Probability in the Engineering and Informational Sciences (2023), **37**:2 442–461
doi:10.1017/S0269964822000481

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RESEARCH ARTICLE

On the dual risk model with Parisian implementation delays under a mixed dividend strategy

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Keywords: Dual risk model, Expected discounted dividend, Expected discounted penalty function, Parisian implementation delays, Threshold dividend

Abstract
In this paper, we consider a mixed dividend strategy in a dual risk model. The mixed dividend strategy is the combination of a threshold dividend and a Parisian implementation delays dividend under periodic observation. Given a series of discrete observation points, when the surplus level is larger than the predetermined bonus barrier at observation point, the Parisian implementation delays dividend is immediately carried out, and the threshold dividend is performed continuously during the delayed period. We study the Gerber-Shiu expected discounted penalty function and the expected discounted dividend payments before ruin in such a dual risk model. Numerical illustrations are given to study the influence of relevant parameters on the ruin-related quantities and the selection of the optimal dividend barrier for a given initial surplus level.

1. Introduction

The dual risk model describes the surplus of a company with a fixed expense rate, and earns a random amount of income at random times. Therefore, it might be appropriate to adopt this model for pharmaceutical, petroleum or any business with random growth. The dual risk model was introduced by Avanzi *et al.* [4], who studied the expected discounted dividends until ruin for the dual model under the barrier strategy, and showed that the optimal value of the dividend barrier under the dual model is independent of the initial surplus. However, if such a barrier strategy is applied, the ultimate ruin probability of the company is always to be 1. Ng [21] proposed a threshold dividend strategy to replace the barrier dividend strategy. In a threshold strategy, excess surplus is paid at a constant rate $c_1 > 0$ instead of a single burst. For more studies on the barrier and threshold dividend strategy, see Cheung and Drekic [7], Gerber and Smith [17], Avanzi *et al.* [5], Albrecher *et al.* [3], Yu *et al.* [31], Peng *et al.* [22], Zhou *et al.* [36], Liu *et al.* [19], Wang *et al.* [24], among others.

Although the surplus flow evolves continuously, it is only checked periodically by the board of directors or tax authority who decide on dividend payments to the shareholders of the insurance company. These led Albrecher *et al.* [1,2] to first consider periodic observation of the classical compound Poisson model. Because ruin and dividend can only be observed at random observation times $\{\nu_i\}_{i=0}^{\infty}$, a lump sum of dividend is payable at such discrete time points. Albrecher *et al.* [1,2] studied the expected discounted dividend payments before ruin and the expected discounted penalty function, respectively. For more related papers on this strategy, see Avanzi and Wong [5], Choi and Cheung [11], Cheung and Zhang [9], Yu *et al.* [30] and the references therein.

In addition to the above-mentioned continuous dividend and periodic dividend, recently, the Parisian implementation delays dividend have become very popular in ruin theory. The Parisian implementation delay idea originates from the concept of Parisian options, see Chesney *et al.* [6]. An example is the owner of which loses the option if the underlying asset price down-crosses the level b remains below this level for a time interval longer than d. In particular, Dassios and Wu [14] first introduced Parisian implementation delays in insurance risk models, if the surplus remains negative for a period of time, then Parisian ruin occurs, and they obtained the Laplace transform of the Parisian ruin time under the diffusion-perturbed classical model with exponentially distributed jumps. For more information on Parisian ruin, we refer to Czarna and Renaud [13], Yang *et al.* [29], Loeffen *et al.* [20], Wang and Zhou [25], Xu *et al.* [28] and so on. The Parisian implementation delays dividend has also attracted a lot of interests recently. Cheung and Wong [8] considered the dual risk model with Parisian implementation delays in dividend payments and derived the expression of the Laplace transform of the time of ruin and the expected discounted dividends paid until ruin. Zhao *et al.* [34] studied a spectrally positive Lévy risk process with Parisian implementation delays in dividend payments and derived the Laplace transform of the ruin time. For more on the Parisian implementation delays in dividend payments, see Wong and Cheung [26], Drekic *et al.* [16], Czarna *et al.* [12] and the references therein.

In principle, during the delay, the company still has access to the random amount of income at random times. Due to the uncontrollability of the delay time, the company may experience significant growth in-between delay dividend times and may wish to distribute a portion of the growth as dividends immediately. It is assumed that dividend is payable only when the process has stayed above the barrier for a certain amount of time $d > 0$. If the process dips below the barrier during that interval, then the decision is revoked and no dividend is paid. As a result, shareholders may never get a dividend. Motivated by this, we propose a class of hybrid dividend strategies that allow continuous dividends within the deferred dividend period. On the basis of Cheung and Wong [8], we add threshold dividend and periodic observation. That is to say, for a pre-specified sequence of random observation times $\{\nu_i\}_{i=0}^{\infty}$, when the surplus is observed above the pre-given barrier level $b > 0$, the Parisian implementation delays is carried out, and dividends will be paid continuously at a fixed rate $c_1 > 0$ in the process of delay (during the delayed period, the surplus drops to b, and the threshold dividend stops correspondingly); if the level of surplus remains above barrier level b throughout the deferral period, the amount exceeding the barrier level at the end of the delay will be paid as a lump sum dividend.

The outline of the paper is organized as follows. Section 2 gives an introduction to the model of this paper, a definition of the function to be studied and also gives some results that will be used in this paper. In Section 3, we provide the general expression of the expected discounted penalty function $\phi_b(u)$, and derive the general result of the function $\phi_b(u)$ by calculating the intermediate function in the expression. In the same way, we give expression and derivation of the expected discounted dividend function $V_b(u)$ in Section 4. In Section 5, some numerical examples are given to analyze the effect of relevant parameters on the ruin-related quantities and the selection of the optimal dividend barrier for a given initial surplus.

2. The model

We consider companies with deterministic expenses and random gains, its available capital can be described by the process $\{U(t)\}_{t\geq 0}$ (in the absence of dividends) defined via

$$U(t) = u - ct + S(t) = u - ct + \sum_{i=1}^{N(t)} X_i, \quad t \geq 0, \tag{2.1}$$

where $U(0) = u \geq 0$ is the initial surplus and $c > 0$ is the constant expense rate per unit time. The premium number process $\{N(t)\}_{t\geq 0}$ is a homogeneous Poisson process with intensity $\lambda > 0$, and the

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gain amounts $\{X_i\}_{i=1}^{\infty}$ are mutually independent and identically distributed (i.i.d.), and also independent of $\{N(t)\}_{t\geq 0}$. The time of ruin is given by $\tau_0 = \inf\{t \geq 0;\ U(t) = 0\}$.

Inspired by the articles mentioned in Section 1, this paper aims to propose a mixed dividend strategy with a Parisian implementation delays in the dual risk model (2.1), which extends the work by Cheung and Wong [8]. Then, the surplus process is denoted by $\{U_b^d\}_{t\geq 0}$. At the observation times $\{\nu_i\}_{i=0}^{\infty}$, $\nu_0 = 0$, if the level of surplus x is observed to exceed the previously given barrier b, then the dividend mixture begins. Defining $T_i = \nu_i - \nu_{i-1}$ for $i = 1, 2, \ldots$, and assumed that the inter-observation times $\{T_i\}_{i=1}^{\infty}$ are i.i.d. with same distribution as T and are independent of $\{N(t)\}_{t\geq 0}$ and $\{X_i\}_{i=1}^{\infty}$. We use $\{V_i\}_{i=1}^{\infty}$ to denote the ith Parisian implementation delay when $\{U_b^d\}_{t\geq 0}$ is observed for the ith time above b. It is assumed that the delays $\{V_i\}_{i=1}^{\infty}$ form a sequence of i.i.d. positive random variables that are independent of $\{N(t)\}_{t\geq 0}$, $\{T_i\}_{i=1}^{\infty}$ and $\{X_i\}_{i=1}^{\infty}$. Now, define the threshold dividend model $U_b(t)$ based on Model 2.1, and the auxiliary process $W_i(t)$, $i = 1, 2, \ldots$

$$U_b(t) = \begin{cases} U(t), & 0 \leq t < \nu_1, \\ U(t), & \nu_j \leq t \leq \nu_1^+,\ j = 1, 2, \ldots, \\ U(\nu_1^+) - (c + c_1)(t - \nu_1^+) + \displaystyle\sum_{i=N(\nu_1^+)+1}^{N(t)} X_i, & \nu_1^+ < t < \eta_1, \end{cases}$$

$$W_i(t) = \begin{cases} U_b(t), & t \geq 0,\ i = 1, \\ b - c(t - \eta_{i-1}) - c_1(t - \nu_i^+) + \displaystyle\sum_{i=N(\eta_{i-1})+1}^{N(t)} X_i, & t \geq \eta_{i-1},\ i = 2, 3, \ldots, \end{cases}$$

where $\nu_i^+ = \inf\{\nu_j \geq \eta_{i-1};\ W_i(\nu_j) > b\}, i = 1, 2, \ldots$ is the first time $\{W_i(t)\}_{t\geq\eta_{i-1}}$ is above the dividend barrier; $\eta_i = (\nu_i^+ + V_i) \wedge \theta_i$ for $i = 1, 2, \ldots$, with the starting point $\eta_0 = 0$ ($x \wedge y = \min(x,\ y)$); whereas $\theta_i = \inf\{t > \nu_i^+;\ W_i(t) = b\}, i = 1, 2, \ldots$ represents the first time $\{W_i(t)\}_{t\geq\nu_i^+}$ down-crosses level b due to the expense rate and threshold dividend. The surplus process $\{U_b^d(t)\}_{t\geq 0}$ can now be characterized by

$$U_b^d(t) = W_i(t), \quad \eta_{i-1} \leq t \leq \eta_i, \quad i = 1, 2, \ldots. \tag{2.2}$$

From the previous hypothesis, we noticed that if $\{W_i(t)\}_{t\geq\nu_i^+}$ stays above the dividend barrier continuously for a period of V_i such that $W_i(\nu_i^+ + V_i) > b$ (or equivalently, $\nu_i^+ + V_i \leq \theta_i$ so that $\eta_i = \nu_i^+ + V_i$), then a dividend of $W_i(\eta_i) - b$ will be paid at time η_i, dragging the process $\{U_b^d(t)\}_{t\geq 0}$ back to level b. On the other hand, if $\{W_i(t)\}_{t\geq\nu_i^+}$ drops below b within a period of length V_i (i.e. $\theta_i \leq \nu_i^+ + V_i$) so that $\eta_i = \theta_i$, then no Parisian dividend will be paid at time η_i. The time of ruin in this modified model $\{U_b^d(t)\}_{t\geq 0}$ is defined as $\tau = \inf\{\nu_i;\ U_b^d(\nu_i) \leq 0\}$. For convenience, we let $c + c_1 = c_2$.

