin{aligned} F_n - f_n &= \sum_{r=1}^{n-1} \binom{n-1}{r-1} F_{n-r} f_r \\ &\leqslant \sum_{r=1}^{n-1} \binom{n-1}{r-1} M_{n-r} M_r \\ &\leqslant \sum_{r=1}^{n-1} \binom{n-1}{r-1} k^n T\{K(n-r)^2 + Kr^2\} \end{aligned}$$

and, again assuming Theorem 1, we have

$$\frac{F_n - f_n}{M_n} \leqslant An^{\frac{1}{2}(k-1)} \sum_{r=1}^{n-1} \binom{n-1}{r-1} T\{-2Kr(n-r)\}.$$

Now

$$\begin{aligned} &\sum_{r=1}^{n-1} \binom{n-1}{r-1} T\{-2Kr(n-r)\} \\ &\qquad \leqslant \sum_{r=1}^{[\frac{1}{2}n]} \binom{n-1}{r-1} T(-Krn) + \sum_{s=1}^{[\frac{1}{2}(n-1)]} \binom{n-1}{s} T(-Ksn) \\ &\qquad < 2^{-Kn}(1 + 2^{-Kn})^{n-1} + (1 + 2^{-Kn})^{n-1} - 1 \\ &\qquad = (1 + 2^{-Kn})^n - 1 < n2^{-Kn}(1 + 2^{-Kn})^{n-1} < An2^{-Kn}. \end{aligned}$$

Hence

$$0 < (F_n - f_n)/M_n < An2^{-Kn} < Ae^{-An}.$$

Theorem 2 follows from this and (3.4).

4. Proof of Theorem 1. Next we prove Theorem 1. We write

$$P = P(s_1, s_2, \ldots, s_k) = \frac{n!}{s_1! \ldots s_h!}.$$

By (1.1) and (3.3), we have

$$T(-Kn^2)M_n = \sum_{(n)} PT(-\tfrac{1}{2}k^{-2}S_2) = \sum{}' + \sum{}'', \tag{4.1}$$

where $\sum'$ includes all those terms for which $|ks_i - n| < n^{\frac{1}{4}}$ for every i. For every term in $\sum''$, we have $|ks_i - n| \geqslant n^{\frac{1}{4}}$ for at least one value of i and so $S_2 \geqslant n^{\frac{1}{2}}$. We have then

$$\begin{aligned} \sum{}'' &\leqslant T(-An^{\frac{1}{2}}) \sum{}'' P(s_1, \ldots, s_h) \\ &< T(-An^{\frac{1}{2}}) \sum_{(n)} P = k^n T(-An^{\frac{1}{2}}). \end{aligned} \tag{4.2}$$

Now we consider any term of $\sum'$, so that $s_i > A_n$ for every i. By Stirling's formula, for $n > A$,

$$\log (n!) = (n + \tfrac{1}{2}) \log n - n + \tfrac{1}{2} \log (2\pi) + \sum_{h=1}^{H-1} c_h n^{-h} + O(n^{-H})$$

and so

$$\begin{aligned} \log P = (n + \tfrac{1}{2}) \log n - \sum (s_i + \tfrac{1}{2}) \log s_i - \tfrac{1}{2}(k-1) \log (2\pi) \\ + \sum_{h=1}^{H-1} c_h (n^{-h} - \sum s_i^{-h}) + O(n^{-H}). \end{aligned}$$

Also

$$\begin{aligned} (n + \tfrac{1}{2}) \log n - \sum (s_i + \tfrac{1}{2}) \log s_i \\ &= (n + \tfrac{1}{2}) \log n - \sum (s_i + \tfrac{1}{2}) \log (n/k) - \sum_1 \\ &= -\tfrac{1}{2}(k-1) \log n + (n + \tfrac{1}{2}k) \log k - \sum_1, \end{aligned}$$

