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Constructing Double Magma on Groups Using Commutation Operations

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Abstract. A *magma* $(M, \star)$ is a nonempty set with a binary operation. A *double magma* $(M, \star, \bullet)$ is a nonempty set with two binary operations satisfying the *interchange law* $(w \star x) \bullet (y \star z) = (w \bullet y) \star (x \bullet z)$. We call a double magma *proper* if the two operations are distinct, and *commutative* if the operations are commutative. A *double semigroup*, first introduced by Kock, is a double magma for which both operations are associative. Given a non-trivial group G we define a system of two magma $(G, \star, \bullet)$ using the commutator operations $x \star y = [x, y] (= x^{-1}y^{-1}xy)$ and $x \bullet y = [y, x]$. We show that $(G, \star, \bullet)$ is a double magma if and only if G satisfies the commutator laws $[x, y; x, z] = 1$ and $[w, x; y, z]^2 = 1$. We note that the first law defines the class of 3-metabelian groups. If both these laws hold in G, the double magma is proper if and only if there exist $x_0, y_0 \in G$ for which $[x_0, y_0]^2 \neq 1$. This double magma is a double semigroup if and only if G is nilpotent of class two. We construct a specific example of a proper double semigroup based on the dihedral group of order 16. In addition, we comment on a similar construction for rings using Lie commutators.

1 Introduction

Definition A *double magma*, $(M, \star, \bullet)$, is a triple consisting of a non-empty set, M, and two binary operations, $\star$ and $\bullet$, defined on M satisfying the *interchange law*:

$$(w \star x) \bullet (y \star z) = (w \bullet y) \star (x \bullet z),$$

for all $w, x, y, z \in M$

We will call a double magma *proper* if its operations are distinct and *improper* if its operations are identical. Our goal here is to construct classes of examples of proper double magma. We refer to a double magma as *commutative*, *associative*, or *unitary* when both of its operations are, respectively, commutative, associative, or unitary (*i.e.*, have an identity element). A *double semigroup* is an associative double magma.

In [3] Kock introduces the notion of a double semigroup in relation to two-fold monoidal categories. He proves that both cancellative double semigroups and inverse double semigroups must be commutative. In this note we will discuss a natural construction of double magma based on a group with operations defined in terms of commutation. We will determine (Theorem 3.7) exact conditions on the group so that our construction yields a proper double magma and a proper double semigroup. It is hoped that the rich source of examples of double magma thus produced will prove useful to researchers studying these objects.

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In [2] Eckmann and Hilton, prove that every unitary double magma is an improper, commutative, double semigroup with the same identity element for each operation. In light of this theorem, if we are to produce proper examples, we must be sure that not both magma are unitary. It is possible, however, for one of the two magma to have an identity element and still produce a proper double magma. Let $D = \{a, b\}$ and define operations on D by the tables below.

$\star$	a	b
a	a	a
b	a	b

$\bullet$	a	b
a	a	a
b	b	b

It is easy to check that $(D, \star)$ is unitary and that $(D, \star, \bullet)$ forms a proper, noncommutative, double semigroup. Probably the most natural example of a double magma in elementary mathematics is the set of integers with addition and subtraction. Note that since subtraction is not associative, this forms a proper, noncommutative, double magma but not a double semigroup. Similarly natural is the set of nonzero rationals with multiplication and division.