To illustrate the features of $\{U_b^d(t)\}_{t\geq 0}$, we plot a sample path of it in Figure 1, where "type 1" and "type 2" represent dividends generated by continuous dividend payments at rate c_1 and Parisian implementation delays dividend, respectively. In this paper, we are interested in the Gerber-Shiu expected discounted penalty function that is defined as (classical risk model)

$$\phi(u) := E^u[e^{-\delta\tau}\omega(U(\tau-), |U(\tau)|)I_{\{\tau<\infty\}}], \quad u \geq 0,$$

where $\delta \geq 0$ is the Laplace transform argument, and E^u is the expectation of the initial surplus u, $I_{\{\tau<\infty\}}$ is an indicator function, $\omega : [0, \infty)\times[0, \infty) \to [0, \infty)$ is a measurable penalty function of the $U(\tau-)$ and $|U(\tau)|$. It has become an important and standard risk measure in ruin theory since various quantities of interests in ruin theory can be obtained for different values of the discount factor δ and different penalty functions ω. For recent research progress on the Gerber-Shiu function, we can refer to work by Lin *et al.* [18], Yuen *et al.* [32], Zhao and Yin [35], Chi and Lin [10], Deng *et al.* [15], Zhang and Su [33], Xie and Zhang [27], among others. In this paper, we consider the Gerber-Shiu expected discounted penalty

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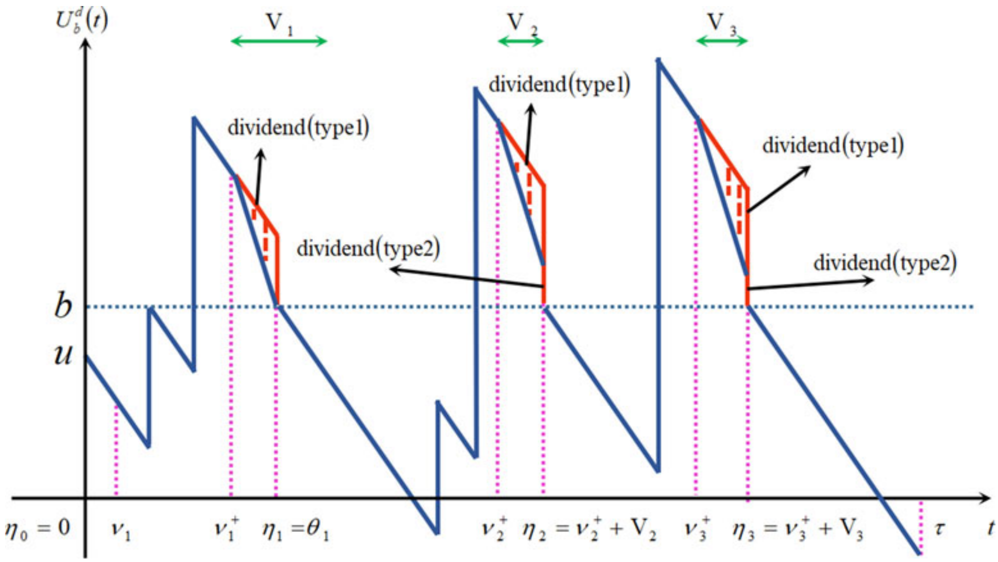

Figure 1. *Sample path of* $\{U_b^d(t)\}_{t\geq 0}$.

function is simply (a constant multiple of) the Laplace transform of the time of ruin given by

$$E^u[e^{-\delta\tau}I_{\{\tau<\infty\}}] = E^u[e^{-\delta\tau}] = \phi_b(u) = \begin{cases} \phi_{Lb}(u), & u < 0, \\ \phi_{Mb}(u), & 0 \leq u \leq b, \\ \phi_{Ub}(u), & b < u, \end{cases} \tag{2.3}$$

where the subscripts "L," "M" and "U" stand for "lower," "middle" and "upper" layers, respectively. Note that we have omitted the indicator $I_{\{\tau<\infty\}}$ of the event $\{\tau < \infty\}$ in the definition (2.3) because ruin occurs with probability one in the presence of a barrier. In addition to the Laplace transform of the time of ruin, we also defined about the expected discounted dividend payments before ruin, which is

$$\begin{aligned} &E^u\left[\sum_{i=1}^{\infty} e^{-\delta\eta_i}[U_b^d(\eta_i^-) - b + c_1\int_{\eta_i^-}^{\nu_i^+} e^{-\delta t}\,dt]I_{\{\eta_i<\tau\}}\right] \\ &= V_b(u) = \begin{cases} V_{Lb}(u), & u < 0, \\ V_{Mb}(u), & 0 \leq u \leq b, \\ V_{Ub}(u), & b < u. \end{cases} \end{aligned} \tag{2.4}$$

Remark 2.1. Theoretically, the classical risk model and the dual risk model have great similarity, the two are mutual reflection in nature. Through the duality principle, studying a problem in one model can often provide ideas or even directly solve the problem in another model. Therefore, the classical risk model and some corresponding existing results are introduced below which will be applied to calculate Eqs. (2.3) and (2.4) later. We first introduce the classical compound Poisson insurance risk process $\{U_S(t)\}_{t\geq 0}$ which is defined by

$$U_S(t) = u + ct - S(t) = u + ct - \sum_{i=1}^{N(t)} X_i, \quad t \geq 0, \tag{2.5}$$

where $U_S(0) = u \geq 0$ is the initial surplus, $c > 0$ is now the incoming premium rate per unit time, $S(t)$ represents the aggregate claim amounts by time t and $\{X_i\}_{i=1}^{\infty}$ is interpreted as the sequence of insurance

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claims. Considering the periodic observation $\{T_i\}_{i=1}^{\infty}$, the model (2.5) becomes

$$U_S(v_i) = U_S(v_{i-1}) + cT_i - [S(v_i) - S(v_{i-1})], \quad i = 1, 2, \dots. \tag{2.6}$$

The time of ruin is defined by $\tau_d = v_{k_b}$, where $k_b = \inf\{k \geq 1 : U_S(v_k) < 0\}$ is the number of observation intervals before ruin. Similarly, we define the moment when the surplus process first crosses the barrier b as $\tau_b = v_{k_b^*}$, where $k_b^* = \inf\{k \geq 1 : U_S(v_k) \geq b\}$ is the number of observation intervals before the first crossing barrier b. To derive the Gerber-Shiu expected discounted penalty function $\phi_b(u)$ and the expected discounted dividend payments before ruin $V_b(u)$, we made the following assumption.

Assumption 1. *We assume that the* $\{T_i\}_{i=1}^{\infty}$ *form an i.i.d. sequence with common density* $f_T(t) = \beta e^{-\beta t}$, $t > 0$, *where the scale parameter* $\beta > 0$.

Assumption 2. *We assume that the distribution X of single claim (gain) amount in this paper is exponential distribution, and its density function is* $p_(x) = a_1 e^{-a_1 x}$, $x > 0$.

Assumption 3. *We assume that each Parisian implementation delay is deterministic such that* $V_i = d$ *for all* $i = 1, 2, \dots$.

Let

$$\lambda[\hat{f}_X(\varepsilon) - 1] + c\varepsilon = \delta + \beta, \tag{2.7}$$

where $f_X(x)$ is the density function of single claim quantity, let $\hat{f}_X(s) = \int_0^{\infty} e^{-sx} f_X(x)\, dx$ denote the Laplace transform of the claim size density. Therefore, Eq. (2.7) can be simplified to

$$\varepsilon^2 + \left(a_1 - \frac{\lambda + \beta + \delta}{c}\right)\varepsilon - \frac{(\beta + \delta)a_1}{c} = 0. \tag{2.8}$$

By Eq. (2.8) in Albrecher *et al.* [2], we known that it has a unique negative solution $\rho_1 < 0$ and a positive solution $\rho_2 > 0$. Note that for $\beta = 0$, Eq. (2.8) reduces to the well-known Lundberg fundamental equation of the compound Poisson risk process. There is also a unique negative root $\rho_1^0 < 0$, and the only positive root $\rho_2^0 > 0$.

