Canad. J. Math. Vol. **62** (6), 2010 pp. 1310–1324
doi:10.4153/CJM-2010-072-x

Iwahori–Hecke Algebras of SL_2 over 2-Dimensional Local Fields

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Abstract. In this paper we construct an analogue of Iwahori–Hecke algebras of SL_2 over 2-dimensional local fields. After considering coset decompositions of double cosets of a Iwahori subgroup, we define a convolution product on the space of certain functions on SL_2, and prove that the product is well-defined, obtaining a Hecke algebra. Then we investigate the structure of the Hecke algebra. We determine the center of the Hecke algebra and consider Iwahori–Matsumoto type relations.

Introduction

Hecke algebras play important roles in the representation theory of p-adic groups. There are two important classes of Hecke algebras. One is the spherical Hecke algebra attached to a maximal compact open subgroup, and the other is the Iwahori–Hecke algebra attached to a Iwahori subgroup. The spherical Hecke algebra is isomorphic to the center of the corresponding Iwahori–Hecke algebra. In the theory of higher dimensional local fields [5], a p-adic field is a 1-dimensional local field. So the theory of p-adic groups and their Hecke algebras is over 1-dimensional local fields.

Recently, the representation theory of algebraic groups over 2-dimensional local fields was initiated by the works of Kapranov, Kazhdan and Gaitsgory [12], [6], [7], [8]. In their development, Cherednik's double affine Hecke algebras ([2]) appear as an analogue of Iwahori–Hecke algebras. The common feature of the works mentioned above is the use of rank one integral structure of a 2-dimensional local field. But there is also a rank two integral structure in a 2-dimensional local field, and it is important in the arithmetic theory to use the rank two integral structure [5]; we refer the reader to Fesenko's article [4] and to the references there for recent developments.

In their paper [13], Kim and Lee constructed an analogue of spherical Hecke algebras of SL_2 over 2-dimensional local fields using rank two integral structure. They also established a Satake isomorphism using Fesenko's $\mathbb{R}((X))$-valued measure defined in [3] (see also [4]). In particular, the algebra is proved to be commutative. A similar result is expected in the case of GL_n and, eventually, in the case of reductive algebraic groups.

In this paper, we construct an analogue of Iwahori–Hecke algebras of SL_2 over 2-dimensional local fields coming from rank two integral structure. Our basic approach will be similar to that of [13]. More precisely, an element of the algebra is an infinite linear combination of characteristic functions of double cosets satisfying

Received by the editors April 1, 2008.
Published electronically September 15, 2010.
The author was supported in part by EPSRC grant on zeta functions and in part by KOSEF Grant # R01-2003-000-10012-0.
AMS subject classification: **22E50**, 20G25.

certain conditions. We define a convolution product of two characteristic functions, using coset decompositions of double cosets of an Iwahori subgroup and imposing a restriction on the support of the product. Since a double coset of the Iwahori subgroup is an uncountable union of cosets in general, it is necessary to prove that the convolution product is well-defined. This will be done in the first part of the paper.

In the second part, we study the structure of the Hecke algebra. It has a natural $\mathbb{Z}$-grading and contains the affine Hecke algebra of SL_2 as a subalgebra. We will find a big commutative subalgebra, and determine its structure completely. Surprisingly, it is different from the group algebra of double cocharacters. We will also calculate the center of the Hecke algebra, and it turns out to be the same as the center of the affine Hecke algebra of SL_2. The classical Iwahori–Hecke algebra has a well-known presentation due to Iwahori and Matsumoto [10], [11]. The relations can be understood as deformations of Coxeter relations of the affine Weyl group. The corresponding Weyl group of a reductive algebraic group over a 2-dimensional local field is not a Coxeter group. In the SL_2 case, A. N. Parshin obtained an explicit presentation of the (double affine) Weyl group [15]. We will investigate Iwahori–Matsumoto type relations of the Hecke algebra in light of Parshin's presentation.

Now that we have spherical Hecke algebras and Iwahori–Hecke algebras of SL_2 over 2-dimensional local fields with respect to rank two integral structure, next natural steps would be considering representations of SL_2 over 2-dimensional local fields, constructing the Hecke algebras for more general reductive algebraic groups, and understanding these algebras in connection with the works of Kapranov, Kazhdan and Gaitsgory mentioned earlier. It seems that another interesting approach to the representation theory over two-dimensional local fields could be obtained from the work of Hrushovski and Kazhdan [9], in which they developed a theory of motivic integration. Actually, in the appendix of the paper [9], given by Avni, an Iwahori–Hecke algebra of SL_2 is constructed using motivic integration. It would be nice if one could find any connection of it to the constructions of this paper.

There are three sections and an appendix in this paper. In the first section, we fix notations and recall the Bruhat decomposition. The next section is devoted to the construction of the Hecke algebra. We will show that the convolution product is well-defined. In the third section, we study the structure of the Hecke algebra, determining the center of the algebra and finding Iwahori–Matsumoto type relations. In the appendix, we present a complete set of formulas for convolution products of characteristic functions. These formulas will be essentially used for many calculations in this paper.

1 Bruhat Decomposition

In this section, we fix notations and collect some results on double cosets and coset decompositions we will use later. We assume that the reader is familiar with basic definitions in the theory of 2-dimensional local fields, which can be found in [18].

Let $F(=F_2)$ be a two dimensional local field with the first residue field F_1 and the last residue field $F_0(=\mathbb{F}_q)$ of q elements. We fix a discrete valuation $\nu\colon F^\times \to \mathbb{Z}^2$ of rank two. Recall that $\mathbb{Z}^2$ is endowed with the lexicographic ordering from the right. Let t_1 and t_2 be local parameters with respect to the valuation ν. We have the ring O

of integers of F with respect to the rank-two valuation v. There is a natural projection $p\colon O \to O/t_1O = F_0$. Note that the ring O is different from the ring O_{21} of integers of F with respect to the rank-one valuation $v_{21}\colon F^\times \to \mathbb{Z}$.

Let G be a connected split semisimple algebraic group defined over $\mathbb{Z}$. We fix a maximal torus T and a Borel subgroup B such that $T \subset B \subset G$, and we have $W_0 = N_G(T)/T$, the Weyl group of G. We write $G = G(F)$ and consider $I = \{x \in G(O) : p(x) \in B(F_0)\}$, the *double Iwahori subgroup*, and $W = N_G(T)/T(O)$, the *double affine Weyl group*, and obtain the following decomposition.

