SOME 𝔖 DIVISION ALGEBRAS

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Introduction. Let K^* be an associative algebra over a field F with identity u, and let $u, e_1, e_2, \ldots,$ be a basis for K^*. Denote by K the linear space, over F, spanned by the $e_i, i = 1, 2, \ldots$. Then for x, y in K, $xy = \alpha u + a$, where $a \in K$. Define $h(x, y) = \alpha$ and $x \cdot y = a$. With respect to the operation $\cdot$ thus defined, K becomes an algebra over F satisfying

$$(1) \qquad (x \cdot y) \cdot z - x \cdot (y \cdot z) = h(y, z)x - h(x, y)z.$$

Further, the bilinear form $h(x, y)$ is associative on K. Any algebra, over a field F, which possesses an associative bilinear form $h(x, y)$ and satisfies (1) will be called a 𝔖 algebra. It is not difficult to show that any 𝔖 algebra K can be obtained from a unique associative algebra K^* with identity by the process described above. The algebra K^* will be called the associated associative algebra of K.

A well-known example of a 𝔖 algebra is the Lie algebra of the Euclidean three-dimensional rotation group (obtained from the real quaternions, with a suitable basis, by the process described above). This paper is concerned with the existence of 𝔖 division algebras which are not associative (every associative algebra is obviously a 𝔖 algebra). It will be shown that several non-associative division algebras which appear in the literature are isotopes of 𝔖 division algebras. The observation that a certain three-dimensional division algebra of Dickson **(3)** is an isotope of a 𝔖 division algebra leads to the construction of a new central division algebra of dimension nine. This is the main result.

It is apparent from the process, described above, for obtaining 𝔖 algebras that two non-equivalent 𝔖 algebras may have equivalent associated associative algebras. It can be shown that if K and L are 𝔖 algebras then K^* is equivalent to L^* if and only if there exist a one to one linear mapping T of K onto L and a linear functional f on K such that

$$(x o y)T = f(x)\ (yT) + f(y)\ (xT) + xT * yT$$

holds for all x, y in K, where o, $*$ are the multiplications in K and L respectively. Two algebras related in this way are said to be pseudo-equivalent.

1. A necessary condition. Theorem 1 gives a necessary condition for a 𝔖 division algebra. It seems appropriate to include Theorem 2, since all of the algebras described in §2 satisfy its hypotheses.

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The following lemmas are used in the proof of Theorem 1.

LEMMA 1. *Let A be an associative algebra of (finite) dimension $n \geqslant 2$ over a field F. If A contains no proper right ideals then A is a division algebra.*

Proof. Suppose A is not a division algebra. Then there exist in A elements $a \neq 0 \neq b$ such that $ab = 0$. Let $B = \{x \in A | ax = 0\}$.

Clearly B is a right ideal and $B \neq 0$, hence $B = A$. Thus $ax = 0$ for all x in A. But then the linear subspace spanned by a is a right ideal of dimension one. This contradiction proves the lemma.

The proof of the following lemma is obtained by a simple computation and is omitted.

LEMMA 2. *Let K be a $\mathfrak{G}$ algebra over a field F with bilinear form $h(x, y)$, and K^* the set of all pairs (ξ, x), $\xi \in F$, $x \in K$. If addition and multiplication are defined for K^* by*

$$(\xi, x) + (\eta, y) = (\xi + \eta, x + y),$$
$$(\xi, x) \cdot (\eta, y) = (\xi\eta + h(x, y), \xi y + \eta x + xy),$$

then K^* is an associative algebra isomorphic to the associated algebra of K.

LEMMA 3. *Let K be a $\mathfrak{G}$ division algebra of dimension $n > 2$ over F and K^* the associated associative algebra. If B^* is a proper right (left) ideal in K^* then the dimension of B^* is either n or 1. Further, K^* contains at most one right (left) ideal of dimension n and at most one of dimension 1.*