The fundamental idea of our constructions is to begin with any group G and a 2-variable word $W(a, b) \in F_2$, the free group of rank two freely generated by a and b, and define two binary operations on G,

$$x \star y = W(x, y) \quad \text{and} \quad x \bullet y = W(y, x),$$

for each $x, y \in G$. To ensure a double magma, we must impose the interchange law, $(w \star x) \bullet (y \star z) = (w \bullet y) \star (x \bullet z)$ in this context. Thus the following law must hold in G,

$$W\big(\, W(y, z), W(w, x)\big) = W\big(\, W(y, w), W(z, x)\big).$$

As a simple example of how our construction works, suppose $W = ab^{-1}$. The interchange law becomes $yz^{-1}xw^{-1} = yw^{-1}xz^{-1}$. Letting x and y be 1, we obtain commutativity of G. Note that if G is commutative, the interchange law holds; thus we conclude that $(G, \star, \bullet)$ is a double magma if and only if G is an abelian group. To determine if this is proper, we must investigate the law $W(x, y) = W(y, x)$. From $xy^{-1} = yx^{-1}$ we deduce that $(xy^{-}1)^2 = 1$, and, letting $y = 1$, we see that G must be of exponent two. Thus we have shown that $W = ab^{-1}$,$(G, \star, \bullet)$ is a double magma if and only if G is abelian, and this is proper exactly when G is not of exponent two. If G is an abelian group and not of exponent two, we note that neither operation of the proper double magma constructed is commutative. Thus we have a class of examples, one for each abelian group not of exponent two, yielding noncommutative proper double magma. If either operation is associative, then G is of exponent two; therefore this construction will not produce a proper double semigroup. To give a specific example, we can select G to be the cyclic group of order three, $C_3 = \langle a; a^3 = 1\rangle$. In this case the operation tables generated by the construction are as follows.

$\star$	1	a	a^2
1	1	a^2	a
a	a	1	a^2
a^2	a^2	a	1

$\bullet$	1	a	a^2
1	1	a	a^2
a	a^2	1	a
a^2	a	a^2	1

The remainder of this note will be an investigation of the case in which $W = [a, b]$, the commutator of a and b. Here we will produce infinitely many nontrivial examples of proper double magma and proper double semigroups.

2 Preliminaries

Let G be a group. For each $x, y \in G$, we represent the action of y on x by conjugation as $x^y = y^{-1}xy$ and we write $x^{-y} = (x^{-1})^y = (x^y)^{-1}$. The commutator of x and y is defined as $[x, y] = x^{-1}y^{-1}xy$. We abbreviate $[[x, y], z]$ by $[x, y, z]$ and, more generally, $[[x_1, \dots, x_{n-1}], x_n]$ by $[x_1, \dots, x_n]$. Further, we denote the commutator $[[w, x], [y, z]]$ by $[w, x; y, z]$ and, more generally,

$$\big[[x_1, y_1; x_2, y_2; \dots; x_{n-1}, y_{n-1}], [x_n, y_n]\big] = \big[[x_1, y_1; x_2, y_2; \dots; x_n, y_n]\big].$$

The subgroup of G generated by $\{[x_1, \dots, x_n] : x_i \in G (1 \le i \le n)\}$ is the n-th term of the lower central series of G, denoted $\gamma_n(G)$ with $\gamma_1(G) = G$. If $\gamma_{n+1}(G) = \{1\}$, we say G is *nilpotent of class n*. Note that $\gamma_2(G) = G'$, the derived group of G. The second derived group of G, denoted G'' is the subgroup of G generated by the set $\{[w, x; y, z] : w, x, y, z \in G\}$. If $[w, x; y, z] = 1$, we refer to G as *metabelian*. Then G is 3-*metabelian* if all of its three generator subgroups are metabelian. In [4] I. D. Macdonald showed that the single law $[x, y; x, z] = 1$ defines the variety (equational class) of 3-metabelian groups precisely. Bachmuth and Lewin [1] proved that the law $[x, y, z][y, z, x][z, x, y] = 1$ also defines the variety of 3-metabelian groups.

The following identities hold for all $x, y, z \in G$:

(Ii) $[x, y] = x^{-1}x^y = y^{-x}y$,
(Iii) $[x, y]^{-1} = [y, x]$,
(Iiii) $[x^{-1}, y] = [x, y]^{-x^{-1}}$ and $[x, y^{-1}] = [x, y]^{-y^{-1}}$,
(Iiv) $[xy, z] = [x, z]^y[y, z] = [x, z][x, z, y][y, z]$,
(Iv) $[x, yz] = [x, z][x, y]^z = [x, z][x, y][x, y, z]$.