Proposition 1.1 ([12], [15]) *We have*

$$G = \coprod_{w\in W} IwI,$$

and the resulting identification $I\backslash G/I \to W$ is independent of the choice of representatives of elements of W.

From now on, in the rest of this paper, we assume that $G = \mathrm{SL}_2$ and B is the subgroup of upper triangular matrices. The following lemma gives explicit formulas for the decomposition in Proposition 1.1. The proof is straightforward, so we omit it.

Lemma 1.2 *Assume that $x = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G$.*
(1) *If $v(a) \le v(b)$ and $v(a) < v(c)$, then $x \in I\begin{pmatrix} a & 0 \\ 0 & a^{-1} \end{pmatrix}I$.*
(2) *If $v(b) < v(a)$ and $v(b) < v(d)$, then $x \in I\begin{pmatrix} 0 & b \\ -b^{-1} & 0 \end{pmatrix}I$.*
(3) *If $v(c) \le v(a)$ and $v(c) \le v(d)$, then $x \in I\begin{pmatrix} 0 & -c^{-1} \\ c & 0 \end{pmatrix}I$.*
(4) *If $v(d) \le v(b)$ and $v(d) < v(c)$, then $x \in I\begin{pmatrix} d^{-1} & 0 \\ 0 & d \end{pmatrix}I$.*

We denote by $C^{(1)}_{i,j}$ and $C^{(2)}_{i,j}$, $(i, j) \in \mathbb{Z}^2$, the double cosets

$$I\begin{pmatrix} t_1^i t_2^j & 0 \\ 0 & t_1^{-i}t_2^{-j} \end{pmatrix}I \quad\text{and}\quad I\begin{pmatrix} 0 & t_1^i t_2^j \\ -t_1^{-i}t_2^{-j} & 0 \end{pmatrix}I, \quad\text{respectively.}$$

In the following lemma, we obtain complete sets of coset representatives in the decomposition of double cosets of the subgroup I into right cosets.

Lemma 1.3 *We have $C^{(a)}_{i,j} = \coprod_z Iz$, where the disjoint union is over z in the following list.*
(1) *If $a = 1$ and $(i, j) \ge (0, 0)$, then*

$$z = \begin{pmatrix} t_1^i t_2^j & 0 \\ 0 & t_1^{-i}t_2^{-j} \end{pmatrix}, \begin{pmatrix} t_1^i t_2^j & 0 \\ t_1^{-i+k}t_2^{-j+l}u & t_1^{-i}t_2^{-j} \end{pmatrix}$$

for $(1, 0) \le (k, l) < (2i + 1, 2j)$, where $u \in O^\times$ are units belonging to a fixed set of representatives of $O/t_1^{2i-k+1}t_2^{2j-l}O$.

(2) *If* $a = 1$ *and* $(i, j) < (0, 0)$, *then*

$$z = \begin{pmatrix} t_1^i t_2^j & 0 \\ 0 & t_1^{-i} t_2^{-j} \end{pmatrix}, \begin{pmatrix} t_1^i t_2^j & t_1^{i+k} t_2^{j+l} u \\ 0 & t_1^{-i} t_2^{-j} \end{pmatrix}$$

for $(0, 0) \le (k, l) < (-2i, -2j)$, *where* $u \in O^\times$ *are units belonging to a fixed set of representatives of* $O/t_1^{-2i-k} t_2^{-2j-l} O$.

(3) *If* $a = 2$ *and* $(i, j) \ge (0, 0)$, *then*

$$z = \begin{pmatrix} 0 & t_1^i t_2^j \\ -t_1^{-i} t_2^{-j} & 0 \end{pmatrix}, \begin{pmatrix} 0 & t_1^i t_2^j \\ -t_1^{-i} t_2^{-j} & -t_1^{-i+k} t_2^{-j+l} u \end{pmatrix}$$

for $(0, 0) \le (k, l) < (2i + 1, 2j)$, *where* $u \in O^\times$ *are units belonging to a fixed set of representatives of* $O/t_1^{2i-k+1} t_2^{2j-l} O$.

(4) *If* $a = 2$ *and* $(i, j) < (0, 0)$, *then*

$$z = \begin{pmatrix} 0 & t_1^i t_2^j \\ -t_1^{-i} t_2^{-j} & 0 \end{pmatrix}, \begin{pmatrix} t_1^{i+k} t_2^{j+l} u & t_1^i t_2^j \\ -t_1^{-i} t_2^{-j} & 0 \end{pmatrix}$$

for $(1, 0) \le (k, l) < (-2i, -2j)$, *where* $u \in O^\times$ *are units belonging to a fixed set of representatives of* $O/t_1^{-2i-k} t_2^{-2j-l} O$.

Proof Since the other cases are similar, we only prove the part (1). Consider

$$z = \begin{pmatrix} t_1^i t_2^j & 0 \\ 0 & t_1^{-i} t_2^{-j} \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad z' = \begin{pmatrix} t_1^i t_2^j & 0 \\ 0 & t_1^{-i} t_2^{-j} \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$$

where $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right), \left(\begin{smallmatrix} a' & b' \\ c' & d' \end{smallmatrix}\right) \in I$. We see that the condition $Iz = Iz'$ is equivalent to

$$c'd - cd' \in t_1^{2i+1} t_2^{2j} O.$$

We write $(c, d) \sim (c', d')$ if $c'd - cd' \in t_1^{2i+1} t_2^{2j} O$. Note that if $\left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in I$ then $c \in t_1 O$ and d is a unit. Let C be the set of pairs $(c, d) \in O^2$ such that $c \in t_1 O$ and d is a unit. Then $\sim$ is an equivalence relation on C. In order to determine different cosets, we need only to determine a set of representatives of the equivalence relation $\sim$, which turn out to be

$$(0, 1) \quad \text{and} \quad (t_1^k t_2^l u, 1) \quad \text{for } (1, 0) \le (k, l) < (2i + 1, 2j),$$

where $u \in O^\times$ are units belonging to a fixed set of representatives of $O/t_1^{2i-k+1} t_2^{2j-l} O$. These yield the elements z in the part (1). ■

Remark 1.4 The disjoint union $C_{i,j}^{(a)} = \coprod_z Iz$ is an uncountable union unless $j = 0$. The same is true for the double cosets of $K = \mathrm{SL}_2(O)$; see [13].