We define the discounted density of the ruin deficit of $\{U_S(t)\}_{t\geq 0}$ at random observation to be $h_\delta^+(y \mid u),\ u \geq 0, h_\delta^-(y \mid u),\ u < 0$. According to the Eqs. (2.16) and (2.17) of Albrecher *et al.* [2]

$$h_\delta^+(y \mid u) = (\rho_1^0 - \rho_1)e^{\rho_1^0 u + \rho_1 y}, \quad u \geq 0, \tag{2.9}$$

$$\begin{aligned} h_\delta^-(y \mid u) = {} & \frac{\beta(a_1 + \rho_2)(\rho_1^0 - \rho_1)}{c(\rho_2 - \rho_1)(\rho_2 - \rho_1^0)} e^{\rho_2 u + \rho_1 y} + \frac{\beta(a_1 + \rho_2)}{c(\rho_2 - \rho_1)} e^{\rho_2(u+y)} I_{\{y \leq -u\}} \\ & + \frac{\beta(a_1 - \rho_1)}{c(\rho_2 - \rho_1)} e^{\rho_1(u+y)} I_{\{y > -u\}}, \quad u < 0. \end{aligned} \tag{2.10}$$

We calculate the dividend function $V_b(x)$ by analyzing the discounted density of the increment of the process $\{U_S(t)\}_{t\geq 0}$ between continuous observation time points (see [2]). Due to the Markovian structure of $\{U_S(t)\}_{t\geq 0}$, this sequence of pairs is i.i.d. with generic distribution $(T, \sum_{i=1}^{N(T)} X_i - cT)$ and joint Laplace transform is

$$E\left[e^{-\delta T - s\left(\sum_{i=1}^{N(T)} X_i - cT\right)}\right] = E[e^{-[\lambda + \delta - cs - \lambda M_X(-s)]T}] = \int_{-\infty}^{\infty} e^{-sy} g_\delta(y)\, dy, \tag{2.11}$$

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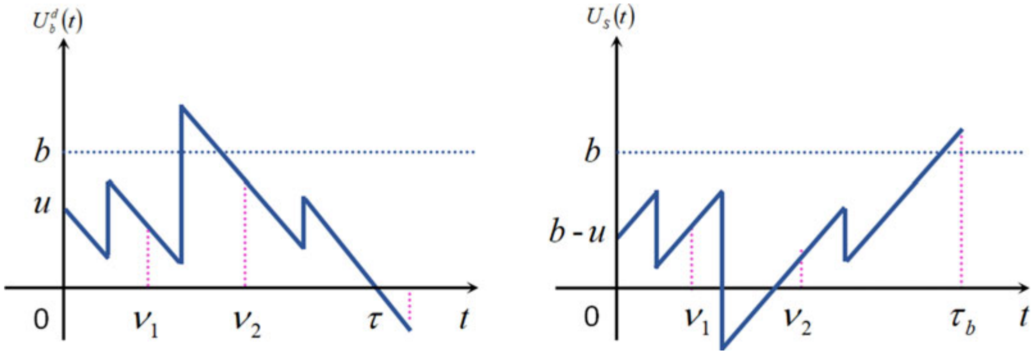

Figure 2. *Sample path before and after folding.*

where $g_\delta(y)(-\infty < y < \infty)$ represents the discounted density of the increment $\sum_{i=1}^{N(T)} X_i - cT$ between successive observation times, $M_X(s)$ is the moment generating function of X, discounted at rate δ with respect to time T. According to the Example (4.1) of Albrecher *et al.* [2], they gives

$$g_\delta^-(y) = \frac{\beta(a_1+\rho_2)}{c(\rho_2-\rho_1)}e^{\rho_2 y}, \quad g_\delta^+(y) = \frac{\beta(a_1+\rho_1)}{c(\rho_2-\rho_1)}e^{\rho_1 y}, \quad y > 0. \tag{2.12}$$

3. The Laplace transform of the time of ruin

In this section, we study the Laplace transform of the time of ruin $\phi_b(u)$. It can be seen from Eq. (2.3) that the form of $\phi_b(u)$ varies with the initial surplus u. Therefore, we first consider the case where the initial surplus $0 \le u \le b$. The case where the initial surplus $u < 0$ and $u > b$ will be resolved later.

For $0 \le u \le b$, we need to distinguish whether ruin occurs before the process was first observed above or below level b (i.e. ruin occurs before or after ν_1^+)

$$\phi_{Mb}(u) = E^u[e^{-\delta\tau} I_{\{\tau<\nu_1^+\}}] + E^u[e^{-\delta\eta_1} I_{\{\nu_1^+<\tau\}}]\phi_{Mb}(b). \tag{3.1}$$

Similarly, let $u = b$ in the above equation and bring back to the above equation to get

$$\phi_{Mb}(u) = E^u[e^{-\delta\tau} I_{\{\tau<\nu_1^+\}}] + \frac{E^u[e^{-\delta\eta_1} I_{\{\nu_1^+<\tau\}}]E^b[e^{-\delta\tau} I_{\{\tau<\nu_1^+\}}]}{1 - E^b[e^{-\delta\eta_1} I_{\{\nu_1^+<\tau\}}]}. \tag{3.2}$$

It can be seen from Eq. (3.2) that the key to calculate the Laplace transform of the ruin time is to calculate $E^u[e^{-\delta\tau} I_{\{\tau<\nu_1^+\}}]$ and $E^u[e^{-\delta\eta_1} I_{\{\nu_1^+<\tau\}}]$ for $0 \le u \le b$. Next, we will calculate these two expressions separately.

3.1. The discussion of $E^u[e^{-\delta\tau} I_{\{\tau<\nu_1^+\}}]$, $0 \le u \le b$

When $0 \le u \le b$, the ruin occurred before the first observed surplus was above b, the surplus process $\{U_b^d(t)\}_{t\ge 0}$ simply behaves like the process $\{U_b(t)\}_{t\ge 0}$ prior to time ν_1^+. As a result, the quantity $E^u[e^{-\delta\tau} I_{\{\tau<\nu_1^+\}}]$ is independent of the distributional assumption on the Parisian implementation delay for $0 \le u \le b$, so that we have Figure 2.

Theorem 1. *When $0 \le u \le b$, we have*

$$E^u[e^{-\delta\tau} I_{\{\tau<\nu_1^+\}}] = \frac{(\rho_2-\rho_2^0)(\rho_2-\rho_1^0)[(\rho_2^0-\rho_1)e^{\rho_2^0(b-u)} - (\rho_1^0-\rho_1)e^{\rho_1^0(b-u)}]}{\rho_2(\rho_2^0-\rho_1)(\rho_2-\rho_1^0)e^{\rho_2^0 b} - (\rho_1^0-\rho_1)(\rho_2-\rho_2^0)e^{\rho_1^0 b}}, \quad 0 \le u \le b. \tag{3.3}$$

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Proof. From Figure 3 and Remark 2.1, we want to calculate the $E^u[e^{-\delta\tau}I_{\{\tau<\nu_1^+\}}]$, which can be observed that the event $\{\tau < \nu_1^+\}$ in the process $\{U_b^d(t)|U_b^d(0) = u\}_{t\geq 0}$ is equivalent to the event that $\{U_S(t)|U_S(0) = b-u\}_{t\geq 0}$ reaches level b before observation dropping below zero. Under such an event, τ in $\{U_b^d(t)|U_d(0) = u\}_{t\geq 0}$ is simply τ_b in $\{U_S(t)|U_b(0) = b-u\}_{t\geq 0}$. We have

$$E^u[e^{-\delta\tau}I_{\{\tau<\nu_1^+\}}] = E[e^{-\delta\tau_b}I_{\{\tau_b<\tau_d\}}|U_S(0) = b-u], \quad 0 \leq u \leq b, \tag{3.4}$$

which can be seen as the Laplace transform of τ_b when $\{U_S(t)\}_{t\geq 0}$ is above b for the first time before ruin. In the spirit of Albrecher *et al.* [1], suppose a penalty function $\omega^*(\cdot)$ is applied to the first overshoot of $\{U_S(t)\}_{t\geq 0}$ over level b avoiding ruin until then and define the quantity

$$\chi(u) = E[e^{-\delta\tau_b}\omega^*(U_S(\tau_b) - b)I_{\{\tau_b<\tau_d\}}|U_S(0) = u], \quad 0 \leq u \leq b. \tag{3.5}$$

According to the Section 4 of Albrecher *et al.* [1], we have

$$\chi(u) = \int_0^\infty \omega^*(y)h_\delta(y\,|\,u)\,dy, \quad 0 \leq u \leq b, \tag{3.6}$$

where $h_\delta(y\,|\,u)$ is the discounted density of the overshoot above level b avoiding ruin. Assume again that both the claim sizes and the observation intervals are exponentially distributed with mean $1/a_1$ and $1/\beta$, we have

$$h_\delta(y\,|\,u) = \frac{e^{-\rho_2 y}(\rho_2-\rho_2^0)(\rho_2-\rho_1^0)[(\rho_2^0-\rho_1)e^{\rho_2^0 u}-(\rho_1^0-\rho_1)e^{\rho_1^0 u}]}{(\rho_2^0-\rho_1)(\rho_2-\rho_1^0)e^{\rho_2^0 b}-(\rho_1^0-\rho_1)(\rho_2-\rho_2^0)e^{\rho_1^0 b}}, \quad y>0;\ 0\leq u\leq b. \tag{3.7}$$

Therefore, when we consider the penalty function $\omega^*(\cdot) = 1$, we obtain

$$\chi(u) = \int_0^\infty \frac{e^{-\rho_2 y}(\rho_2-\rho_2^0)(\rho_2-\rho_1^0)[(\rho_2^0-\rho_1)e^{\rho_2^0 u}-(\rho_1^0-\rho_1)e^{\rho_1^0 u}]}{(\rho_2^0-\rho_1)(\rho_2-\rho_1^0)e^{\rho_2^0 b}-(\rho_1^0-\rho_1)(\rho_2-\rho_2^0)e^{\rho_1^0 b}}\,dy, \quad 0\leq u\leq b. \tag{3.8}$$