3 Construction

Given a group G, we define two binary operations $\star$ and $\bullet$ on G as follows. For each $x, y \in G$,

$$x \star y = [x, y] \quad \text{and} \quad x \bullet y = [y, x].$$

Proposition 3.1 *For any group G the following statements are equivalent.*

(i) *$(G, \star)$ is commutative.*
(ii) *$(G, \bullet)$ is commutative.*
(iii) *For each $x, y \in G$, $x \star y = x \bullet y$.*
(iv) *For each $x, y \in G$, $[x, y]^2 = 1$.*

Proof We will establish that each of the first three statements is equivalent to the fourth. Note first that $\star$ is commutative if and only if $[x, y] = [y, x]$ for each $x, y \in G$. Applying (Iii) we obtain $[x, y] = [y, x] = [x, y]^{-1}$, which is equivalent to $[x, y]^2 = 1$ in G. The other two equivalences are derived similarly. ∎

Note that if $(G, \star, \bullet)$ is a double magma, then the equivalence of statements (iii) and (iv) in Proposition 3.1 can be interpreted as saying that this double magma is proper if and only if there are $x_0, y_0 \in G$ such that $[x_0, y_0]^2 \neq 1$.

Proposition 3.2 *Each of* $\star$ *and* $\bullet$ *is associative on G if and only if* $[x, y, z][y, z, x] = 1$, *for each* $x, y, z \in G$.

Proof Associativity of the first operation, $(x \star y) \star z = x \star (y \star z)$, translates into commutators as $[[x, y], z] = [x, [y, z]]$. By (Iii) we have $[x, [y, z]] = [[y, z], x]^{-1}$, which is equivalent to $[x, y, z][y, z, x] = 1$. The result follows in the same manner for "$\bullet$". ■

Note that in both propositions the conditions being investigated turn out to be "varietal" in the group G. Something close to this will occur for the interchange law, which, according to the definitions of our operations, translates into the commutator law

(CI) $$[w, x; y, z] = [x, y; x, z].$$

We will refer to this law as the *commutator interchange law*. If this holds for a group G, its impact on the structure of G is interesting. We will show first that (CI) implies that G is 3-metabelian. But (CI) is not equivalent to 3-metabelian. We will identify the class of groups determined by (CI) precisely in Theorem 3.6. This will be proved by commutator calculations, for which we will need to establish some preliminary lemmas.

Lemma 3.3 *For any group G, the following laws are equivalent.*

(3Mi) $[x, y; x, z] = 1$.
(3Mii) $[x, y; y, z] = 1$.
(3Miii) $[x, y; z, y] = 1$.
(3Miv) $[x, y; [x, z]^u] = 1$.

Proof We will show that (3Mi) is equivalent to (3Mii) and (3Mii) is equivalent to (3Miii). Then we will show that (3Mi) is equivalent to (3Miv). By identities (Iii) and (Iiii) we have $[x, y; x, z] = [[y, x]^{-1}; x, z] = [y, x; x, z]^{-[y,x]^{-1}}$. Thus, if either (3Mi) or (3Mii) holds, it implies the other. Similarly, if we begin with (3Mii) and apply (Iii) and (Iiii), we obtain $[x, y; y, z] = [x, y; [z, y]^{-1}] = [x, y; z, y]^{-[z,y]^{-1}}$. Thus, (3Mii) and (3Miii) are equivalent. To see that (3Mi) is equivalent to (3Miv), note that (3Mi) follows from (3Miv) letting $u = 1$. Next suppose that $[x, y; x, z] = 1$. Substituting zu for z and applying (Iv) twice, we obtain

$$1 = [x, y; x, zu] = \big[x, y; [x, u][x, z]^u\big] = \big[x, y; [x, z]^u\big][x, y; x, u]^{[x,z]^u}.$$

Note that (3Mi) implies that $[x, y; x, u]^{[x,z]^u} = 1$; thus, (3Miv) follows. ■

The labels on the laws above are to remind the reader of Macdonald's result that each of these laws defines the 3-metabelian variety.