Proof. Let $e_1, \ldots, e_n$ be a basis for K. Then, by Lemma 2, $(1, 0)$, $(0, e_1)$, $\ldots$, $(0, e_n)$ is a basis for K^*. If B^* is a proper right ideal then it has a basis (α_1, f_1), (α_2, f_2), $\ldots$, (α_t, f_t), where $t \leqslant n$. Suppose all of the $\alpha_i = 0$, then,

$$(0, f_i)(0, a) = (h(f_i, a), f_i a) = \left(0, \sum_{j=1}^{t} \gamma_{ij} f_j\right),$$

for $i = 1, \ldots, t$, and all a in K. Thus the subspace of K spanned by the f_i is a right ideal in K. It follows that $t = n$, and incidentally that K is associative. If $t < n$ then not all of the $\alpha_i = 0$. Without loss of generality suppose that $\alpha_1 = 1$, then $(1, g_1)$, $(0, g_2)$, $\ldots$, $(0, g_t)$ is a basis for B^*, where $g_1 = f_1$, $g_i = f_i - \alpha_i f_1$, $i = 2, \ldots, t$. Since $t < n$ there is an x in K independent of $g_1, \ldots, g_t$. Suppose $t > 1$, and let z be the solution in K of the equation $g_2 z = x$. Clearly $(0, g_2)$ $(0, z) = (h(g_2, z), x)$ is an element of B^*. Thus

$$x = \sum_{i=1}^{t} \mu_i g_i,$$

contrary to the choice of x. Hence $t = 1$, and this completes the first part of the lemma.

If K^* has two right ideals of dimension 1, say B^*_1 and B^*_2 then the sum $B^*_1 + B^*_2$ is a right ideal of dimension 2, which is not possible since $n > 2$.

Further, if K^* has two right ideals of dimension n, say C^*_1 and C^*_2 then since $C^*_1 + C^*_2 = K^*$ it follows that the right ideal $C^*_1 \cap C^*_2$ has dimension $n - 1$. Again this is impossible since $n > 2$. This completes the proof of the lemma.

LEMMA 4. *With K and K^* defined as in Lemma 3 let B^* be a right ideal of dimension n in K^*. If C^* is a proper right ideal in B^* then C^* is a two-sided ideal in K^*.*

Proof. First note that any right ideal in B^* is a right ideal in K^*. Thus, since C^* has dimension $< n$, it follows from Lemma 3 that the dimension of C^* is 1. Let $v^*, w^*_2, \ldots, w_n^*$ be a basis for B^*, where v^* is a basis for C^*. Since $n > 2$ there is a b^* in B^*, $b^* \notin C^*$ such that $v^*b^* = 0$. Let $Z^* = \{z^* \in B^* | v^*z^* = 0\}$, then Z^* is a right ideal in B^* and hence in K^*. Clearly $Z^* \neq 0$, $Z^* \neq K^*$, and hence by Lemma 3 has dimension 1 or n. Now, since $b^* \in Z^*$, $b^* \notin C^*$, $Z^* \neq C^*$. Thus, by Lemma 3, Z^* does not have dimension 1. It follows that $Z^* = B^*$ or $v^*x^* = 0$ for all x^* in B^*. Consider now the left ideal $K^* v^*$ of K^*. Any element a^* in K^* can be written $a^* = \alpha 1^* + a^*_1$, where $a^*_1 \in B^*$. Thus, $v^*a^* = \alpha v^*$ or $(K^*v^*)\, a^* \subseteq K^*v^*$ and hence K^*v^* is a two-sided ideal in K^*. Clearly, $C^* \subseteq K^*v^*$ and $K^*v^* \neq K^*$, for otherwise there is an a^* in K^* such that $a^*v^* = 1^*$ or $b^* = a^*v^*b^* = 0$, a contradiction. Thus, by Lemma 3, K^*v^* is either C^* or B^*. Suppose $K^*v^* = B^*$, then $B^* \cdot B^* = K^*v^*B^* = 0$. But this implies that K contains divisors of zero, a contradiction. Thus, $C^* = K^*v^*$ is a two-sided ideal in K^*.