Consolidating the above observations, we arrive at Eq. (3.3). So replace u in Eq. (3.8) with $b-u$. □

3.2. *The discussion of* $E^u[e^{-\delta\eta_1}I_{\{\nu_1^+<\tau\}}], 0 \leq u \leq b$

Theorem 2. *When* $0 \leq u \leq b$*, we have*

$$E^u[e^{-\delta\eta_1}I_{\{\nu_1^+<\tau\}}] = \int_0^\infty \left(h_\delta^+(y\,|\,b-u) - \int_0^\infty E[e^{-\delta\tau_b}I_{\{\tau_b<\tau_d\}}; U_S(\tau_b)\in b+dx\,|\,U_S(0)=b-u]\, h_\delta^+(y\,|\,b+x)\right) E^{b+y}[e^{-\delta\eta_1}]\,dy, \tag{3.9}$$

where

$$h_\delta^+(y\,|\,u) = (\rho_1^0-\rho_1)e^{\rho_1^0 u+\rho_1 y}, \quad y>0. \tag{3.10}$$

Proof. In model (2.1), when the level of surplus crosses barrier b, it can be analogous to the deficit at the moment of ruin in model (2.5). Therefore, the deficit of ruin of surplus process $\{U_b^d(t)\}_{t\geq 0}$ is equivalent

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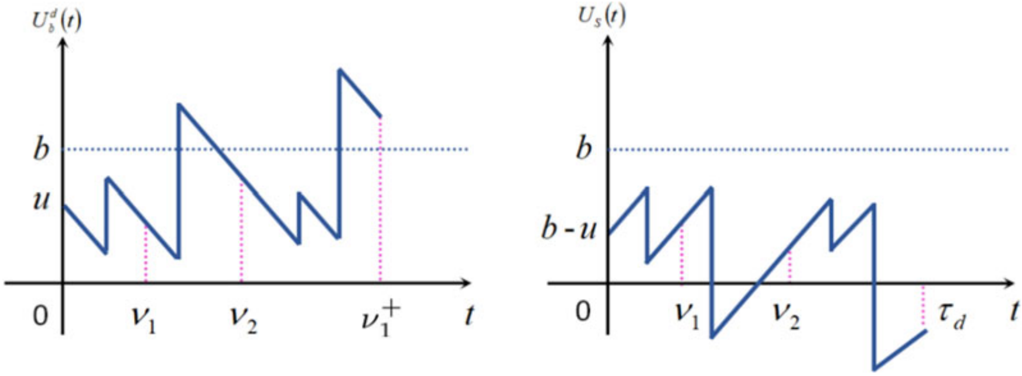

Figure 3. *Sample path before and after folding.*

to the part that the surplus process of $\{U_S(t)\}_{t\geq 0}$ exceeds level b. It can be known from Eq. (2.9) that

$$h_\delta^+(y\,|\,u) = (\rho_1^0 - \rho_1)e^{\rho_1^0 u + \rho_1 y}, \quad y > 0.$$

Note that for surplus process $\{U_S(t)\}_{t\geq 0}$, we consider the discounted density of the ruin deficit $h_\delta^+(y\,|\,u)$. One of the cases is that the time when the surplus level exceeds b is observed for the first time before the ruin, which is equivalent to that in the surplus process $\{U_b^d(t)\}_{t\geq 0}$, the ruin occurs before the surplus level exceeds b is observed for the first time which needs to be deducted. The sample trajectories of surplus process $\{U_b^d(t)\}_{t\geq 0}$ and surplus process $\{U_S(t)\}_{t\geq 0}$ are shown in Figure 3.

Such contribution can be removed by subtracting $\int_0^\infty E[e^{-\delta\tau_b} I_{\{\tau_b<\tau_d\}}; U_S(\tau_b) \in b + dx\,|\,U_S(0) = b - u]h_\delta^+(y\,|\,b + x)$ from $h_\delta^+(y\,|\,b - u)$. Thus, we can obtain Eq. (3.9). Since the functions $\int_0^\infty E[e^{-\delta\tau_b} I_{\{\tau_b<\tau_d\}}; U_S(\tau_b) \in b + dx\,|\,U_S(0) = b - u]h_\delta^+(y\,|\,b + x)$ can be obtained from the calculation of $\chi(u)$, refer to Section 4 of Albrecher *et al.* [1] for details, it suffices to derive $E^u[e^{-\delta\eta_1}]$ for $u > b$ in order to have a full characterization of $\phi_{Mb}(u)$. Finally, we consider $E^u[e^{-\delta\eta_1}]$, $u > b$.

According to the previous definition, when $U_b^d(0) = u > b$, since the surplus was observed above b at the initial moment, one has that $\eta_1 = \theta_1 \wedge d$. We found that during the delayed period, due to the threshold dividend, the surplus level can only be continuously observed until the surplus again recover to the obstacle level b. For the results of the $E^u[e^{-\delta\eta_1}]$, $u > b$, can directly be used, where Cheung and Wong [8] Eqs. (4.5), (4.9) and (4.10)

$$E^u[e^{-\delta\eta_1}] = E^{u-b}[e^{-\delta(\tau_0\wedge d)}]. \tag{3.11}$$

If ruin occurs before the first gain, then $\tau_0 = u/c_2$ with probability $e^{-\lambda(u/c_2)}$. In contrast, if there is at least one gain before ruin, then $\tau_0 > u/c_2$. From the Eq. (4.38) of Seal [23], we know that the density function $f_U(t\,|\,u)$ at the time of ruin τ_0 is

$$f_U(t\,|\,u) = \sum_{k=1}^{\infty} \frac{\lambda^k t^{k-1} e^{-\lambda t}(u)}{k!} p^{*k}(c_2 t - u), \quad t > u/c_2, \tag{3.12}$$

where $p^{*k}(\cdot)$ is the k-fold convolution density of the gain density $p(\cdot)$ with itself. when $u \geq c_2 d$, ruin happens after u/c_2, which is $\tau_0 \geq d$, and we arrive at

$$E^u[e^{-\delta(\tau_0\wedge d)}] = e^{-\delta d}, \quad u \geq c_2 d. \tag{3.13}$$

For $0 < u < c_2 d$, conditioning on τ_0 leads to

$$\begin{aligned} E^u[e^{-\delta(\tau_0\wedge d)}] &= E^u[e^{-\delta\tau_0} I_{\{\tau_0<d\}}] + E^u[e^{-\delta d} I_{\{\tau_0\geq d\}}] \\ &= e^{-(\delta+\lambda)u/c_2} + \int_{u/c_2}^{d} e^{-\delta t} f_U(t\,|\,u)\,dt \\ &\quad + e^{-\delta d}\left(1 - e^{-\lambda u/c_2} - \int_{u/c_2}^{d} f_U(t\,|\,u)\,dt\right), \quad 0<u<c_2 d. \end{aligned} \tag{3.14}$$

Note that the density function $p(y) = a_1 e^{-a_1 y}$, $p^{*k}(\cdot)$ is an Erlang(k) density and hence Eq. (3.12) becomes

$$f_U(t\,|\,u) = \sum_{k=1}^{\infty} \frac{\lambda^k t^{k-1} e^{-\lambda t} u}{k!}\left(\frac{a_1^k (c_2 t - u)^{k-1} e^{-a_1(c_2 t - u)}}{(k-1)!}\right), \quad t > u/c_2. \tag{3.15}$$

Then, the first integral in Eq. (3.14) is found to be

$$\begin{aligned} &\int_{u/c_2}^{d} e^{-\delta t} f_U(t\,|\,u)\,dt \\ &= \sum_{k=1}^{\infty} \frac{(\lambda a_1)^k u}{k!(k-1)!} \int_{u/c_2}^{d} t^{k-1} e^{-(\lambda+\delta)t} (c_2 t - u)^{k-1} e^{-a_1(c_2 t - u)}\,dt \\ &= \sum_{k=1}^{\infty} \frac{(\lambda a_1/c_2)^k}{k!} \sum_{i=0}^{k-1} \frac{(k+i-1)!}{(i!(k-i-1)!)(\frac{\lambda+\delta}{c_2}+a_1)^{k+i}} \\ &\quad \times \left(u^{k-i} e^{-\frac{\lambda+\delta}{c_2}u} - \sum_{j=0}^{k+i-1} \frac{[(\frac{\lambda+\delta}{c_2}+a_1)(c_2 d - u)]^j e^{-((\lambda+\delta)/c_2+a_1)c_2 d}}{j!} u^{k-i} e^{a_1 u}\right). \end{aligned} \tag{3.16}$$

The remaining integral $\int_{u/c_2}^{d} f_U(t\,|\,u)\,dt$ in Eq. (3.14) is simply a special case of the above expression with $\delta = 0$, and because that

$$\begin{aligned} &h_\delta^+(y\,|\,b-u) - \int_0^\infty E[e^{-\delta\tau_b} I_{\{\tau_b<\tau_d\}}; U_S(\tau_b)\in b+dx\,|\,U_S(0)=b-u]\,h_\delta^+(y\,|\,b+x) \\ &= (\rho_1^0-\rho_1)e^{\rho_1^0 b+\rho_1 y}\left(e^{-\rho_1^0 u} + \frac{(\rho_2-\rho_2^0)(\rho_2-\rho_1^0)[(\rho_2^0-\rho_1)e^{\rho_2^0(b-u)} - (\rho_1^0-\rho_1)e^{\rho_1^0(b-u)}]}{\rho_1^0\rho_2(\rho_2^0-\rho_1)(\rho_2-\rho_1^0)e^{\rho_2^0 b} - (\rho_1^0-\rho_1)(\rho_2-\rho_2^0)e^{\rho_1^0 b}}\right). \end{aligned} \tag{3.17}$$