Lemma 3.4 *If G is* 3*-metabelian, then the following laws hold in G.*

(L1) $[x, y, z; x, u] = 1$.
(L2) $[x, y; x, u, v] = 1$.
(L3) $[x, y, z; [x, u]]^v = 1$.
(L4) $[x, y, z; y, u] = 1$.
(L5) $[x, y, z; x, u, v] = 1$.

Proof To establish (L1) we begin with (3Mi) in the form $1 = [x, y; x, u]$ and substitute yz for y. Applying (Iiv) twice, we obtain

$$\begin{aligned}1 &= [x, yz; x, u] = \big[[x, z][x, y][x, y, z]; x, u\big]\\ &= [x, z; x, u]^{[x,y][x,y,z]}[x, y; x, u]^{[x,y,z]}[x, y, z; x, u].\end{aligned}$$

By (3Mi) the first two factors are trivial, therefore our result follows. Given (L1) we can show (L2) is an equivalent law using (Iii):

$$1 = [x, y, z; x, u] = [x, u; x, y, z]^{-1}.$$

If we begin with (L1) in the form, $[x, y, z; x, uv] = 1$, and apply (Iv) twice we have,

$$1 = [x, y, z; x, uv] = \big[x, y, z; [x, v][x, u]^v\big] = \big[x, y, z; [x, u]^v\big][x, y, z; x, v]^{[x,u]^v}.$$

Since the second commutator is trivial by (L1), we have established (L3). To establish (L4) we apply (Ii) and then apply (Iiv) and (Iiii) repeatedly:

$$\begin{aligned}[x, y, z; y, u] &= [y^{-x}y, z; y, u] = \big[[y^{-x}, z]^y[y, z]; y, u\big]\\ &= \big[[y^{-x}, z]^y; y, u\big]^{[y,z]}[y, z; y, u].\end{aligned}$$

The second commutator is trivial by (3Mi), therefore we have

$$\begin{aligned}[x, y, z; y, u] &= \big[[y^{-x}, z]^y; y, u\big]^{[y,z]} = \big[[y^x, z]^{-y^{-x}y}; y, u\big]^{[y,z]}\\ &= \big[[y^x, z]^{-[x,y]}; y, u\big]^{[y,z]} = \big[[y, z^{x^{-1}}]^{x[x,y]}; y, u\big]^{-c[y,z]}\\ &= \big[y, u; [y, z^{x^{-1}}]^{x[x,y]}\big]^{c[y,z]},\end{aligned}$$

where $c = [y, z^{x^{-1}}]^{-x[x,y]}$. Note that this last commutator is a conjugate of an instance of (3Miv) and is therefore trivial. This establishes (L4). We derive (L5) from (L1), with u replaced by uv, applying (Iv) repeatedly.

$$\begin{aligned}1 &= [x, y, z; x, uv] = \big[x, y, z; [x, v][x, u][x, u, v]\big]\\ &= [x, y, z; x, u, v]\big[x, y, z; [x, v][x, u]\big]^{[x,u,v]}\\ &= [x, y, z; x, u, v][x, y, z; x, u]^{[x,u,v]}[x, y, z; x, v]^{[x,u][x,u,v]}.\end{aligned}$$

Since the last two factors are conjugates of instances of (L1), we obtain (L5). ∎

Lemma 3.5 *If G is 3-metabelian, then the law $[w, x; y, z][w, y; x, z] = 1$ holds in G.*