THEOREM 1. *If K is a $\mathfrak{G}$ division algebra of (finite) dimension $n > 2$ over a field F and K^* the associated associative algebra then either K^* is a division algebra or $K^* = F u^* \oplus A^*$ where A^* is a division algebra. In the latter case K is pseudo-equivalent to A^*.*

Proof. Suppose first that K^* contains no right ideals of dimension n. Let C^* be a right ideal of dimension 1 with basis v^*. Clearly there exists a $y^* \notin C^*$ such that $v^*y^* = 0$. Let $Z^* = \{z^* \in K^* | v^*z^* = 0\}$, then Z^* is a right ideal in K^*. Further, $Z^* \neq C^*$, $Z^* \neq 0$, $Z^* \neq K^*$, and hence by Lemma 3 has dimension n contrary to the assumption that K^* contains no right ideals of dimension n. Thus, in this case, K^* is a division algebra.

Suppose next that K^* is not a division algebra and hence contains exactly one right ideal A^* of dimension n. It will be shown that A^* is a division algebra. Thus, suppose that A^* contains a proper right ideal C^*. By Lemma 3 the dimension of C^* is 1. Let $(\alpha_1, f_1), (\alpha_2, f_2), \ldots, (\alpha_n, f_n)$ be a basis for A^* where (α_1, f_1) is a basis for C^*. As in the proof of Lemma 4, $(\alpha_1, f_1)\ (\xi, x) = (0, 0)$ for all (ξ, x) in A^*. Further, since $(\alpha_1, f_1)^2 = (0, 0)$, it follows that $\alpha_1 \neq 0$. Hence A^* has a basis $(1, f), (0, g_2), \ldots, (0, g_n)$ where $(1, f)$ is a basis for C^* and $f, g_2, \ldots, g_n$ span K. Now, by Lemma 4, C^* is a two-sided ideal in A^*. Thus, since A^* contains no proper right ideals other than C^*, it follows that A^*/C^*

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contains no proper right ideals. Hence by Lemma 1 A^*/C^* is a division algebra, and has a basis

$$[(0, g_2) + C^*], \ldots, [(0, g_n) + C^*].$$

Clearly A^*/C^* has an identity, say, $[(0, e) + C^*]$, and hence

$$[(0, e) + C^*]\cdot[(0, x) + C^*] = [(0, x) + C^*]$$

for all $[(0, x) + C^*]$ in A^*/C^*. Thus, $(h(e, x), e\cdot x) = (0, x) + \eta(1, f)$, or $ex = x + \eta f$ for all x in the subspace of K spanned by $g_2, \ldots, g_n$. Since $n > 2$, $eg_i = g_i + \eta_i f$, $(i = 2, 3)$, or $ey = y$, where

$$y = \eta_2^{-1}g_2 - \eta_3^{-1}g_3 \neq 0.$$

But, $(1, f)\ (0, y) = (0, 0)$ or $y + fy = 0$, whence $(-f)y = y$. Thus, $e + f = 0$ contrary to the linear independence of $f, g_2, \ldots, g_n$ over F. This proves that A^* contains no proper right ideals and hence, by Lemma 1, is a division algebra. If e^* is the identity in A^* and 1^* the identity in K^*, then $u^* = 1^* - e^*$ is a non-zero idempotent orthogonal to every element of A^* and $u^*, e^*, e^*_2, \ldots, e^*_n$ is a basis for K^* where $e^*, e^*_2, \ldots, e^*_n$ is a basis for A^*. Thus, $K^* = Fu^* \oplus A^*$. Since $1^*, e^*, e^*_2, \ldots, e^*_n$ is also a basis for K^*, it follows that K is pseudo equivalent to A^*.

In connection with Theorem 1 there are two open questions: (i) Is the theorem true when the dimension of K over F is not finite? (ii) Are there any $\mathfrak{G}$ division algebras K for which K^* is a division algebra? This second question indicates that all of the examples of $\mathfrak{G}$ division algebras described in this paper are pseudo-equivalent images of associative division algebras. With this in mind the next theorem is proved.