where

$$\begin{aligned} \sum_1 &= \frac{n}{k} \sum \left(1 + \frac{u_i + \frac{1}{2}k}{n}\right) \log \left(1 + \frac{u_i}{n}\right) \\ &= \sum_{m=2}^{\infty} \frac{(-1)^{m-2}}{n^{m-1}} \left\{\frac{S_{m-1}}{2(m-1)} + \frac{S_m}{km(m-1)}\right\}, \end{aligned}$$

since $S_1 = 0$ by (3.1). Again

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$$n^h(n^{-h} - \sum s_i^{-h}) = 1 - k^h \sum \{1 + (u_i/n)\}^{-h}$$
$$= 1 - k^h\left\{k + \sum_{m=1}^{\infty} (-1)^m \binom{m+h-1}{h-1} \frac{S_m}{n^m}\right\}$$

and so

$$n^{-h} - \sum s_i^{-h} = -\frac{k^{h+1} - 1}{n^h} - \sum_{m=1}^{\infty} (-1)^m \binom{m+h-1}{h-1} \frac{S_m}{n^{h+m}}.$$

Hence, if we take H odd,

$$\log P = (n + \tfrac{1}{2}k) \log k - \tfrac{1}{2}(k-1) \log (2\pi n) + \sum_{h=1}^{H-1} d_h n^{-h} + O\{n^{-H}(1 + S_{H+1})\}, \tag{4.3}$$

where

$$d_h = \sum_{m=0}^{h+1} v(k, h, m) S_m$$

is a polynomial in the u_i of degree at most $h + 1$. Now

$$\exp\left(\sum_{h=1}^{H-1} d_h n^{-h}\right) = 1 + \sum_{h=1}^{\infty} D_h n^{-h},$$

where

$$D_h = \sum_{\Sigma m_c c = h} \prod_c \left(\frac{d_c^{m_c}}{m_c!}\right),$$

the sum being taken over all partitions of h, a typical partition being into m_1 parts 1, m_2 parts 2, and so on, and the product over every different part c in the partition. Thus D_h is a polynomial in the u_i of degree at most $2h$ and

$$D_h \leqslant A_h(1 + S_{2h}). \tag{4.4}$$

Hence, by (4.3),

$$\sum {}'PT\left(-\frac{S_2}{2k^2}\right) = \frac{k^{n+\frac{1}{2}k}}{(2\pi n)^{\frac{1}{2}(k-1)}}\left\{\sum_{h=o}^{H-1} \frac{J'_h}{n^h} + O\left(\frac{J'_0 + V'_{2H}}{n^H}\right)\right\}, \tag{4.5}$$

where

$$V'_m = \sum {}'S_m T(-\tfrac{1}{2}k^{-2}S_2)$$

and

$$J'_0 = V'_0,\quad J'_h = \sum {}'D_h T(-\tfrac{1}{2}k^{-2}S_2).$$

We write

$$V_m = V_m(n) = \sum_{((n))} S_m T(-\tfrac{1}{2}k^{-2}S_2)$$

with the notation introduced in (2.3). The sum is certainly convergent and

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$$(4.6)\qquad |V_m - V'_m| \leqslant \sum_{S_2 \geqslant n^{\frac{1}{2}}} |S_m| T(-\tfrac{1}{2}k^{-2}S_2)$$
$$< A_m \sum_{t \geqslant n^{\frac{1}{2}}} t^{m+k} T(-At) < A_m T(-An^{1/4}).$$

We see that, when $m > 0$,

$$S_m(s_1, s_2, \ldots, s_h; n) = \sum (ks_i - n)^m = \sum \{k(s_i - 1) - (n-k)\}^m$$
$$= S_m(s_1 - 1, s_2 - 1, \ldots, s_k - 1; n - k)$$

and so, when $m \geqslant 0$,

$$(4.7)\qquad V_m(n) = V_m(n-k),$$

from which it follows that

$$(4.8)\qquad |V_m(n)| \leqslant \max_{|a| \leqslant \frac{1}{2}k} |V_m(a)| < A.$$