Proof Assuming G is 3-metabelian, we start from our variant of Macdonald's law, (3Mii) $[w, x; x, z] = 1$, and substitute xy for x. Thus by identities (Iiv) and (Iv),

$$1 = [w, xy; xy, z] = \big[[w, y][w, x][w, x, y]; [x, z][x, z, y][y, z]\big].$$

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To make the calculation easier to view, we let $X = [x,z][x,z,y][y,z]$ and apply (Iiv) and (Iv), to obtain

$$\begin{aligned}1 &= \big[[w,y][w,x][w,x,y];X\big] = [w,y;X]^{[w,x][w,x,y]}\big[[w,x][w,x,y];X\big]\\ &= [w,y;X]^{[w,x][w,x,y]}[w,x;X]^{[w,x,y]}[w,x,y;X].\end{aligned}$$

We will consider each of these three factors separately, calling them A, B, and C, respectively. To complete the proof, we will show that $A = [w,x;y,z]$, $B = [w,y;x,z]$, and $C = 1$.

Consideration of A. Reintroducing X and applying (Iv) repeatedly, we obtain

$$\begin{aligned}A &= [w,y;X]^{[w,x][w,x,y]} = \big[w,y;[x,z][x,z,y][y,z]\big]^{[w,x][w,x,y]}\\ &= [w,y;y,z]^{[w,x][w,x,y]}\big[w,y;[x,z][x,z,y]\big]^{[y,z][w,x][w,x,y]}.\end{aligned}$$

The first factor is trivial by (3Mii); therefore, applying (Iv) to the second factor, we get

$$A = [w,y;x,z,y]^{[y,z][w,x][w,x,y]}[w,y;x,z]^{[x,z,y][y,z][w,x][w,x,y]}.$$

The first factor is trivial by (3Miii), thus

$$A = [w,y;x,z]^{[x,z,y][y,z][w,x][w,x,y]}.$$

We will now argue that each of these four conjugating elements commutes with $[w,y;x,z]$ and, therefore has trivial action. By identities (Iii) and (Iiii),

$$\begin{aligned}\big[[w,y;x,z];[x,z,y]\big] &= \big[[x,z;w,y]^{-1};[x,z,y]\big]\\ &= \big[[x,z;w,y];[x,z,y]\big]^{-[x,z;w,y]^{-1}} = 1.\end{aligned}$$

The last equality follows follows from (3Mi) by replacing x by $[x,z]$, y by $[w,y]$, and z by y. Thus, $[x,z,y]$ commutes with $[w,y;x,z]$ and the first conjugation is trivial (*i.e.*, $[w,y;x,z]^{[x,z,y]} = [w,y;x,z]$). Next we consider $[[w,y;x,z];[y,z]]$. This is trivial, since it is an instance of (L4) with x, y, z, and u replaced by w, y, $[x,z]$, and z, respectively. Thus the second conjugate also has trivial action. The third conjugate has trivial action, since $[w,y;x,z;w,x] = 1$ by (L1). Similarly, $[w,y;x,z;w,y,x] = 1$ by (L5). Thus, all four conjugates have been shown to commute with $[w,y;x,z]$, and it follows that $A = [w,y;x,z]$.

Consideration of B. Applying arguments similar to those used for A, we have,

$$\begin{aligned}B &= [w,x;X]^{[w,x,y]} = \big[w,x;[x,z][x,z,y][y,z]\big]^{[w,x,y]}\\ &= [w,x;y,z]^{[w,x,y]}\big[w,x;[x,z][x,z,y]\big]^{[y,z][w,x,y]}\\ &= [w,x;y,z]^{[w,x,y]}[w,x;x,z,y]^{[y,z][w,x,y]}[w,x;x,z]^{[x,z,y][y,z][w,x,y]}.\end{aligned}$$

The last factor is trivial by (3Mii). The middle factor is trivial by applying (Iii) to $[w,x]$ and moving the inverse to the outside of the commutator using (Iiii), thus obtaining an instance of (L2). Addressing the first factor, note that $[[x,y;w,z],[x,y,w]] = 1$ by (L5); therefore, we have shown that $B = [x,y;w,z]$.