2 Iwahori–Hecke Algebras

In this section, we define the convolution product of two characteristic functions of double cosets of the subgroup I, and we show that the product is well-defined. Then we construct an analogue of the Iwahori–Hecke algebra of SL_2.

We fix a set of representatives R of the double affine Weyl group W to be

$$(2.1)\quad R = \left\{ \eta^{(1)}_{i,j} := \begin{pmatrix} t_1^i t_2^j & 0 \\ 0 & t_1^{-i} t_2^{-j} \end{pmatrix}, \quad \eta^{(2)}_{i,j} := \begin{pmatrix} 0 & t_1^i t_2^j \\ -t_1^{-i} t_2^{-j} & 0 \end{pmatrix} : (i, j) \in \mathbb{Z}^2 \right\}.$$

We define a map $\eta \colon G \to R$ so that $x \in I\eta(x)I$ for each $x \in G$. That is, we assign to an element x of G its representative $\eta(x) \in R$ in the decomposition $G = \coprod_{w \in W} IwI$ of Proposition 1.1.

We put

$$(2.2)\qquad C_j = \coprod_{m \in \mathbb{Z}} C^{(1)}_{m,j} \cup C^{(2)}_{m,j}, \quad j \in \mathbb{Z}.$$

We denote by $\chi^{(1)}_{i,j}$ and $\chi^{(2)}_{i,j}$ the characteristic functions of the double cosets $C^{(1)}_{i,j}$ and $C^{(2)}_{i,j}$, respectively. We will consider the following types of functions:

$$(2.3)\qquad \sum_{r \leq i} c_r \chi^{(a)}_{r,j}\ (j > 0), \quad \sum_{r \geq i} c_r \chi^{(a)}_{r,j}\ (j < 0), \quad \text{and} \quad \sum_{i \leq r \leq i'} c_r \chi^{(a)}_{r,0}$$

for $i, i', j \in \mathbb{Z}$, $a = 1, 2$ and $c_r \in \mathbb{C}$. Now we define the convolution product of two characteristic functions.

Definition 2.4 For $a, b = 1, 2$ and $(i, j), (k, l) \in \mathbb{Z}^2$, we define

$$(2.5)\qquad (\chi^{(a)}_{i,j} * \chi^{(b)}_{k,l})(x) = \begin{cases} q^{-1} \sum_z \chi^{(a)}_{i,j}\big(\eta(x) z^{-1}\big) & \text{if } x \in C_{j+l} \text{ and } jl \geq 0, \\ 0 & \text{otherwise,} \end{cases}$$

where the sum is over the representatives z of the decomposition $C^{(b)}_{k,l} = \coprod_z Iz$.

Remark 2.6 One can construct a certain invariant $\mathbb{R}((X))$-valued measure $d\gamma$ on G. Then we could define the convolution product of two functions f and g on G by

$$(f * g)(x) = \int_G f(xy^{-1}) g(y)\, d\gamma(y).$$

The definition of the convolution product given above is derived from this formula.

Since the cardinality of the set of the representatives z in the union is uncountable (Remark 1.4) in general, we need to prove the following.

Theorem 2.7 *The convolution product $\chi^{(a)}_{i,j} * \chi^{(b)}_{k,l}$ yields a well-defined function of one of the types in (2.3) for any $a, b = 1, 2$ and $(i, j), (k, l) \in \mathbb{Z}^2$.*

Proof We write

$$\chi_{i,j}^{(a)} * \chi_{k,l}^{(b)} = \sum_m c_m^{(1)} \chi_{m,j+l}^{(1)} + \sum_m c_m^{(2)} \chi_{m,j+l}^{(2)}, \quad c_m^{(1)}, c_m^{(2)} \in \mathbb{C}, m \in \mathbb{Z}.$$

(1) Assume that $b = 1$, $j \geq 0$ and $(k, l) \geq (0, 0)$. By Lemma 1.3, we need only to consider

$$z = \begin{pmatrix} t_1^k t_2^l & 0 \\ t_1^{-k+k'} t_2^{-l+l'} u & t_1^{-k} t_2^{-l} \end{pmatrix}$$

for $(1, 0) \leq (k', l') < (2k + 1, 2l)$, where $u \in O^\times$ are units belonging to a fixed set of representatives of $O/t_1^{2k-k'+1} t_2^{2l-l'} O$.

(a) If $\eta(x) = \eta_{m,j+l}^{(1)}$, then

$$\eta(x) z^{-1} = \begin{pmatrix} t_1^{m-k} t_2^j & 0 \\ -t_1^{-m-k+k'} t_2^{-j-2l+l'} u & t_1^{-m+k} t_2^{-j} \end{pmatrix}.$$

(i) If $a = 1$, then we have $m - k = i$, and either $(-m-k+k', -j-2l+l') > (m-k, j)$ or $(-m-k+k', -j-2l+l') > (-m+k, -j)$ by Lemma 1.2. The first case gives $(k', l') > (2m, 2j + 2l) = (2i + 2k, 2j + 2l)$, and so $j = 0$, $l' = 2l$ and $2i + 2k < k' < 2k + 1$. Counting the number of possible representative units in $O/t_1^{2k-k'+1} O$, we see that $c_{i+k}^{(1)}$ is finite and $c_m^{(1)} = 0$ for all $m \neq i + k$. The second case gives $(k', l') > (2k, 2l)$, but $(k', l') < (2k + 1, 2l)$ from the assumption, and so $c_m^{(1)} = 0$ for all $m \in \mathbb{Z}$.

(ii) If $a = 2$, then we must have $l' = 2l$ and $-m - k + k' = -i$. Then $k' < 2k + 1$ implies $m < i + k + 1$, and counting the number of possible representative units in $O/t_1^{2k-k'+1} O$, we see that $c_m^{(2)}$ is finite for each m.