Application of Eqs. (3.11), (3.13) and (3.14) to Eq. (3.9) leads to

$$\begin{aligned} &E^u[e^{-\delta\eta_1} I_{\{\nu_1^+<\tau\}}] \\ &= (\rho_1^0-\rho_1)e^{\rho_1^0 b}\left(e^{-\rho_1^0 u} + \frac{(\rho_2-\rho_2^0)(\rho_2-\rho_1^0)[(\rho_2^0-\rho_1)e^{\rho_2^0(b-u)} - (\rho_1^0-\rho_1)e^{\rho_1^0(b-u)}]}{\rho_1^0\rho_2(\rho_2^0-\rho_1)(\rho_2-\rho_1^0)e^{\rho_2^0 b} - (\rho_1^0-\rho_1)(\rho_2-\rho_2^0)e^{\rho_1^0 b}}\right) \\ &\quad \times \left\{\frac{1}{\frac{\lambda+\delta}{c_2}-\rho_1}\left(1-e^{-(\frac{\lambda+\delta}{c_2}-\rho_1)c_2 d}\right) - e^{-\delta d}\left[\frac{1}{\rho_1} + \frac{1}{\frac{\lambda}{c_2}-\rho_1}(1-e^{-(\frac{\lambda}{c_2}-\rho_1)c_2 d})\right]\right. \\ &\quad \left. + \int_0^{c_2 d} e^{\rho_1 y}\int_{y/c_2}^{d} e^{-\delta t} f_U(t|y)\,dt\,dy - e^{-\delta d}\int_0^{c_2 d} e^{\rho_1 y}\int_{\frac{y}{c_2}}^{d} f_U(t|y)\,dt\,dy\right\}, \quad 0\leq u\leq b. \end{aligned} \tag{3.18}$$

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Using Eq. (3.15), the first double integral above is evaluated as

$$\begin{aligned}\int_0^{c_2 d} & e^{\rho_1 y} \int_{y/c_2}^{d} e^{-\delta t} f_U(t|y)\, dt\, dy \\ &= \sum_{k=1}^{\infty} \frac{(-\frac{\lambda\rho_1}{c_2})^k}{k!} \sum_{i=0}^{k-1} \frac{(k+i-1)!}{i!(k-i-1)!} \frac{1}{(\frac{\lambda+\delta}{c_2}-\rho_1)^{k+i}} \left\{ \frac{(k-i)!}{(\frac{\lambda+\delta}{c_2}-\rho_1)^{k-i+1}} \right. \\ &\times \left(1 - \sum_{j=0}^{k-i} \frac{[(\frac{\lambda+\delta}{c_2}-\rho_1)c_2 d]^j e^{-(\frac{\lambda+\delta}{c_2}-\rho_1)c_2 d}}{j!}\right) - \sum_{j=0}^{k+i-1} \left(\frac{\lambda+\delta}{c_2}-\rho_1\right)^j \\ &\left. \times e^{-(\frac{\lambda+\delta}{c_2}-\rho_1)c_2 d} \frac{(c_2 d)^{j+k-i+1}(k-i)!}{(j+k-i+1)!} \right\}, \end{aligned} \tag{3.19}$$

and the second double integral in Eq. (3.18) can be directly set as $\delta = 0$ in Eq. (3.19). With the Eqs. (3.18) and (3.3) derived, $\phi_{Mb}(u),\ 0 \le u \le b$ are determined by Eq. (3.2). □

The following two Remarks respectively give the results satisfied by $\phi_{Ub}(u)$ and $\phi_{Lb}(u)$.

Remark 3.1. When $u > b$, at the time of the first observation, there are two scenarios for surplus $\phi_{Ub}(u)$

1. When the first observation is made, surplus $U_b^d(t)$ is already ruin ($U_b^d(\nu_1) < 0$),

$$\chi_{U_1}(u) = \int_u^{\infty} g_\delta^-(y)\, dy. \tag{3.20}$$

2. When the first observation is occurred, the surplus $U_b^d(t)$ is above 0 ($U_b^d(\nu_1) > 0$),

$$\begin{aligned}\chi_{U_2}(u) &= \int_{u-b}^{u} g_\delta^-(y)\phi_{Mb}(u-y)\, dy + \int_0^{u-b} g_\delta^-(y) E^{u-y-b}[e^{-\delta(\tau_0 \wedge d)}]\phi_{Mb}(b)\, dy \\ &+ \int_0^{\infty} g_\delta^+(y) E^{u+y-b}[e^{-\delta(\tau_0 \wedge d)}]\phi_{Mb}(b)\, dy. \end{aligned} \tag{3.21}$$

So $\phi_{Ub}(u)$ can be expressed as

$$\begin{aligned}\phi_{Ub}(u) &= \chi_{U_1}(u) + \chi_{U_2}(u) \\ &= \int_{u-b}^{u} g_\delta^-(y)\phi_{Mb}(u-y)\, dy + \int_u^{\infty} g_\delta^-(y)\, dy + \int_0^{u-b} g_\delta^-(y) \\ &\times E^{u-y-b}[e^{-\delta(\tau_0 \wedge d)}]\phi_{Mb}(b)\, dy \\ &+ \int_0^{\infty} g_\delta^+(y) E^{u+y-b}[e^{-\delta(\tau_0 \wedge d)}]\phi_{Mb}(b)\, dy, \quad u > b. \end{aligned} \tag{3.22}$$

Remark 3.2. When $u < 0$, at the time of the first observation, there are also two scenarios for surplus $\phi_{Lb}(u)$.

1. When the first observation is made, surplus $U_b^d(t)$ is already ruin ($U_b^d(\nu_1) < 0$),

$$\chi_{L_1}(u) = \int_0^{\infty} g_\delta^-(y)\, dy + \int_0^{-u} g_\delta^+(y)\, dy. \tag{3.23}$$

2. When the first observation is occurred, the surplus $U_b^d(t)$ is above 0 ($U_b^d(\nu_1) > 0$),

$$\chi_{L_2}(u) = \int_{-u}^{b-u} g_\delta^+(y)\phi_{Mb}(u+y)\,dy + \int_{b-u}^{\infty} g_\delta^+(y)E^{u+y-b}[e^{-\delta(\tau_0\wedge d)}]\phi_{Mb}(b)\,dy. \tag{3.24}$$

Then $\phi_{Lb}(u)$ can be expressed as

$$\begin{aligned}\phi_{Lb}(u) &= \chi_{L_1}(u) + \chi_{L_2}(u)\\ &= \int_0^{\infty} g_\delta^-(y)\,dy + \int_0^{-u} g_\delta^+(y)\,dy + \int_{-u}^{b-u} g_\delta^+(y)\phi_{Mb}(u+y)\,dy\\ &\quad + \int_{b-u}^{\infty} g_\delta^+(y)E^{u+y-b}[e^{-\delta(\tau_0\wedge d)}]\phi_{Mb}(b)\,dy, \quad u<0.\end{aligned} \tag{3.25}$$

4. The expected discounted dividends until ruin

In this section, we study the expected discounted dividend payments before ruin $V_{Mb}(u)$. Similar to Section 3, we first consider the case where the initial surplus $0 \le u \le b$. Since it is a Parisian implementation delays in a dividend model with periodic observations, we note that dividend payment is possible only if the process $\{U_b^d(t)\}_{t\ge 0}$ is observed above barrier b before ruin, which leads to

$$\begin{aligned}V_{Mb}(u) &= E^u\left[e^{-\delta\eta_1}\left[U_b^d(\eta_1^-) - b + c_1\int_{\eta_1^-}^{\nu_1^+} e^{-\delta t}\,dt\right] I_{\{\nu_1^+<\tau\}}\right]\\ &\quad + E^u[e^{-\delta\eta_1}I_{\{\nu_1^+<\tau\}}]V_{Mb}(b), \quad 0\le u\le b.\end{aligned} \tag{4.1}$$

Besides, Eq. (4.1) at $u=b$ implies

$$\begin{aligned}V_{Mb}(u) &= E^u\left[e^{-\delta\eta_1}\left[U_b^d(\eta_1^-) - b + c_1\int_{\eta_1^-}^{\nu_1^+} e^{-\delta t}\,dt\right] I_{\{\nu_1^+<\tau\}}\right]\\ &\quad + \frac{E^u[e^{-\delta\eta_1}I_{\{\nu_1^+<\tau\}}]E^b[e^{-\delta\eta_1}[U_b^d(\eta_1^-) - b + c_1\int_{\eta_1^-}^{\nu_1^+} e^{-\delta t}\,dt]I_{\{\nu_1^+<\tau\}}]}{1 - E^b[e^{-\delta\eta_1}I_{\{\nu_1^+<\tau\}}]}.\end{aligned} \tag{4.2}$$

The calculation of Eq. (4.2) is same as the calculation of Laplace transform of the time of ruin $\phi_{Mb}(u)$, and the final result is directly given here. The case where the initial surplus $u<0$ and $u>b$ will be resolved later. Since $E^u[e^{-\delta\eta_1}I_{\{\nu_1^+<\tau\}}]$ is given in Eq. (3.18), we only need to calculate $E^u[e^{-\delta\eta_1}[U_b^d(\eta_1^-) - b + c_1\int_{\eta_1^-}^{\nu_1^+} e^{-\delta t}\,dt]I_{\{\nu_1^+<\tau\}}]$, $0\le u\le b$.