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Consideration of C. First note that

$$\begin{aligned}C &= [x,y,w;X] = \big[x,y,w;[y,z][y,z,w][w,z]\big]\\ &= [x,y,w;w,z]\big[x,y,w;[y,z][y,z,w]\big]^{[w,z]}.\end{aligned}$$

By (3Mii) we see that the first factor is trivial. Therefore,

$$C = \big[x,y,w;[y,z][y,z,w]\big]^{[w,z]} = [x,y,w;y,z,w]^{[w,z]}[x,y,w;y,z]^{[y,z,w][w,z]}.$$

The first factor is trivial by (3Miii); thus, applying (Iii) and (Iiii),

$$\begin{aligned}C &= [w,x,y;X] = \big[w,x,y;[x,z][x,z,y][y,z]\big]\\ &= [w,x,y;y,z]\big[w,x,y;[x,z][x,z,y]\big]^{[y,z]}.\end{aligned}$$

By (3Mii) the first commutator is trivial. Applying (Iv) we obtain,

$$C = \big[w,x,y;[x,z][x,z,y]\big]^{[y,z]} = [w,x,y;x,z,y]^{[y,z]}[w,x,y;x,z]^{[x,z,y][y,z]}.$$

Note that the first commutator is trivial by (3Miii). We claim that the second commutator is also trivial. Applying (Iii) and (Iiii) repeatedly, we have

$$\begin{aligned}[w,x,y;x,z] &= \big[[x,w]^{-1},y;x,z\big] = \big[[x,w,y]^{-[x,w]^{-1}};x,z\big]\\ &= \big[[x,w,y]^{-1};[x,z]^{[x,w]}\big]^{[x,w]^{-1}} = \big[x,w,y;[x,z]^{[x,w]}\big]^{-[x,w,y]^{-1}[x,w]^{-1}}.\end{aligned}$$

Since $[x,w,y;[x,z]^{[x,w]}]$ is an instance of (L4), thus we have shown that $C = 1$.

In summary, we have shown that $1 = AB = [w,x;y,z][w,y;x,z]$, as required. ■

We now collect our preliminary results together to derive the following theorem.

Theorem 3.6 *The commutator interchange law, $[w,x;y,z] = [w,y;x,z]$, holds in a group if and only if the group is 3-metabelian with every commutator of the form $[w,x;y,z]$ either trivial or of order two. That is, in varieties of groups, the law $[w,x;y,z] = [w,y;x,z]$ is logically equivalent to the union of the laws $[x,y;x,z] = 1$ and $[w,x;y,z]^2 = 1$.*

Proof Suppose first that $[w,x;y,z] = [w,y;x,z]$ holds in a group G. Replacing w by x we obtain $1 = [x,x;y,z] = [x,y;x,z]$, thus we conclude that G is 3-metabelian. Knowing this, it follows by Lemma 3.5 that $[w,x;y,z][w,y;x,z] = 1$ holds in G. Combining this with the interchange law, we have

$$1 = [w,x;y,z][w,y;x,z] = [w,x;y,z]^2.$$

We conclude that the interchange law implies both laws stated in the theorem. Conversely, if G is assumed to be 3-metabelian, then, by Lemma 3.5, we have $[w,x;y,z][w,y;x,z] = 1$ in G. Thus we have $[w,x;y,z] = [w,y;x,z]^{-1}$. Since our hypothesis states that $[w,x;y,z]^2 = 1$ in G, we conclude that $[w,x;y,z] = [w,y;x,z]$ in G. Thus the interchange laws holds in G. ■

We can now combine the information we have gathered to reflect on what kinds of examples of double objects we can construct from groups using the operations of left and right commutation.

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Theorem 3.7 *The following statements are true for any group G.*

(i) $(G, \star, \bullet)$ *is a double magma if and only if G satisfies the laws* $[x, y; x, z] = 1$ *and* $[w, x; y, z]^2 = 1$.