(b) If $\eta(x) = \eta_{m,j+l}^{(2)}$, then

$$\eta(x) z^{-1} = \begin{pmatrix} -t_1^{m-k+k'} t_2^{j+l'} u & t_1^{m+k} t_2^{j+2l} \\ -t_1^{-m-k} t_2^{-j-2l} & 0 \end{pmatrix}.$$

(c) If $a = 1$, then $m - k + k' = i$ and $l' = 0$, and it follows from Lemma 1.2 that $j \leq -l$. Since $j \geq 0$ and $l \geq 0$, we have $j = l = l' = 0$. The condition $1 \leq k' < 2k + 1$ gives $i - k - 1 < m \leq i + k - 1$ and $c_m^{(2)}$ is finite for each m.

(d) If $a = 2$, then $l = l' = 0$, $m + k = i$ and $1 \leq k' < 2k + 1$. Thus $c_{i-k}^{(2)}$ is finite and $c_m^{(2)} = 0$ for $m \neq i - k$.

Assume that $b = 1$, $j \leq 0$ and $(k, l) < (0, 0)$. By Lemma 1.3, we need only to consider

$$z = \begin{pmatrix} t_1^k t_2^l & t_1^{k+k'} t_2^{l+l'} u \\ 0 & t_1^{-k} t_2^{-l} \end{pmatrix}$$

for $(0, 0) \leq (k', l') < (-2k, -2l)$, where $u \in O^\times$ are units belonging to a fixed set of representatives of $O/t_1^{-2k-k'} t_2^{-2l-l'} O$.

(a) If $\eta(x) = \eta^{(1)}_{m,j+l}$, then

$$\eta(x)z^{-1} = \begin{pmatrix} t_1^{m-k}t_2^{j} & -t_1^{m+k+k'}t_2^{j+2l+l'}u \\ 0 & t_1^{-m+k}t_2^{-j} \end{pmatrix}.$$

(i) If $a = 1$, then we have $m-k = i$, and either $(m-k, j) \le (m+k+k', j+2l+l')$ or $(-m+k, -j) \le (m+k+k', j+2l+l')$ by Lemma 1.2. The first case yields $(k', l') \ge (-2k, -2l)$, but $(k', l') < (-2k, -2l)$ from the assumption. Thus $c^{(1)}_m = 0$ for all $m \in \mathbb{Z}$. The second case gives $(k', l') \ge (-2m, -2j - 2l) = (-2i - 2k, -2j - 2l)$, and we must have $j = 0$, $l' = -2l$ and $-2i - 2k \le k' < -2k$. Thus $c^{(1)}_{i+k}$ is finite and $c^{(1)}_m = 0$ for all $m \ne i + k$.

(ii) If $a = 2$, then we have $l' = -2l$ and $m+k+k' = i$. The condition $k' < -2k$ leads to $i + k < m$, and $c^{(2)}_m$ is finite for each m.

(b) If $\eta(x) = \eta^{(2)}_{m,j+l}$, then

$$\eta(x)z^{-1} = \begin{pmatrix} 0 & t_1^{m+k}t_2^{j+2l} \\ -t_1^{-m-k}t_2^{-j-2l} & t_1^{-m+k+k'}t_2^{-j+l'}u \end{pmatrix}.$$

(i) If $a = 1$, then $-m + k + k' = -i$ and $l' = 0$, and we have $-j \le l$ by Lemma 1.2. Since $j \ge 0$ and $l \ge 0$, we have $j = l = l' = 0$. The condition $0 \le k' < -2k$ yields $i + k \le m < i - k$ and $c^{(2)}_m$ is finite for each m.

(ii) If $a = 2$, then $l = l' = 0$, $m + k = i$ and $0 \le k' < -2k$. Thus $c^{(2)}_{i-k}$ is finite and $c^{(2)}_m = 0$ for $m \ne i - k$.

Assume that $b = 2$, $j \ge 0$ and $(k, l) \ge (0, 0)$. By Lemma 1.3, we need only to consider

$$z = \begin{pmatrix} 0 & t_1^{k}t_2^{l} \\ -t_1^{-k}t_2^{-l} & -t_1^{-k+k'}t_2^{-l+l'}u \end{pmatrix}$$

for $(0, 0) \le (k', l') < (2k + 1, 2l)$, where $u \in O^\times$ are units belonging to a fixed set of representatives of $O/t_1^{2k-k'+1}t_2^{2l-l'}O$.

(i) If $\eta(x) = \eta^{(1)}_{m,j+l}$, then

$$\eta(x)z^{-1} = \begin{pmatrix} -t_1^{m-k+k'}t_2^{j+l'}u & -t_1^{m+k}t_2^{j+2l} \\ t_1^{-m-k}t_2^{-j-2l} & 0 \end{pmatrix}.$$

(i) If $a = 1$, then it is similar to (1)(b)(i).

(ii) If $a = 2$, then it is similar to (1)(b)(ii).

(ii) If $\eta(x) = \eta^{(2)}_{m,j+l}$, then

$$\eta(x)z^{-1} = \begin{pmatrix} t_1^{m-k}t_2^{j} & 0 \\ t_1^{-m-k+k'}t_2^{-j-2l+l'}u & t_1^{-m+k}t_2^{-j} \end{pmatrix}.$$

(i) If $a = 1$, then it is similar to (1)(a)(i).

(ii) If $a = 2$, then it is similar to (1)(a)(ii).

Assume that $b = 2$, $j \le 0$ and $(k, l) < (0, 0)$. By Lemma 1.3, we need only to consider

$$z = \begin{pmatrix} t_1^{k+k'} t_2^{l+l'} u & t_1^k t_2^l \\ -t_1^{-k} t_2^{-l} & 0 \end{pmatrix}$$

for $(1, 0) \le (k', l') < (-2k, -2l)$, where $u \in O^\times$ are units belonging to a fixed set of representatives of $O/t_1^{-2k-k'} t_2^{-2l-l'} O$.

(a) If $\eta(x) = \eta^{(1)}_{m,j+l}$, then

$$\eta(x) z^{-1} = \begin{pmatrix} 0 & -t_1^{m+k} t_2^{j+2l} \\ t_1^{-m-k} t_2^{-j-2l} & t_1^{-m+k+k'} t_2^{-j+l'} u \end{pmatrix}.$$

(i) If $a = 1$, then it is similar to (2)(b)(i).

(ii) If $a = 2$, then it is similar to (2)(b)(ii).