We write

$$J_0 = V_0, \quad J_h = J_h(n) = \sum_{((n))} D_h T(-\tfrac{1}{2}k^{-2}S_2),$$

the convergence of the last sum following from that of V_0 and V_{2h} by (4.4). Also

$$|J_h - J'_h| < A_h T(-An^{1/4})$$

by (4.4) and (4.6). Hence, by (4.1), (4.5), and (4.8),

$$(4.9)\qquad M_n = (2\pi n)^{-\frac{1}{2}(k-1)} k^{n+\frac{1}{2}k} T(Kn^2) \left\{ \sum_{h=0}^{H-1} J_h n^{-h} + O(n^{-H}) \right\}.$$

We can show, just as for $V_m(n)$ in the last paragraph that $J_h(n) = J_h(n-k)$ and so $|J_h(n)| < A_h$. (4.9) is now Theorem 1 with

$$C_h = C_h(k, n) = k^{\frac{1}{2}} (\log 2)^{\frac{1}{2}(k-1)} (2\pi)^{-\frac{1}{2}(k-1)} J_h$$

and $C_h(k, n) = C_h(k, a)$, if a is any residue of $n \pmod k$.

By (3.2) and (2.3),

$$J_0(n) = V_0(n) = L(n).$$

Hence, by (4.7), $L(n) = L(a)$ and (2.2) follows.

5. Proof of Theorem 3. We now evaluate $C_0(k, a)$ and $L(a)$, which are defined by (2.2) and (2.3). We suppose (2.4) satisfied and write

$$T_r = a - \sum_{i=r+1}^{k-1} s_i \qquad (0 \leqslant r \leqslant k-2), \qquad T_{k-1} = a.$$

We can prove by induction on r that, for $1 \leqslant r \leqslant h - 1$, we have

$$(5.1)\qquad s_k^2 + \sum_{i=1}^{r} s_i^2 = \sum_{i=1}^{r} \frac{\{(i+1)s_i - T_i\}^2}{i(i+1)} + \frac{T_r^2}{r+1}.$$

For $r = 1$, (5.1) reduces to

$$s_k^2 + s_1^2 = \tfrac{1}{2}(2s_1 - T_1)^2 + \tfrac{1}{2}T_1^2,$$

which is true since

$$T_1 = a - \sum_{i=2}^{k-1} s_i = s_1 + s_k.$$

If we assume (5.1) true for $r = R - 1$, its truth for $r = R$ follows provided that

$$R(R+1)s_R^2 = \{(R+1)s_R - T_R\}^2 + RT_R^2 - (R+1)T_{R-1}^2$$

and this is a trivial consequence of the fact that $T_R = T_{R-1} + s_R$.

If we put $r = k - 1$ in (5.1), we have

$$\sum s_i^2 = \sum_{i=1}^{k-1} 2A_i(s_i + y_i)^2 + (a^2/k),$$

where

$$A_i = \frac{i+1}{2i}, \qquad y_i = -\frac{T_i}{i+1} = \frac{1}{i+1}\left(\sum_{r=i+1}^{k-1} s_r - a\right).$$

Hence

$$L(a) = \sum_{((a))} T\{-\tfrac{1}{2} \sum s_i^2 + \tfrac{1}{2}(a^2/k)\} \tag{5.2}$$

$$= \sum_{s_1,\ldots,s_{k-1}=-\infty}^{\infty} T\left\{-\sum_{i=1}^{k-1} A_i(s_i + y_i)^2\right\}.$$

We have thus eliminated s_k.