(ii) $(G, \star, \bullet)$ *is a double semigroup if and only if G is nilpotent of class two (*i.e., *satisfies the law* $[x, y, z] = 1$*).*

(iii) *A double magma or double semigroup,* $(G, \star, \bullet)$*, is proper if and only if there exist* $x_0, y_0 \in G$ *such that* $[x_0, y_0]^2 \neq 1$.

Proof Part (i) follows immediately from Theorem 3.6. Part (iii) follows from the equivalence of (iii) and (iv) in Proposition 3.1 (see the comment following the proof of Proposition 3.1). We will now establish part (ii).

Suppose first that $(G, \star, \bullet)$ is a double semigroup. Since G satisfies (CI), Theorem 3.6 implies that G must be 3-metabelian and satisfy the law $[w, x; y, z]^2 = 1$. Recall that Bachmuth and Lewin [1] prove that the Jacobi identity, $[x, y, z][y, z, x][z, x, y] = 1$, defines the variety of 3-metabelian groups. Thus we know that the Jacobi identity holds in G. Proposition 3.2 states that for $(G, \star, \bullet)$ to be associative, it is necessary and sufficient that the law $[x, y, z][y, z, x] = 1$ hold in G. Combining this with the Jacobi identity, it follows that $[x, y, z] = 1$ holds in G and, hence, G is nilpotent of class two. Conversely, if G is of class two, then all commutators of weights three and higher are trivial. Therefore, (CI) holds in G and it follows that $(G, \star, \bullet)$ is a double magma. Since all weight three commutators are trivial, the law $[x, y, z][y, z, x] = 1$ holds in G, and, by Proposition 3.2, both " $\star$ " and " $\bullet$ " are associative. Thus, $(G, \star, \bullet)$ is a double semigroup. ∎

We should note that the double systems constructed here all have a zero element, the identity element of G, which we denote "1". Since $[x, 1] = [1, x] = 1$ for each $x \in G$, "1" acts as an annihilator for both operations, that is, $1 \star x = x \star 1 = 1$ and $1 \bullet x = x \bullet 1 = 1$ for each $x \in G$

The most general example of a proper double magma constructed in this manner would be based on the relatively free group of the subvariety of 3-metabelian groups determined by the law $[w, x; y, z]^2 = 1$. B. H. Neumann [5] gives an example of a 3-metabelian that is not metabelian. His group is of order 2^{14} and is 3-metabelian but not metabelian. Its derived group is not of exponent two, and satisfies the identity $[w, x; y, z]^2 = 1$. Thus, if a nonmetabelian example were desirable, Neumann's group could be used to construct a proper double magma on a group of the solvability length three. This complexity might prove useful in some contexts.

Practically speaking, to construct a proper double magma in our manner, G could be chosen to be any metabelian group, or a group that is nilpotent of class three, as long as the square of some commutator is nontrivial. This could be realized, among other ways, by letting G be a finite metacyclic group of odd order. Alternately, one could select any dihedral group of order not equal to 1, 2, 4, or 8. For example if we were to select the dihedral group of order six,

$$D_3 = \langle a, b; a^3 = 1, b^2 = 1, a^b = a^2 \rangle,$$

the group is metabelian and $[a, b]^2 = a^2 \neq 1$. Thus $(D_3, \star, \bullet)$ is a proper, noncommutative double magma.

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To construct a proper double semigroup, we require that G be a nonabelian class two group with some commutator not of order two. In the most general case, we could take G to be the relatively free group of class two. A simpler example would be any nonabelian group of order p^3, with p an odd prime. These groups are of class two and have no elements of even order. For a group containing 2-elements we could select $D_8 = \langle a, b; a^8, b^2, ba = a^7 b\rangle$, the dihedral group of order 16. This is a metabelian group and $[a, b]^2 = a^4 \neq 1$. Thus, $(D_8, \star, \bullet)$ is a proper noncommutative double semigroup. To be completely explicit, we give the Cayley table for $(D_8, \star)$ below. The table for $(D_8, \bullet)$ results by replacing each entry in the table below by its inverse in D_8.