(b) If $\eta(x) = \eta^{(2)}_{m,j+l}$, then

$$\eta(x) z^{-1} = \begin{pmatrix} t_1^{m-k} t_2^{j} & t_1^{m+k+k'} t_2^{j+2l+l'} u \\ 0 & t_1^{-m+k} t_2^{-j} \end{pmatrix}.$$

(i) If $a = 1$, then it is similar to (2)(a)(i).

(ii) If $a = 2$, then it is similar to (2)(a)(ii). ∎

A complete set of formulas for the convolution product $\chi^{(a)}_{i,j} * \chi^{(b)}_{k,l}$ can be found in the Appendix, and we obtain the following result.

Corollary 2.8

$$\chi^{(a)}_{i,j} * \chi^{(b)}_{k,l} = \begin{cases} \sum_{m \le i+k} c_m \chi^{(b)}_{m,j+l}, & c_m \in \mathbb{C}, \quad \text{if } a = 2,\ j > 0 \text{ and } l > 0, \\ \sum_{m > i+k} c_m \chi^{(b)}_{m,j+l}, & c_m \in \mathbb{C}, \quad \text{if } a = 2,\ j < 0 \text{ and } l < 0, \\ \text{a finite sum} & \qquad\qquad\quad \text{otherwise.} \end{cases} \tag{2.9}$$

We denote by $\mathcal{H}(G, I)$ the $\mathbb{C}$-vector space generated by the functions of types in (2.3). We linearly extend the convolution product $*$ defined in (2.5) to the whole space $\mathcal{H}(G, I)$. It follows from Corollary 2.8 that it is well-defined. Thus we have obtained a $\mathbb{C}$-algebra structure on the space $\mathcal{H}(G, I)$.

Definition 2.10 The $\mathbb{C}$-algebra $\mathcal{H}(G, I)$ will be called the *Iwahori–Hecke algebra* of G $(= \mathrm{SL}_2)$.

It can be easily checked that $\iota := q\chi^{(1)}_{0,0}$ is the identity element of the algebra $\mathcal{H}(G, I)$. Furthermore, we have:

Proposition 2.11 *The algebra* $\mathcal{H}(G, I)$ *is an associative algebra.*

https://doi.org/10.4153/CJM-2010-072-x Published online by Cambridge University Press

Proof Assume that $\alpha = \chi^{(a)}_{i,j}$, $\beta = \chi^{(b)}_{k,l}$ and $\gamma = \chi^{(c)}_{m,n}$. We fix the sets of representatives $\{z_1\}$, $\{z_2\}$ and $\{z_3\}$ in the decompositions $C^{(a)}_{i,j} = \coprod_{z_1} Iz_1$, $C^{(b)}_{k,l} = \coprod_{z_2} Iz_2$ and $C^{(c)}_{m,n} = \coprod_{z_3} Iz_3$, respectively. We write

$$\alpha * \beta = \sum_{\sigma} c(\alpha, \beta; \sigma)\sigma, \quad \text{where } \sigma = \chi^{(d)}_{p,j+l},\ d = 1, 2,\ p \in \mathbb{Z}.$$

Then

$$c(\alpha, \beta; \sigma) = \operatorname{Card}\{z_2 : \eta^{(d)}_{p,j+1} z_2^{-1} \in C^{(a)}_{i,j}\} = \operatorname{Card}\{(z_1, z_2) : I\eta^{(d)}_{p,j+l} = Iz_1z_2\}.$$

The coefficient $c(\alpha, \beta; \sigma)$ is finite for any σ by Theorem 2.7. Similarly, we write

$$\sigma * \gamma = \sum_{\tau} c(\sigma, \gamma; \tau)\tau, \quad \text{where } \tau = \chi^{(e)}_{r,j+l+n},\ e = 1, 2,\ r \in \mathbb{Z}.$$

We define

$$c(\alpha, \beta, \gamma; \tau) = \operatorname{Card}\{(z_1, z_2, z_3) : I\eta^{(e)}_{r,j+l+n} = Iz_1z_2z_3\}.$$

Since $(\alpha * \beta) * \gamma$ is defined, the number $\sum_{\sigma} c(\alpha, \beta; \sigma)c(\sigma, \gamma; \tau)$ is finite. Now it is not difficult to see that

$$\sum_{\sigma} c(\alpha, \beta; \sigma)c(\sigma, \gamma; \tau) = c(\alpha, \beta, \gamma; \tau).$$

Similarly, one can show that

$$\sum_{\sigma'} c(\alpha, \sigma'; \tau)c(\beta, \gamma; \sigma') = c(\alpha, \beta, \gamma; \tau),$$

where σ' is defined with regard to $\alpha*(\beta*\gamma)$. It proves the assertion of the proposition. ■

Remark 2.12 The argument is essentially the same as in the case of Hecke operators on the space of modular forms; see [1], [17].

3 The Structure of $\mathcal{H}(G, I)$

In this section, we investigate the structure of the Hecke algebra $\mathcal{H}(G, I)$. We will see that it has a big commutative subalgebra. After that, the center will be determined and Iwahori–Matsumoto type relations will be found.

For each $j \in \mathbb{Z}$, we let $\mathcal{H}_j$ be the subspace of $\mathcal{H}(G, I)$, consisting of the functions with their supports contained in C_j, where the set C_j is defined in (2.2). We put

$$\mathcal{H}_- = \bigoplus_{j<0} \mathcal{H}_j \quad \text{and} \quad \mathcal{H}_+ = \bigoplus_{j>0} \mathcal{H}_j.$$

Then, clearly, we have

$$\mathcal{H}(G, I) = \mathcal{H}_- \oplus \mathcal{H}_0 \oplus \mathcal{H}_+.$$

https://doi.org/10.4153/CJM-2010-072-x Published online by Cambridge University Press

It is easy to see that $\mathcal{H}_0$ is isomorphic to the (usual) affine Hecke algebra of SL_2 and that each $\mathcal{H}_j$ ($j \in \mathbb{Z}$) is a right and left $\mathcal{H}_0$-module.