Theorem 3. *When $0 \le u \le b$, we have*

$$\begin{aligned}&E^u\left[e^{-\delta\eta_1}\left[U_b^d(\eta_1^-) - b + c_1\int_{\eta_1^-}^{\nu_1^+} e^{-\delta t}\,dt\right] I_{\{\nu_1^+<\tau\}}\right]\\ &= \int_0^{\infty}\left(h_\delta^+(y\,|\,b-u) - \int_0^{\infty} E[e^{-\delta\tau_b}I_{\{\tau_b<\tau_d\}}; U_S(\tau_b)\in b+dx\,|\,U_S(0)=b-u]h_\delta^+(y\,|\,b+x)\right)\\ &\quad\times\left(E^y\left[e^{-\delta d}\left[U(d) - c_1\int_0^d e^{-\delta t}\,dt\right]\right] - E^y\left[e^{-\delta\tau_0}c_1\int_0^{\tau_0} e^{-\delta t}\,dt I_{\{\tau_0<d\}}\right]\right.\\ &\quad\left.+E^y\left[e^{-\delta d}\left[U(d) - c_1\int_0^d e^{-\delta t}\,dt\right] I_{\{\tau_0\ge d\}}\right]\right)dy.\end{aligned}$$

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Proof. Analogous to Eq. (3.9), the former quantity can be expressed in terms of the latter one via

$$E^u\left[e^{-\delta\eta_1}[U_b^d(\eta_1^-) - b + c_1\int_{\eta_1^-}^{\nu_1^+} e^{-\delta t}\,dt]I_{\{\nu_1^+<\tau\}}\right]$$
$$= \int_0^\infty \left(h_\delta^+(y\,|\,b-u) - \int_0^\infty E[e^{-\delta\tau_b}I_{\{\tau_b<\tau_d\}}; U_S(\tau_b) \in b+dx\,|\,U_S(0)=b-u]h_\delta^+(y\,|\,b+x)\right)$$
$$\times E^{b+y}\left[e^{-\delta\eta_1}[U_b^d(\eta_1^-) - b + c_1\int_{\eta_1^-}^{\nu_1^+} e^{-\delta t}\,dt]\right]dy. \tag{4.3}$$

According to the definitions in Section 3.2, when $U_b^d(0) = U(0) = u > b$, it gives that $\eta_1 = \theta_1 \wedge d$ as the surplus was observed above b at the initial moment. Hence, $E^u[e^{-\delta\eta_1}[U_b^d(\eta_1^-) - b + c_1\int_{\eta_1^-}^{\nu_1^+} e^{-\delta t}\,dt]]$ can be written as

$$E^u\left[e^{-\delta\eta_1}[U_b^d(\eta_1^-) - b + c_1\int_{\eta_1^-}^{\nu_1^+} e^{-\delta t}\,dt]\right]$$
$$= E^{u-b}\left[e^{-\delta(\tau_0\wedge d)}[U(\tau_0\wedge d) - c_1\int_0^{\tau_0\wedge d} e^{-\delta t}\,dt]\right], \quad u > b. \tag{4.4}$$

In order to obtain $V_{Mb}(u)$, we need to compute $E^u[e^{-\delta(\tau_0\wedge d)}[U(\tau_0\wedge d) - c_1\int_0^{\tau_0\wedge d} e^{-\delta t}\,dt]]$, $u > 0$. It is similar to Eq. (3.13), ruin occurs after u/c_2 when $u \geq c_2 d$, which is $\tau_0 \geq d$, and we arrive at

$$E^u\left[e^{-\delta(\tau_0\wedge d)}[U(\tau_0\wedge d) - c_1\int_0^{\tau_0\wedge d} e^{-\delta t}\,dt]\right] = E^u\left[e^{-\delta d}[U(d) - c_1\int_0^d e^{-\delta t}\,dt]\right]$$
$$= e^{-\delta d}\left[u + (\lambda E[X_1] - c)d - \frac{c_1}{\delta}(1 - e^{-\delta d})\right]. \tag{4.5}$$

For $0 < u < c_2 d$, conditioning on τ_0 leads to

$$E^u\left[e^{-\delta(\tau_0\wedge d)}[U(\tau_0\wedge d) - c_1\int_0^{\tau_0\wedge d} e^{-\delta t}\,dt]\right] = E^u\left[e^{-\delta d}[U(d) - c_1\int_0^d e^{-\delta t}\,dt]I_{\{\tau_0\geq d\}}\right]$$
$$- E^u\left[e^{-\delta\tau_0}c_1\int_0^{\tau_0} e^{-\delta t}\,dt I_{\{\tau_0<d\}}\right]. \tag{4.6}$$

For the first term to the right of Eq. (4.6), we have

$$E^u\left[e^{-\delta d}[U(d) - c_1\int_0^d e^{-\delta t}\,dt]I_{\{\tau_0\geq d\}}\right] = E^u\left[e^{-\delta d}[U(d) - c_1\int_0^d e^{-\delta t}\,dt]\right]$$
$$- E^u\left[e^{-\delta d}[U(d) - c_1\int_{d-\tau_0}^d e^{-\delta t}\,dt]I_{\{\tau_0<d\}}\right]$$
$$= e^{-\delta d}\left(u + (\lambda E[X_1] - c)\left\{d - \int_{u/c_2}^d (d-t)f_U(t\,|\,u)\,dt - \left(d - \frac{u}{c_2}\right)e^{-\lambda(\frac{u}{c_2})}\right\}\right.$$
$$\left. -\frac{c_1}{\delta}(1 - e^{-\delta d}) + c_1\int_{d-u/c_2}^d e^{-\delta t}e^{-\lambda\frac{u}{c_2}}\,dt + c_1\int_{u/c_2}^d f_U(t\,|\,u)\int_{d-t}^d e^{-\delta z}\,dz\,dt\right). \tag{4.7}$$

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The other term in Eq. (4.6)

$$E^u\left[e^{-\delta\tau_0}c_1\int_0^{\tau_0}e^{-\delta t}\,dtI_{\{\tau_0<d\}}\right]$$
$$=e^{-\delta\frac{u}{c_2}}\frac{c_1}{\delta}(1-e^{-\delta\frac{u}{c_2}})+\int_{u/c_2}^{d}e^{-\delta t}c_1\int_0^t e^{-\delta z}\,dzf_U(t\,|\,u)\,dt. \tag{4.8}$$

Then

$$E^u\left[e^{-\delta\eta_1}\left[U_b^d(\eta_1^-)-b+c_1\int_{\eta_1^-}^{\nu_1^+}e^{-\delta t}\,dt\right]I_{\{\nu_1^+<\tau\}}\right]$$
$$=e^{-\delta d}(\rho_1^0-\rho_1)e^{\rho_1^0 b}\left(e^{-\rho_1^0 u}+\frac{(\rho_2-\rho_2^0)(\rho_2-\rho_1^0)[(\rho_2^0-\rho_1)e^{\rho_2^0(b-u)}-(\rho_1^0-\rho_1)e^{\rho_1^0(b-u)}]}{\rho_1^0\rho_2(\rho_2^0-\rho_1)(\rho_2-\rho_1^0)e^{\rho_2^0 b}-(\rho_1^0-\rho_1)(\rho_2-\rho_2^0)e^{\rho_1^0 b}}\right)$$
$$\times\left\{\frac{c_1e^{-(\frac{\delta}{c_2}-\rho_1)c_2d}-c_1}{\delta(\frac{\delta}{c_2}-\rho_1)}+\frac{c_1-c_1e^{-(\frac{2\delta}{c_2}-\rho_1)c_2d}}{\delta(\frac{2\delta}{c_2}-\rho_1)}+\frac{e^{\rho_1c_2d}}{\rho_1}e^{-\delta d}\frac{c_1}{\delta}(1-e^{-\delta d})\right.$$
$$-\int_0^{c_2d}e^{\rho_1y}\int_{\frac{y}{c_2}}^{d}e^{-\delta t}c_1\left(\frac{1}{\delta}-\frac{e^{-\delta}}{\delta}\right)f_U(t\,|\,u)\,dt\,dy-\int_0^{c_2d}e^{\rho_1y}\left\{\frac{c_1}{\delta}(1-e^{-\delta d})\right.$$
$$\left.+c_1\int_0^{d-\frac{y}{c_2}}e^{-\delta t}e^{-\lambda\frac{y}{c_2}}\,dt+c_1\int_{\frac{y}{c_2}}^{d}f_U(t\,|\,u)\int_0^{d-t}e^{-\delta z}\,dz\,dt\right\}dy+\frac{1}{\rho_1^2}-\left(\frac{\lambda}{\rho_1}+c_2\right)$$
$$\times\left[-\frac{d}{\rho_1}-d\int_0^{c_2d}e^{\rho_1y}\int_{\frac{y}{c_2}}^{d}f_U(t|y)\,dt\,dy+\int_0^{c_2d}e^{\rho_1y}\int_{\frac{y}{c_2}}^{d}tf_U(t|y)\,dt\,dy-\frac{1}{\lambda-c_2\rho_1}\right.$$
$$\left.\left.\times\left(c_2d-\frac{1}{\frac{\lambda}{c_2}-\rho_1}\left(1-e^{(\rho_1-\frac{\lambda}{c_2})c_2d}\right)\right)\right]\right\},\quad 0\le u\le b. \tag{4.9}$$

Using Eq. (3.19), the double integral above is evaluated as

$$\int_0^{c_2d}e^{\rho_1y}\int_{\frac{y}{c_2}}^{d}te^{-\delta t}f_U(t|y)\,dt\,dy=\sum_{k=1}^{\infty}\frac{(-\lambda\rho_1)^k}{(k-1)!c_2^{k+1}}\sum_{i=0}^{k}\frac{(k+i-1)!}{i!(k-i)!}\frac{1}{(\frac{\lambda+\delta}{c_2}-\rho_1)^{k+i}}$$
$$\times\left\{\frac{(k+1-i)!}{(\frac{\lambda+\delta}{c_2}-\rho_1)^{k-i+2}}\left(1-\sum_{j=0}^{k+1-i}\frac{[(\frac{\lambda+\delta}{c_2}-\rho_1)c_2d]^je^{-(\frac{\lambda+\delta}{c_2}-\rho_1)c_2d}}{j!}\right)-\sum_{j=0}^{k+i-1}\left(\frac{\lambda+\delta}{c_2}-\rho_1\right)^j\right.$$
$$\left.\times e^{-(\frac{\lambda+\delta}{c_2}-\rho_1)c_2d}\frac{(c_2d)^{j+k-i+2}(k-i)!}{(j+k-i+2)!}\right\}, \tag{4.10}$$

and the second double integral in Equation (4.9) can be directly set as $\delta=0$ in Eq. (3.19), the third double integral in Eq. (4.9) can be directly set as $\delta=0$ in Eq. (4.10). With Eqs. (3.18) and (4.9) derived, $V_{Mb}(u), 0\le u\le b$ are determined by Eq. (4.2). □

The following two Remarks respectively give the results satisfied by $V_{Ub}(u)$ and $V_{Lb}(u)$.