$\star$	1	a	a^2	a^3	a^4	a^5	a^6	a^7	b	ab	a^2b	a^3b	a^4b	a^5b	a^6b	a^7b
1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a	1	1	1	1	1	1	1	1	a^6	a^6	a^6	a^6	a^6	a^6	a^6	a^6
a^2	1	1	1	1	1	1	1	1	a^4	a^4	a^4	a^4	a^4	a^4	a^4	a^4
a^3	1	1	1	1	1	1	1	1	a^2	a^2	a^2	a^2	a^2	a^2	a^2	a^2
a^4	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1	1
a^5	1	1	1	1	1	1	1	1	a^6	a^6	a^6	a^6	a^6	a^6	a^6	a^6
a^6	1	1	1	1	1	1	1	1	a^4	a^4	a^4	a^4	a^4	a^4	a^4	a^4
a^7	1	1	1	1	1	1	1	1	a^2	a^2	a^2	a^2	a^2	a^2	a^2	a^2
b	1	a^2	a^4	a^6	1	a^2	a^4	a^6	1	a^6	a^4	a^2	1	a^6	a^4	a^2
ab	1	a^2	a^4	a^6	1	a^2	a^4	a^6	a^2	1	a^6	a^4	a^2	1	a^6	a^4
a^2b	1	a^2	a^4	a^6	1	a^2	a^4	a^6	a^4	a^2	1	a^6	a^4	a^2	1	a^6
a^3b	1	a^2	a^4	a^6	1	a^2	a^4	a^6	a^6	a^4	a^2	1	a^6	a^4	a^2	1
a^4b	1	a^2	a^4	a^6	1	a^2	a^4	a^6	1	a^6	a^4	a^2	1	a^6	a^4	a^2
a^5b	1	a^2	a^4	a^6	1	a^2	a^4	a^6	a^2	1	a^6	a^4	a^2	1	a^6	a^4
a^6b	1	a^2	a^4	a^6	1	a^2	a^4	a^6	a^4	a^2	1	a^6	a^4	a^2	1	a^6
a^7b	1	a^2	a^4	a^6	1	a^2	a^4	a^6	a^6	a^4	a^2	1	a^6	a^4	a^2	1

4 A Note on Ring Constructions

If $(R, +, \cdot)$ is a ring, then we define the ring commutator (or Lie commutator) for each $x, y \in R$ as $\langle x, y\rangle = x \cdot y - y \cdot x$. We can then construct double systems from rings as we did from groups by defining two binary operations on the ring R as follows. For each $x, y \in R$, let $x \star y = \langle x, y\rangle$ and let $x \bullet y = \langle y, x\rangle$. The *ring commutator interchange law* takes the form

(RCI) $$\langle w, x; y, z\rangle = \langle w, y; x, z\rangle.$$

Calculations, much simpler than those for groups, show that $(R, +, \cdot)$ is a proper magma precisely when (RCI) holds in R and there exist $x_0, y_0 \in R$ such that $2\langle x, y\rangle \neq 0$. Analogous to the group case, it is true that (RCI) is equivalent to the laws

$$\langle x, y; x, z\rangle = 0 \quad \text{and} \quad 2\langle w, x; y, z\rangle = 0$$

in equational classes of rings. To obtain a proper double semigroup, the commutator identity $\langle x, y, z\rangle = 0$ must hold in R and there must exist $x_0, y_0 \in R$ such that $2\langle x, y\rangle \neq 0$. The law (RCI) is a polynomial identity holding on R, and this fact has a significant impact on the structure of R. We ask if there is a nice, structural characterization of those rings in which (RCI) holds?

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