We define

$$\begin{aligned}
&\Theta_{1,0} = \chi^{(1)}_{1,0},\\
&\Theta_{-1,0} = \chi^{(1)}_{-1,0} - (q-1)\chi^{(2)}_{-1,0} - (q-1)\chi^{(2)}_{0,0} + q(q+q^{-1}-2)\chi^{(1)}_{0,0},\\
&\Theta_{0,1} = \chi^{(1)}_{0,1},\\
&\Theta_{0,-1} = \chi^{(1)}_{0,-1} - (q-1)\textstyle\sum_{i\geq 0} q^i \chi^{(2)}_{i,-1}.
\end{aligned}$$

One can check $\Theta^{-1}_{1,0} = \Theta_{-1,0}$. (Recall that $\iota = q\chi^{(1)}_{0,0}$ is the identity.) The elements $\Theta_{1,0}$ and $\Theta_{-1,0}$ are the same as appear in Bernstein's presentation of the affine Hecke algebra $\mathcal{H}_0$.

Lemma 3.1 *The elements $\Theta_{1,0}$, $\Theta_{-1,0}$, $\Theta_{0,1}$and $\Theta_{0,-1}$ commute with each other. Moreover, we have:*
(1) $\Theta^i_{1,0} * \Theta^j_{0,1} = q^{-(i+j-1)}\chi^{(1)}_{i,j}$ for $i \in \mathbb{Z}$ and $j \geq 0$,
(2) $\Theta^i_{-1,0}*\Theta^j_{0,-1} = q^{-(i+j-1)}\chi^{(1)}_{-i,-j} - (q-1)q^{-(i+j-1)}\chi^{(2)}_{-i,-j} + \sum_{m>-ia=1,2} c^{(a)}_m \chi^{(a)}_{m,-j}$ for $i \in \mathbb{Z}$ and $j > 0$ and for $c^{(a)}_m \in \mathbb{C}$.

Proof We use the formulas in the Appendix and inductions on i and j to obtain (1) and (2). ∎

Let us denote by $\mathcal{A}$ the commutative subalgebra of $\mathcal{H}(G,I)$ generated by the elements $\Theta_{1,0}$, $\Theta_{-1,0}$, $\Theta_{0,1}$ and $\Theta_{0,-1}$. The structure of $\mathcal{A}$ is described by the following proposition.

Proposition 3.2 *The algebra $\mathcal{A}$ is isomorphic to the quotient of $\mathbb{C}[X, X^{-1}, Y, Z]$ by the relation $YZ = 0$.*

Proof We have the surjective homomorphism $\phi\colon \mathbb{C}[X, X^{-1}, Y, Z]/(YZ) \to \mathcal{A}$ defined by

$$\phi(X) = \Theta_{1,0}, \quad \phi(X^{-1}) = \Theta_{-1,0}, \quad \phi(Y) = \Theta_{0,1}, \quad \phi(Z) = \Theta_{0,-1}.$$

It follows from (1) and (2) of Lemma 3.1 that ϕ is also injective. ∎

Our next task is to determine the center of $\mathcal{H}(G,I)$.

Theorem 3.3 *The center of $\mathcal{H}(G,I)$ is the same as the center of $\mathcal{H}_0$ generated by the element $\Theta_{1,0} + \Theta_{-1,0}$.*

Proof It is well known that the element $\Theta_{1,0}+\Theta_{-1,0}$ generates the center of $\mathcal{H}_0$ [14]. Let us check if it commutes with $\chi^{(a)}_{i,j}$, $j \neq 0$. First, we assume $j > 0$. Since $\chi^{(1)}_{i,j} \in \mathcal{A}$, it commutes with $\Theta_{1,0} + \Theta_{-1,0}$. We have $\chi^{(2)}_{i,j} = q\chi^{(1)}_{i,j} * \chi^{(2)}_{0,0}$. Since $\chi^{(2)}_{0,0} \in \mathcal{H}_0$, we get $\chi^{(2)}_{i,j} * (\Theta_{1,0} + \Theta_{-1,0}) = (\Theta_{1,0} + \Theta_{-1,0}) * \chi^{(2)}_{i,j}$. Next, we assume $j < 0$. We obtain, using the formulas in the Appendix,

$$\chi^{(1)}_{i,j} * (\Theta_{1,0} + \Theta_{-1,0}) = q\chi^{(1)}_{i+1,j} + q^{-1}\chi^{(1)}_{i-1,j} = (\Theta_{1,0} + \Theta_{-1,0}) * \chi^{(1)}_{i,j}.$$

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Since $\chi^{(2)}_{i,j} = \chi^{(1)}_{i,j} * \chi^{(2)}_{0,0} - (1-q^{-1})\chi^{(1)}_{i,j}$, the element $\chi^{(2)}_{i,j}$ also commutes with $\Theta_{1,0} + \Theta_{-1,0}$. Therefore, the element $\Theta_{1,0} + \Theta_{-1,0}$ is in the center of the algebra $\mathcal{H}(G, I)$.

Suppose that ζ is an element in the center of $\mathcal{H}(G, I)$. We write $\zeta = \sum_{j\in\mathbb{Z}} \zeta_j$, $\zeta_j \in \mathcal{H}_j$. Since $\chi^{(2)}_{0,0} * \mathcal{H}_j \subset \mathcal{H}_j$ and $\mathcal{H}_j * \chi^{(2)}_{0,0} \subset \mathcal{H}_j$ for each j, the equality $\zeta * \chi^{(2)}_{0,0} = \chi^{(2)}_{0,0} * \zeta$ yields $\zeta_j * \chi^{(2)}_{0,0} = \chi^{(2)}_{0,0} * \zeta_j$ for each j. First, we assume $j > 0$. Suppose that we choose the largest i so that

$$\zeta_j = c_1\chi^{(1)}_{i,j} + c_2\chi^{(2)}_{i,j} + \sum_{m<ia=1,2} c^{(a)}_m \chi^{(a)}_{m,j}, \quad c_1 \neq 0 \text{ or } c_2 \neq 0.$$

We get

$$\chi^{(2)}_{0,0} * \zeta_j = c_1(1-q^{-1})\chi^{(1)}_{i,j} + c_2(1-q^{-1})\chi^{(2)}_{i,j} + \sum_{m<ia=1,2} c'^{(a)}_m \chi^{(a)}_{m,j}.$$

On the other hand,

$$\zeta_j * \chi^{(2)}_{0,0} = c_1 q^{-1}\chi^{(2)}_{i,j} + c_2\chi^{(1)}_{i,j} + c_2(1-q^{-1})\chi^{(2)}_{i,j} + \sum_{m<ia=1,2} c''^{(a)}_m \chi^{(a)}_{m,j}.$$

Thus we have $c_1 = c_2 = 0$, a contradiction. It implies that $\zeta_j = 0$.