We now take $x > 0$ and write

$$W(x, y) = \sum_{s=-\infty}^{\infty} e^{-x(s+y)^2}, \qquad W(x) = W(x, 0).$$

An application of Poisson's formula **(2)** gives us

$$W(x, y) = \left(\frac{\pi}{x}\right)^{\frac{1}{2}}\left\{1 + 2\sum_{t=1}^{\infty} \exp\left(-\frac{\pi^2 t^2}{x}\right)\cos 2\pi t y\right\}. \tag{5.3}$$

If we put $y = 0$ in this, we have

$$W(x) = (\pi/x)^{\frac{1}{2}} W(\pi^2/x).$$

It follows that

$$\left|\left(\frac{x}{\pi}\right)^{\frac{1}{2}} W(x, y) - 1\right| \leqslant 2\sum_{t=1}^{\infty} \exp\left(-\frac{\pi^2 t^2}{x}\right)|\cos 2\pi t y| \tag{5.4}$$

$$\leqslant 2\sum_{t=1}^{\infty} \exp\left(-\frac{\pi^2 t^2}{x}\right) = W\left(\frac{\pi^2}{x}\right) - 1.$$

We now write

$$x_i = A_i \log 2, \quad B_i = W(\pi^2/x_i) - 1,$$

so that, by (5.4), we have

$$1 - B_i < (x_i/\pi)^{\frac{1}{2}} W(x_i, y_i) \leqslant 1 + B_i.$$

We have then, by (5.2),

$$L(a) = \sum_{s_2, s_3, \ldots, s_{k-1} = -\infty}^{\infty} W(x_1, y_1) T\left\{ - \sum_{i=2}^{k-1} A_i (s_i + y_i)^2 \right\}$$
$$\leqslant \left(\frac{\pi}{x_1}\right)^{\frac{1}{2}} (1 + B_1) \sum_{s_1, \ldots, s_{k-1} = -\infty}^{\infty} T\left\{ - \sum_{i=2}^{k-1} A_i (s_i + y_i)^2 \right\},$$

since B_1 is independent of y_1 and so of $s_2, \ldots, s_{k-1}$ and all the terms in the last sum are positive. Continuing this process step by step, we find that

$$L(a) \leqslant \pi^{\frac{1}{2}(k-1)} \prod_{i=1}^{k-1} \{x_i^{-\frac{1}{2}} (1 + B_i)\}.$$

Now

$$\prod_{i=1}^{k-1} x_i = (\log 2)^{k-1} \prod_{i=1}^{k-1} A_i = \frac{k (\log 2)^{k-1}}{2^{k-1}}$$

and so

$$C_0(k, a) = k^{\frac{1}{2}} \left(\frac{\log 2}{2\pi}\right)^{\frac{1}{2}(k-1)} L(a) \leqslant \prod_{i=1}^{k-1} (1 + B_i).$$

A precisely similar argument, with inequality signs reversed, shows that

$$C_0(k, a) \geqslant \prod_{i=1}^{k-1} (1 - B_i).$$

The B_i are very easy to compute. We find that

$$B_1 = 1.3097 \times 10^{-6}, \qquad B_2 = 1.1374 \times 10^{-8}$$

and so on; in particular,

$$B_{200} < 10^{-12}.$$

Theorem 3 follows quite simply from the calculations.

On the other hand, if $k = 2$ and $s_1 + s_2 = a$,

$$s_1^2 + s_2^2 - \tfrac{1}{2} a^2 = \tfrac{1}{2}(s_1 - s_2)^2 = \tfrac{1}{2}(2s_1 - a)^2$$

and so

$$L(a) = \sum_{s=-\infty}^{\infty} 2^{-(s - \frac{1}{2}a)^2} = W(\log 2, -\tfrac{1}{2}a).$$

Hence

$$C_0(2, a) = \left(\frac{\log 2}{\pi}\right)^{\frac{1}{2}} W(\log 2, -\tfrac{1}{2}a) = 1 + 2 \sum_{m=1}^{\infty} \exp\left(-\frac{m^2 \pi^2}{\log 2}\right) \cos m\pi a$$

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by (5.3) and so

$$C_0(2, 0) - C_0(2, 1) = 4\sum_{m=0}^{\infty} \exp\left(-\frac{(2m+1)^2\pi^2}{\log 2}\right)$$
$$> 4\exp(-\pi^2/\log 2) > 2.6194 \times 10^{-6}.$$

(2.5) follows and we observe also that $C_0(2, 0)$ and $C_0(2, 1)$ differ by very nearly as much as Theorem 3 allows.

References

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