THEOREM 2. *Let A be an associative division algebra of dimension >2 over a field F and $f(x)$ a non-trivial linear mapping of A into F, with $f(1) = \alpha \neq -1, -\frac{1}{2}$. Let $A(\text{o})$ be the pseudo-equivalent image of A with multiplication defined by $x\text{o}y = f(x)y + f(y)x + xy$. The isotope $A(*)$ of $A(\text{o})$ defined by $x * y = xU^{-1}\,\text{o}y\,U^{-1}$, where U is the non-singular linear transformation $x \to x\text{o}1$, is central over F.*

Proof. It follows from the restriction $f(1) = \alpha \neq -1, -\frac{1}{2}$ (in case the characteristic of F is 2, $f(1) \neq 1$ is the requirement) that the linear transformation U is non-singular. Let $\gamma = (2\alpha + 1)^{-1}$, and then a simple computation shows that the product $x * y$ in terms of the multiplication xy in A is given by

$$(2) \qquad x * y = (\alpha + 1)^{-2}[xy + \alpha\gamma(f(x)y + f(y)x) - \gamma f(x)f(y)].$$

Note that $1\text{o}1 = (2\alpha + 1)$ is the identity for $A(*)$.

Suppose first that $\alpha = 0$. Then (2) becomes

$$x * y = xy - f(x)f(y),$$

and the proof that $A(*)$ is central can be found in Albert (**1**, p. 298). Albert's proof is given for an algebra A of finite dimension > 2 over a field F of characteristic 2, but it is easily seen to be valid in this more general case.

To complete the proof, suppose $\alpha \neq 0$, and let c be any element in the centre of $A(*)$. Since the dimension of $A(*) > 2$, there exists an $x \neq 0$ with $f(x) = f(cx) = 0$. Clearly there exists a basis $1, \{e_\alpha\}$ for A with $f(e_\alpha) = 0$ for all α. Then, since A is a division algebra, $x, \{x\ e_\alpha\}$ is also a basis for A. Suppose that $f(xy) = 0$ for all y in A. Then $f(xe_\alpha) = 0$ for all α, and hence, since $f(x) = 0$, it follows that $f(z) = 0$ for all z in A, a contradiction. Thus, there exists a y such that $f(xy) \neq 0$. With this choice for c, x, y equate $(c * x) * y$ and $c * (x * y)$ to obtain $\alpha\gamma f(xy)c = \gamma f(c)f(xy)$. This implies that $c = \alpha^{-1}f(c)$ is a scalar multiple of the identity in $A(*)$.

2. Some 𝔖 division algebras. Each of the five division algebras described below is obtained in the following way: start with an associative division algebra A over a field F and let $A(\text{o})$ be the pseudo-equivalent algebra with multiplication defined by $x\text{o}y = f(x)y + f(y)x + xy$ for a suitable linear functional $f(x)$ defined on A. Denote by U the linear transformation of A defined by $xU = x\text{o}1$ (1 the identity in A), and let $A(*)$ be the isotope of $A(\text{o})$ defined by $x * y = x\ U^{-1}\text{o}y\ U^{-1}$. In the following examples A and $f(x)$ will be chosen so that $A(\text{o})$ is a division algebra (without identity) and hence $A(*)$ a division algebra with identity $1\text{o}1 = 1 + 2f(1)$. It follows from Theorem 2 that each $A(*)$ is central. The algebras numbered (ii) and (v) appear to be new. References are given for the other three. The following lemma will be used in the construction of each of the five algebras.

LEMMA 5. *Let A be an associative division algebra and $A(\text{o})$ the pseudo-equivalent image with multiplication $x\text{o}y = f(x)y + f(y)x + xy$, where $f(1) \neq -1$. If $A(\text{o})$ contains proper divisors of zero then there exist x, y in $A(\text{o})$ with $f(x) = f(y) = 1$ such that $x\text{o}y = 0$.*

Proof. Choose $x' \neq 0 \neq y'$ so that $x'\text{o}y' = 0$. Clearly $f(x'), f(y')$ cannot both be zero. Suppose $f(x') = 0$, then $f(y')x' + x'y' = 0$, or $x'(f(y') + y') = 0$. This implies $y' = -f(y'), f(y') = -f(y')\,f(1)$, or $f(1) = -1$, a contradiction. Thus $f(x') \neq 0$ and similarly $f(y') \neq 0$. Let $x = [f(x')]^{-1}x'$, $y = [f(y')]^{-1}y'$ so that $f(x) = f(y) = 1$. Clearly $x\text{o}y = 0$.