Remark 4.1. When $u>b$, at the time of the first observation, there are two scenarios for surplus $V_{Ub}(u)$

1. When the first observation is made, surplus $U_b^d(t)$ is already ruin ($U_b^d(\nu_1)<0$),

$$\chi_{U_1}^*(u)=\int_u^{\infty}g_\delta^-(y)\,dy. \tag{4.11}$$

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2. When the first observation is occurred, the surplus $U_b^d(t)$ is above 0 ($U_b^d(\nu_1) > 0$),

$$\chi_{U_2}^*(u) = \int_{u-b}^{u} g_\delta^-(y) V_{Mb}(u-y)\, dy + \int_0^{u-b} g_\delta^-(y) \left\{ E^{u-y-b}\left[e^{-\delta(\tau_0\wedge d)}\left[U(\tau_0 \wedge d) - c_1 \int_0^{\tau_0\wedge d} e^{-\delta t}\, dt \right]\right] + E^{u-y-b}[e^{-\delta(\tau_0\wedge d)}] V_{Mb}(b) \right\} dy + \int_0^{\infty} g_\delta^+(y) \left\{ E^{u+y-b}\left[e^{-\delta(\tau_0\wedge d)}\left[U(\tau_0 \wedge d) - c_1 \int_0^{\tau_0\wedge d} e^{-\delta t}\, dt \right]\right] + E^{u+y-b}[e^{-\delta(\tau_0\wedge d)}] V_{Mb}(b) \right\} dy. \tag{4.12}$$

So $V_{Ub}(u)$ can be expressed as

$$\begin{aligned} V_{Ub}(u) &= \chi_{U_1}^*(u) \times 0 + \chi_{U_2}^*(u) \\ &= \int_0^{u-b} g_\delta^-(y) \left\{ E^{u-y-b}\left[e^{-\delta(\tau_0\wedge d)}\left[U(\tau_0 \wedge d) - c_1 \int_0^{\tau_0\wedge d} e^{-\delta t}\, dt \right]\right] + E^{u-y-b}[e^{-\delta(\tau_0\wedge d)}] V_{Mb}(b) \right\} dy + \int_0^{\infty} g_\delta^+(y) \left\{ E^{u+y-b}\left[e^{-\delta(\tau_0\wedge d)}\left[U(\tau_0 \wedge d) \right.\right.\right. \\ &\quad \left.\left.\left. - c_1 \int_0^{\tau_0\wedge d} e^{-\delta t}\, dt \right]\right] + E^{u+y-b}[e^{-\delta(\tau_0\wedge d)}] V_{Mb}(b) \right\} dy \\ &\quad + \int_{u-b}^{u} g_\delta^-(y) V_{Mb}(u-y)\, dy, \quad u > b. \end{aligned} \tag{4.13}$$

Remark 4.2. When $u < 0$, at the time of the first observation, there are two scenarios for surplus $V_{Lb}(u)$.

1. When the first observation is made, surplus $U_b^d(t)$ is already ruin ($U_b^d(\nu_1) < 0$),

$$\chi_{L_1}^*(u) = \int_0^{\infty} g_\delta^-(y)\, dy + \int_0^{-u} g_\delta^+(y)\, dy. \tag{4.14}$$

2. When the first observation is occurred, the surplus $U_b^d(t)$ is above 0 ($U_b^d(\nu_1) > 0$),

$$\chi_{L_2}^*(u) = \int_{-u}^{b-u} g_\delta^+(y) V_{Mb}(u+y)\, dy + \int_{b-u}^{\infty} g_\delta^+(y) \left\{ E^{u+y-b}\left[e^{-\delta(\tau_0\wedge d)}\left[U(\tau_0 \wedge d) - c_1 \int_0^{\tau_0\wedge d} e^{-\delta t}\, dt \right]\right] + E^{u+y-b}[e^{-\delta(\tau_0\wedge d)}] V_{Mb}(b) \right\} dy. \tag{4.15}$$

So $V_{Lb}(u)$ can be expressed as

$$\begin{aligned} V_{Lb}(u) &= \chi_{L_1}^*(u) \times 0 + \chi_{L_2}^*(u) \\ &= \int_{b-u}^{\infty} g_\delta^+(y) \{ E^{u+y-b}\left[e^{-\delta(\tau_0\wedge d)}\left[U(\tau_0 \wedge d) - c_1 \int_0^{\tau_0\wedge d} e^{-\delta t}\, dt \right]\right] \\ &\quad + E^{u+y-b}[e^{-\delta(\tau_0\wedge d)}] V_{Mb}(b) \Big\} dy + \int_{-u}^{b-u} g_\delta^+(y) V_{Mb}(u+y)\, dy, \quad u < 0. \end{aligned} \tag{4.16}$$

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Remark 4.3. The key computational procedure for $\phi_b(u)$ and $V_b(u)$ is summarized in below:

1. Since the initial surplus $U_b^d(0) = u$ can be any value, $\phi_b(u)$ can be expressed as $\phi_{Lb}(u)$ for $u < 0$, $\phi_{Mb}(u)$ for $0 \le u \le b$ and $\phi_{Ub}(u)$ for $b < u$. In the same way, $V_b(u)$ can be expressed as $V_{Lb}(u)$ for $u < 0$, $V_{Mb}(u)$ for $0 \le u \le b$ and $V_{Ub}(u)$ for $b < u$.
2. Through simplification of Eqs. (3.2) and (4.2), it is found that $\phi_{Mb}(u)$ and $V_{Mb}(u)$ is only related to $E^u[e^{-\delta\tau} I_{\{\tau<\nu_1^+\}}]$, $E^u[e^{-\delta\eta_1} I_{\{\nu_1^+<\tau\}}]$ and $E^u[e^{-\delta\eta_1}[U_b^d(\eta_1^-) - b + c_1 \int_{\eta_1^-}^{\nu_1^+} e^{-\delta t}\, dt] I_{\{\nu_1^+<\tau\}}]$ for $0 \le u \le b$ as well as $E^u[e^{-\delta\eta_1}]$ for $u > b$.
3. Use Eq. (3.3) to get $E^u[e^{-\delta\tau} I_{\{\tau<\nu_1^+\}}]$ for $0 \le u \le b$.
4. Determine $E^u[e^{-\delta\eta_1} I_{\{\nu_1^+<\tau\}}]$ for $0 \le u \le b$ by Eq. (3.18), where the first double integrals therein is evaluated by Eq. (3.19) (with the second double integral in Eq. (3.18) can be directly set as $\delta = 0$ in Eq. (3.19)).
5. Determine $E^u[e^{-\delta\eta_1}]$ for $u > b$ by Eq. (3.11), it can be broken down into Eqs. (3.13) and (3.14), where the two integrals therein are evaluated by Eq. (3.16) (the remaining integral $\int_{u/c_2}^{d} f_U(\cdot)\, dt$ in Eq. (3.14) is simply a special case of the above expression with $\delta = 0$).
6. Get $E^u[e^{-\delta\eta_1}[U_b^d(\eta_1^-) - b + c_1 \int_{\eta_1^-}^{\nu_1^+} e^{-\delta t}\, dt] I_{\{\nu_1^+<\tau\}}]$ for $0 \le u \le b$ by Eq. (4.9), where the first double integral is Eq. (4.10), and the second double integral in Eq. (4.9) can be directly set as $\delta = 0$ in Eq. (3.19), the third double integral in Eq. (4.9) can be directly set as $\delta = 0$ in Eq. (4.10).

Remark 4.4. When the observation interval T is $Erlang(n)$ distributed and the single claim (gain) amount X has a rational Laplace transform, we can also obtain the results of $\phi_b(u)$ and $V_b(u)$. As can be seen from step 2 of Remark 4.3, to calculate $\phi_b(u)$ and $V_b(u)$, we must know the values of $h_\delta^+(y\,|\,u)$, $h_\delta^-(y\,|\,u)$, $g_\delta^+(y)$, $g_\delta^-(y)$ and $\chi(u)$. When the observation interval T is $Erlang(n)$ distributed and the single claim (gain) amount X has a rational Laplace transform, we can refer to Section 4 of Albrecher *et al.* [1], Sections 3.2 and 4 of Albrecher *et al.* [2] and Remark 3 of Cheung and Wong [8] for the results of these formulas. After the solutions of the above five expressions are obtained, we proceed with the steps of the Remark 4.3 and the results of $\phi_b(u)$ and $V_b(u)$ after the relaxed Assumption 1 and Assumption 2 are obtained. For detailed calculations, interested readers can refer to references Albrecher *et al.* [1], Albrecher *et al.* [2] and Cheung and Wong [8].