A similar argument also works for the case $j < 0$, and we have $\zeta_j = 0$ in this case, too. Thus $\zeta = \zeta_0 \in \mathcal{H}_0$. It completes the proof. ∎

The double affine Weyl group W is not a Coxeter group, but it has a similar presentation as one can see in the following proposition due to A. N. Parshin. It is also related to Kyoji Saito's elliptic Weyl groups [16].

Proposition 3.4 **([15])** *The group W has a presentation given by*

$$W \cong \langle s_0, s_1, s_2 \mid s_0^2 = s_1^2 = s_2^2 = e, (s_0 s_1 s_2)^2 = e\rangle. \tag{3.5}$$

We can easily determine elements of W corresponding to the generators in the presentation. For example, we can take, using the same notation,

$$s_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \quad s_1 = \begin{pmatrix} 0 & t_1^{-1} \\ -t_1 & 0 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 0 & t_2^{-1} \\ -t_2 & 0 \end{pmatrix}.$$

We define

$$\phi_0 = q^{\frac12}\chi^{(2)}_{0,0} = q^{\frac12}\chi_{Is_0I}, \quad \phi_1 = q^{\frac12}\chi^{(2)}_{-1,0} = q^{\frac12}\chi_{Is_1I}, \quad \phi_2 = q^{\frac12}\chi^{(2)}_{0,-1} = q^{\frac12}\chi_{Is_2I}. \tag{3.6}$$

The elements ϕ_0 and ϕ_1 have the special property

$$\phi_0 * \mathcal{H}_- = 0 \quad \text{and} \quad \phi_1 * \mathcal{H}_+ = 0,$$

which follows from the formulas (2)(f) in the Appendix.

Proposition 3.7 *The following identities hold in* $\mathcal{H}(G, I)$*:*

$$\begin{aligned} \phi_0 * \phi_0 &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\phi_0 + \iota, \\ \phi_1 * \phi_1 &= (q^{\frac{1}{2}} - q^{-\frac{1}{2}})\phi_1 + \iota, \\ \phi_0 * \phi_1 * \phi_2 * \phi_0 * \phi_1 &= \phi_2. \end{aligned} \tag{3.8}$$

Proof We check all the relations using the formulas in the Appendix. ∎

Remark 3.9 We can consider the relations in (3.8) as Iwahori–Matsumoto type relations. The first two relations in (3.8) are the usual deformation. The last one in (3.8) reflects the structure of the group algebra of W. However, we have

$$\phi_2 * \phi_2 = (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) \sum_{m>0} q^{m-\frac{1}{2}} \chi^{(2)}_{m,-2},$$

which reveals a new feature of the Hecke algebra $\mathcal{H}(G, I)$.

Appendix

(1) (a) If $(i, j) \geq (0, 0)$ and $(k, l) \geq (0, 0)$, or if $(i, j) < (0, 0)$ and $(k, l) < (0, 0)$, then

$$\chi^{(1)}_{i,j} * \chi^{(1)}_{k,l} = q^{-1}\chi^{(1)}_{i+k,j+l}, \quad \chi^{(1)}_{i,j} * \chi^{(2)}_{k,l} = q^{-1}\chi^{(2)}_{i+k,j+l}.$$

(b) If $i \geq 0$, $j = 0$ and $l < 0$, or if $i < 0$, $j = 0$ and $l > 0$, then

$$\chi^{(1)}_{i,0} * \chi^{(1)}_{k,l} = q^{2|i|-1}\chi^{(1)}_{i+k,l}, \quad \chi^{(1)}_{i,0} * \chi^{(2)}_{k,l} = q^{2|i|-1}\chi^{(2)}_{i+k,l}.$$

(c) If $j > 0$, $k < 0$ and $l = 0$, then

$$\chi^{(1)}_{i,j} * \chi^{(1)}_{k,0} = q^{-2k-1}\chi^{(1)}_{i+k,j} + (1 - q^{-1}) \sum_{m=i+k}^{i-k-1} q^{i-k-m-1}\chi^{(2)}_{m,j},$$

$$\chi^{(1)}_{i,j} * \chi^{(2)}_{k,0} = (1 - q^{-1}) \sum_{m=i+k+1}^{i-k-1} q^{i-k-m-1}\chi^{(1)}_{m,j} + q^{-2k-2}\chi^{(2)}_{i+k,j}.$$

(d) If $j < 0$, $k \geq 0$ and $l = 0$, then

$$\chi^{(1)}_{i,j} * \chi^{(1)}_{k,0} = q^{2k-1}\chi^{(1)}_{i+k,j} + (1 - q^{-1}) \sum_{m=i-k}^{i+k-1} q^{-i+k+m}\chi^{(2)}_{m,j},$$

$$\chi^{(1)}_{i,j} * \chi^{(2)}_{k,0} = (1 - q^{-1}) \sum_{m=i-k}^{i+k} q^{-i+k+m}\chi^{(1)}_{m,j} + q^{2k}\chi^{(2)}_{i+k,j}.$$

(e) If $i \geq 0$, $j = 0$, $k < 0$ and $l = 0$, then

$$\chi^{(1)}_{i,0} * \chi^{(1)}_{k,0} = q^{\min\{2i-1,-2k-1\}} \chi^{(1)}_{i+k,0} + (1-q^{-1}) \sum_{m=\max\{i+k,-i-k\}}^{i-k-1} q^{i-k-m-1} \chi^{(2)}_{m,0},$$

$$\chi^{(1)}_{i,0} * \chi^{(2)}_{k,0} = (1-q^{-1}) \sum_{m=\max\{i+k+1,-i-k\}}^{i-k-1} q^{i-k-m-1} \chi^{(1)}_{m,0} + q^{\min\{2i-1,-2k-2\}} \chi^{(2)}_{i+k,0}.$$

(f) If $i < 0$, $j = 0$, $k \geq 0$ and $l = 0$, then

$$\chi^{(1)}_{i,0} * \chi^{(1)}_{k,0} = q^{\min\{-2i-1,2k-1\}} \chi^{(1)}_{i+k,0} + (1-q^{-1}) \sum_{m=i-k}^{\min\{i+k-1,-i-k-1\}} q^{-i+k+m} \chi^{(2)}_{m,0},$$

$$\chi^{(1)}_{i,0} * \chi^{(2)}_{k,0} = (1-q^{-1}) \sum_{m=i-k}^{\min\{i+k,-i-k-1\}} q^{-i+k+m} \chi^{(1)}_{m,0} + q^{\min\{-2i-1,2k\}} \chi^{(2)}_{i+k,0}.$$