(i) Let F be an ordered field and A a quaternion division algebra over F in which the norm $N(x) = \bar{x}x = x\bar{x}$ is a positive definite quadratic form. Let $f(x) = x + \bar{x}$ and $A(\text{o})$ the pseudo-equivalent image described in Lemma 5. Since $f(1) = 2 \neq -1$, it follows from Lemma 5 that either $A(\text{o})$ is a division algebra or $x\text{o}y = 0$ for some x, y with $f(x) = f(y) = 1$. The latter assumption implies $(1 + x)(1 + y) = 1$. Further, $f(1 + x) = f(1 + y) = 3$, so that $N(1 + x) \geqslant 9/4$, $N(1 + y) \geqslant 9/4$. Hence

$$1 = N[(1 + x)(1 + y)] = N(1 + x)\cdot N(1 + y) \geqslant 81/16,$$

a contradiction. Thus $A(\circ)$ is a division algebra. The isotope $A(*)$, described at the beginning of this section, closely resembles one of the quasigroup division algebras obtained by Bruck (**2**, p. 179) using the four group.

(ii) Let F be a field of characteristic $\neq 2, 3$, and $F(x)$ the field obtained by adjoining a single indeterminate. Any element $r(x)/q(x)$, where r and q are relatively prime polynomials, may be written in the form

$$\frac{r}{q} = t + \frac{p}{q},$$

where $\deg p < \deg q$ or $p = 0$.

Define

$$f\left(\frac{r}{q}\right) = t(0),$$

the constant term of the polynomial $t(x)$ and note that

$$\frac{r}{q} \rightarrow f\left(\frac{r}{q}\right)$$

is a linear mapping of $F(x)$ onto F, with $f(1) = 1$. With this linear functional and $A = F(x)$, define $A(\circ)$ as in Lemma 5. Since $f(1) = 1 \neq -1$, it follows from Lemma 5 that if $A(\circ)$ has proper divisors of zero then $a \circ b = 0$ for some $a, b \in A$ with $f(a) = f(b) = 1$. Then $(1 + a)(1 + b) = 1$ and $f(1 + a) = f(1 + b) = 2$. Hence

$$1 + a = x\, g(x) + 2 + \frac{p}{q} = t + \frac{p}{q}.$$

But

$$(1 + b) = (1 + a)^{-1} = \frac{q}{tq + p},$$

and if $g(x) \neq 0$, then $\deg q < \deg(tq + p)$ which implies $f(1 + b) = 0$, a contradiction. If $g(x) = 0$, then

$$1 + b = \frac{q}{2q + p}$$

and $f(1 + b) = \frac{1}{2}$. But $2 = \frac{1}{2}$ implies $3 = 0$, a contradiction. Thus, $A(\circ)$ contains no proper divisors of zero.

To see that the equation $a \circ y = b$, $0 \neq a$, $b \in A(\circ)$, has a solution, first suppose $f(a) = 0$. A simple computation shows that $y = b/a - \frac{1}{2}[f(b/a)]$ is a solution. If $f(a) \neq 0$ it may be assumed that $f(a) = 1$. Then $a \neq -1$ so that $a(1 + a)^{-1}$ exists. Let

$$a = xg(x) + 1 + \frac{p}{q} = t + \frac{p}{q},$$

then

$$1 + a = (1 + t) + \frac{p}{q} = \frac{(1 + t)q + p}{q}$$

and

$$\frac{a}{1+a} = \frac{tq+p}{(1+t)q+p}.$$

Now

$$\gamma = f\left(\frac{a}{1+a}\right) = \begin{cases} 1, & g(x) \neq 0, \\ \frac{1}{2}, & g(x) = 0. \end{cases}$$

In either case $\gamma \neq -1$, so that $(1+\gamma)^{-1}$ exists in F. A simple computation shows that

$$y = \frac{b - (1+\gamma)^{-1} f\left(\frac{b}{1+a}\right) a}{1+a}$$

is a solution of the equation $aoy = b$. Since $A(o)$ is commutative and contains no divisors of zero it follows that $A(o)$ is a division algebra. The isotope $A(*)$, as defined above, has an identity $1o1 = 3$.