(2) (a) If $j > 0$ and $l > 0$, then

$$\chi^{(2)}_{i,j} * \chi^{(1)}_{k,l} = (1-q^{-1}) \sum_{m \leq i+k} q^{i+k-m} \chi^{(1)}_{m,j+l},$$

$$\chi^{(2)}_{i,j} * \chi^{(2)}_{k,l} = (1-q^{-1}) \sum_{m \leq i+k} q^{i+k-m} \chi^{(2)}_{m,j+l}.$$

(b) If $j < 0$ and $l < 0$, then

$$\chi^{(2)}_{i,j} * \chi^{(1)}_{k,l} = (1-q^{-1}) \sum_{m > i+k} q^{-i-k+m-1} \chi^{(1)}_{m,j+l},$$

$$\chi^{(2)}_{i,j} * \chi^{(2)}_{k,l} = (1-q^{-1}) \sum_{m > i+k} q^{-i-k+m-1} \chi^{(2)}_{m,j+l}.$$

(c) If $(i, j) \geq (0, 0)$, $k < 0$ and $l = 0$, or if $(i, j) < (0, 0)$, $k \geq 0$ and $l = 0$, then

$$\chi^{(2)}_{i,j} * \chi^{(1)}_{k,0} = q^{-1} \chi^{(2)}_{i-k,j}, \quad \chi^{(2)}_{i,j} * \chi^{(2)}_{k,0} = q^{-1} \chi^{(1)}_{i-k,j}.$$

(d) If $j > 0$, $k \geq 0$ and $l = 0$, then

$$\chi^{(2)}_{i,j} * \chi^{(1)}_{k,0} = (1-q^{-1}) \sum_{m=i-k+1}^{i+k} q^{i+k-m} \chi^{(1)}_{m,j} + q^{2k-1} \chi^{(2)}_{i-k,j},$$

$$\chi^{(2)}_{i,j} * \chi^{(2)}_{k,0} = q^{2k} \chi^{(1)}_{i-k,j} + (1-q^{-1}) \sum_{m=i-k}^{i+k} q^{i+k-m} \chi^{(2)}_{m,j}.$$

(e) If $j < 0$, $k < 0$ and $l = 0$, then

$$\chi^{(2)}_{i,j} * \chi^{(1)}_{k,0} = (1 - q^{-1}) \sum_{m=i+k+1}^{i-k} q^{-i-k+m-1} \chi^{(1)}_{m,j} + q^{-2k-1} \chi^{(2)}_{i-k,j},$$

$$\chi^{(2)}_{i,j} * \chi^{(2)}_{k,0} = q^{-2k-2} \chi^{(1)}_{i-k,j} + (1 - q^{-1}) \sum_{m=i+k+1}^{i-k-1} q^{-i-k+m-1} \chi^{(2)}_{m,j}.$$

(f) If $i \geq 0$, $j = 0$ and $l < 0$, or if $i < 0$, $j = 0$ and $l > 0$, then

$$\chi^{(2)}_{i,0} * \chi^{(1)}_{k,l} = 0, \quad \chi^{(2)}_{i,0} * \chi^{(2)}_{k,l} = 0.$$

(g) If $i \geq 0$, $j = 0$ and $l > 0$, then

$$\chi^{(2)}_{i,0} * \chi^{(1)}_{k,l} = (1 - q^{-1}) \sum_{m=-i+k}^{i+k} q^{i+k-m} \chi^{(1)}_{m,l},$$

$$\chi^{(2)}_{i,0} * \chi^{(2)}_{k,l} = (1 - q^{-1}) \sum_{m=-i+k}^{i+k} q^{i+k-m} \chi^{(2)}_{m,l}.$$

(h) If $i < 0$, $j = 0$ and $l < 0$, then

$$\chi^{(2)}_{i,0} * \chi^{(1)}_{k,l} = (1 - q^{-1}) \sum_{m=i+k+1}^{-i+k-1} q^{-i-k+m-1} \chi^{(1)}_{m,l},$$

$$\chi^{(2)}_{i,0} * \chi^{(2)}_{k,l} = (1 - q^{-1}) \sum_{m=i+k+1}^{-i+k-1} q^{-i-k+m-1} \chi^{(2)}_{m,l}.$$

(i) If $i \geq 0$, $j = 0$, $k \geq 0$ and $l = 0$, then

$$\chi^{(2)}_{i,0} * \chi^{(1)}_{k,0} = (1 - q^{-1}) \sum_{m=\max\{i-k+1,-i+k\}}^{i+k} q^{i+k-m} \chi^{(1)}_{m,0} + q^{\min\{2i,2k-1\}} \chi^{(2)}_{i-k,0},$$

$$\chi^{(2)}_{i,0} * \chi^{(2)}_{k,0} = q^{\min\{2i,2k\}} \chi^{(1)}_{i-k,0} + (1 - q^{-1}) \sum_{m=\max\{i-k,-i+k\}}^{i+k} q^{i+k-m} \chi^{(2)}_{m,0}.$$

(j) If $i < 0$, $j = 0$, $k < 0$ and $l = 0$, then

$$\chi^{(2)}_{i,0} * \chi^{(1)}_{k,0} = (1 - q^{-1}) \sum_{m=i+k+1}^{\min\{i-k,-i+k-1\}} q^{-i-k+m-1} \chi^{(1)}_{m,0} + q^{\min\{-2i-2,-2k-1\}} \chi^{(2)}_{i-k,0},$$

$$\chi^{(2)}_{i,0} * \chi^{(2)}_{k,0} = q^{\min\{-2i-2,-2k-2\}} \chi^{(1)}_{i-k,0} + (1 - q^{-1}) \sum_{m=i+k+1}^{\min\{-i+k-1,i-k-1\}} q^{-i-k+m-1} \chi^{(2)}_{m,0}.$$

Acknowledgments I would like to thank Henry Kim and Ivan Fesenko for their encouragement and useful comments during preparation of this paper.

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