(iii) Let F be a field of characteristic two such that there exists a purely inseparable extension field A of dimension $2^r > 2$ and degree 2 over F. Let $x \to f(x)$ be any non trivial linear mapping of A into F such that $f(1) = 0$, and define $A(o)$ as in Lemma 5. Suppose $A(o)$ is not a division algebra. Then, by Lemma 5, $A(o)$ contains x, y with $f(x) = f(y) = 1$ and $xoy = 0$, that is,

$$x + y + x \cdot y = 0. \tag{3}$$

From (3) it follows that $f(xy) = f(x) + f(y) = 1 + 1 = 0$. Multiply (3) on the left by x, and note that $x^2 = \alpha \in F$, to obtain

$$\alpha + xy + \alpha y = 0. \tag{4}$$

From (4) it follows that

$$\alpha = f(\alpha y) = f(\alpha + xy + \alpha y) = f(0) = 0,$$

a contradiction. Thus $A(o)$ is a division algebra. In this example $xU = xo1 = f(x) + x$, $xU^2 = x$, whence $xU^{-1} = f(x) + x$ and the multiplication $x * y$ in $A(*)$ in terms of the multiplication in A is given by

$$x * y = xU^{-1}oyU^{-1} = (f(x) + x)\,o\,(f(y) + y) = f(x)f(y) + xy.$$

The algebra $A(*)$ was constructed by Albert **(1)** who showed that it is a central division algebra, thereby establishing the existence of central commutative division algebras of degree two and characteristic two.

For the next two algebras the following lemma will be needed.

LEMMA 6. *Let A be an associative division algebra of degree 3 over a field F of characteristic $\neq 2$. For $x \in A$ let $\lambda^3 - \tau(x)\lambda^2 + \alpha(x)\lambda - \nu(x) = 0$ be the equation satisfied by x. If $\tau(x) = 1$ then $\tau(x^{-1}) \neq 1$.*

Proof. If $\tau(x) = 1$, then $x^3 - x^2 + \alpha(x)x - \nu(x) = 0$ where $\nu(x) \neq 0$. Multiply by $-[\nu(x)]^{-1}x^{-3}$ to obtain

$$-[\nu(x)]^{-1} + [\nu(x)]^{-1} x^{-1} - \alpha(x)[\nu(x)]^{-1}x^{-2} + x^{-3} = 0.$$

Thus x^{-1} satisfies the equation

$$\lambda^3 - \alpha(x)[\nu(x)]^{-1}\lambda^2 + [\nu(x)]^{-1}\lambda - [\nu(x)]^{-1} = 0,$$

so that $\tau(x^{-1}) = \alpha(x)[\nu(x)]^{-1}$. Suppose $\tau(x^{-1}) = 1$, then $\alpha(x) = \nu(x)$ and the equation satisfied by x is $\lambda^3 - \lambda^2 + \nu(x)\lambda - \nu(x) = 0$. This implies that $x = 1$, whence $\tau(x) = 3$ or $1 = 3$, a contradiction, since the characteristic of F is not 2. Thus $\tau(x^{-1}) \neq 1$.

(iv) Let F be a field of characteristic $\neq 2$, which has a cubic extension A. With $\tau(x)$ defined as in Lemma 6, let $f(x) = -\frac{1}{2}[\tau(x)]$, and note that $f(1) = -3/2$. With this $f(x)$ define A(o) as in Lemma 5. If A(o) is not a division algebra then, by Lemma 5, A(o) contains x and y with $f(x) = f(y) = 1$ and $xoy = 0$. This implies that $(1 + x)(1 + y) = 1$. But $f(1 + x) = f(1 + y) = -\frac{1}{2}$, so that $\tau(1 + x) = \tau(1 + y) = 1$, contrary to Lemma 6 since $(1 + y) = (1 + x)^{-1}$. The isotope $A(*)$ has an identity $1o1 = -2$ and has been studied extensively by Dickson **(3)**.

(v) Let A be a cyclic division algebra of degree 3 over a field F of characteristic $\neq 2$. As in the preceeding example let $f(x) = -\frac{1}{2}[\tau(x)]$, $\tau(x)$ defined as in Lemma 6. Again define A(o) as in Lemma 5. The proof that A(o) is a division algebra is the same as the proof above in (iv). The isotope $A(*)$ contains a subalgebra isomorphic to Dickson's algebra of dimension 3 described in (iv).

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