Remark 4.5. Note that with $\beta \to \infty$, $c_1 = 0$, the model in this paper becomes the dual risk model with Parisian implementation delays in dividend payments (the surplus process is observed continuously). Similarly, $h_\delta^+(y\,|\,u)$ also degenerates into the expected discounted deficit at ruin for the classical compound Poisson risk model with continuous observation. Both the Laplace transform of the time of ruin and the Expected discounted dividends until ruin can degenerate to the result of Cheung and Wong [8].

5. Numerical illustrations

This section aims at providing some numerical examples to study the effect of Parisian implementation delays under a mixed dividend strategy on the Gerber-Shiu expected discounted penalty function and the expected discounted dividend payments before ruin. The optimal dividend barrier that maximizes $V_b(u)$ with respect to b will also be discussed. In all examples, it is assumed that the constant expense rate is $c = 0.5$, the fixed continuous dividend interest rate is $c_1 = 0.25$, the periodic observation with $\beta = 1$ and the Laplace transform argument is $\delta = 0.1$. All numbers are generated using the software package Mathematica (Specific reference Cheung and Wong [8]).

The functions $\phi_b(u)$ and $V_b(u)$ are computed for the initial surplus levels $u = 2, 6, 8, 10$ and fixed barrier level $b = 10$, and that's the case of $0 < u \le b$. When the initial surplus u, the amount of income is subject to the exponential parameter a_1 and the deterministic delays $d = 10$ are also fixed, the higher the revenue intensity parameter λ of Poisson arrival rate, the higher the total $V_b(u)$ of the expected

Table 1. $c = 0.5,\ c_1 = 0.25,\ \beta = 1,\ a_1 = 1,\ d = 2,\ b = 10,\ \delta = 0.1.$

	$\phi_{10}(u)$				$V_{10}(u)$			
λ	$u = 2$	$u = 6$	$u = 8$	$u = 10$	$u = 2$	$u = 6$	$u = 8$	$u = 10$
1	0.043389	0.000198	0.000014	0.000002	1.51882	3.51092	4.72591	6.35823
0.9	0.064976	0.000060	0.000062	0.000011	1.29783	1.58336	2.23002	3.13839
0.8	0.095988	0.001754	0.000232	0.000024	0.22002	0.72001	1.07673	1.60664
0.7	0.139022	0.021035	0.000878	0.000121	0.12266	0.49653	0.80351	1.29533

Table 2. $c = 0.5,\ c_1 = 0.25,\ \beta = 1,\ \lambda = 1,\ d = 2,\ b = 10,\ \delta = 0.1.$

	$\phi_{10}(u)$				$V_{10}(u)$			
a_1	$u = 2$	$u = 6$	$u = 8$	$u = 10$	$u = 2$	$u = 6$	$u = 8$	$u = 10$
1	0.043389	0.000198	0.000014	0.000002	1.51882	3.51092	4.72591	6.35823
1.2	0.061386	0.000507	0.000048	0.000008	0.72883	2.04065	3.04647	4.54500
1.4	0.084452	0.001199	0.000140	0.000012	0.32592	1.15171	1.95184	3.30516
1.6	0.112468	0.015661	0.000389	0.000041	0.05704	0.26698	0.52773	1.04242

Table 3. $c = 0.5,\ c_1 = 0.25,\ \beta = 1,\ a_1 = 1,\ \lambda = 1,\ b = 10,\ \delta = 0.1.$

	$\phi_{10}(u)$				$V_{10}(u)$			
d	$u = 2$	$u = 6$	$u = 8$	$u = 10$	$u = 2$	$u = 6$	$u = 8$	$u = 10$
1	0.043391	0.000198	0.000014	0.000002	1.43257	3.31155	4.45754	5.99716
2	0.043389	0.000198	0.000014	0.000002	1.51882	3.51092	4.72591	6.35823
3	0.043389	0.000198	0.000014	0.000002	1.51989	3.51339	4.72923	6.36269
4	0.043389	0.000197	0.000013	0.000001	1.53849	3.55638	4.78710	6.44055

Table 4. $c = 0.5,\ c_1 = 0.25,\ \beta = 1,\ a_1 = 1,\ d = 2,\ b = 10,\ \delta = 0.1.$

	$\phi_{10}(u)$				$\phi_{10}(u)$			
λ	$u = 10$	$u = 11$	$u = 15$	$u = 20$	$u = -1$	$u = -3$	$u = -5$	$u = -10$
1	0.0000019	0.0000015	0.0000014	0.0000014	0.76063	0.86271	0.89460	0.90830
0.9	0.0000109	0.0000089	0.0000081	0.0000081	0.78218	0.87158	0.89801	0.90857
0.8	0.0000236	0.0000201	0.0000175	0.0000175	0.80367	0.87976	0.90093	0.90876
0.7	0.0001211	0.0001035	0.0000901	0.0000656	0.81026	0.88159	0.90144	0.90878

discounted dividend is. Because more positive earnings a company has, the more dividends it pays. Similarly, the expected discounted penalty function $\phi_b(u)$ will decrease as λ increases.

The findings on Table 2 is same as that in Table 1. The income intensity parameter λ in Table 1 is replaced by the exponential parameter a_1 of the single income amount, and then the relationship among a_1 and the expected discounted penalty function $\phi_b(u)$ and the expected discounted dividend function $V_b(u)$ is discussed. As it can be seen from the results in Tables 1 and 2, both λ and a_1 influence the total amount of earnings $S(t)$ at time t. Therefore, the impact of the change in parameter a_1 on $\phi_b(u)$ and $V_b(u)$ of the single return is the same as that in Table 1.

Table 5. $c = 0.5,\ c_1 = 0.25,\ \beta = 1,\ \lambda = 1,\ d = 2,\ b = 10,\ \delta = 0.1.$

	$\phi_{10}(u)$				$\phi_{10}(u)$			
a_1	$u = 10$	$u = 11$	$u = 15$	$u = 20$	$u = -1$	$u = -3$	$u = -5$	$u = -10$
1	0.0000019	0.0000015	0.0000014	0.0000014	0.76063	0.86271	0.89460	0.90830
1.2	0.0000080	0.0000065	0.0000060	0.0000060	0.79497	0.88158	0.90246	0.90890
1.4	0.0000119	0.0000099	0.0000088	0.0000088	0.82283	0.89317	0.90615	0.90905
1.6	0.0000405	0.0000337	0.0000302	0.0000301	0.84487	0.90009	0.90783	0.90908

Table 6. $c = 0.5,\ c_1 = 0.25,\ \beta = 1,\ a_1 = 1,\ \lambda = 1,\ b = 10,\ \delta = 0.1.$

	$\phi_{10}(u)$				$\phi_{10}(u)$			
d	$u = 10$	$u = 11$	$u = 15$	$u = 20$	$u = -1$	$u = -3$	$u = -5$	$u = -10$
1	0.0000021	0.0000018	0.0000018	0.0000018	0.76063	0.86271	0.89460	0.90830
2	0.0000019	0.0000015	0.0000014	0.0000014	0.76063	0.86271	0.89460	0.90830
3	0.0000016	0.0000012	0.0000012	0.0000012	0.76063	0.86271	0.89460	0.90830
4	0.0000007	0.0000011	0.0000010	0.0000010	0.76063	0.86271	0.89460	0.90830

With the extension of the certainty time d of the delay, the Parisian implementation delays dividend becomes more stringent, which leads to a later ruin time and hence a smaller value of $\phi_b(u)$. As the delay gets longer, there are three effects on the expected discounted dividend function $V_b(u)$:

1. A longer delay time makes the duration of threshold dividend longer. However, when d reaches a certain length, the surplus process has passed through the barrier b and dropped below b, which will no longer affect the total threshold dividend.
2. The longer the delay, the lower the probability of ruin. The longer the company survives, the more potential dividend opportunities will be generated by the company's long surplus growth.
3. The longer delay means that early Parisian implementation delays dividend are more unlikely.

The numerical results in Table 3 suggest that the above first two effects dominates.

The left side of Tables 4–6 considers the case where the initial surplus are greater than the dividend barrier $b = 10$. It can be seen from the table that similar findings of Tables 1–3 are still retained. In each of Tables, for fixed λ, a_1 and d, it is observed that $\phi_b(u)$ decreases and converges as u increases. Intuitively, when u is considerably larger than the dividend barrier $b = 10$, although the threshold dividend is being paid, the initial surplus is so large that it is likely the surplus process will remain above the dividend barrier b during the entire delay period. In such a case, the surplus will drop to dividend barrier b upon payment of a dividend at the end of the delay period, and the time remaining until ruin is just the time of ruin with initial surplus level b. Therefore, any further increase in u virtually does not affect the ruin time but merely increases the amount of the first dividend.

Similarly, the right side of Tables 4– 6 considers the case where the initial surplus are less than the dividend barrier b, which is also shown in Tables 1–3. By comparing Tables 1 and 4, it can be seen that the smaller the initial surplus u is, the smaller the impact of revenue intensity parameter λ on the expected penalty function $\phi_b(u)$ is. Besides, from Tables 2 and 5, it can be seen that the smaller the initial earnings u is, the smaller the impact of exponential parameter a_1 of the single income amount on the expected penalty function $\phi_b(u)$ is. This is because the smaller the initial surplus is, the more likely ruin will occur and it will converges to a constant of 1.

In addition to the effects of various parameters on the expected discounted penalty function $\phi_b(u)$ and the expected discounted dividend function $V_b(u)$, we are also interested in the optimal barrier

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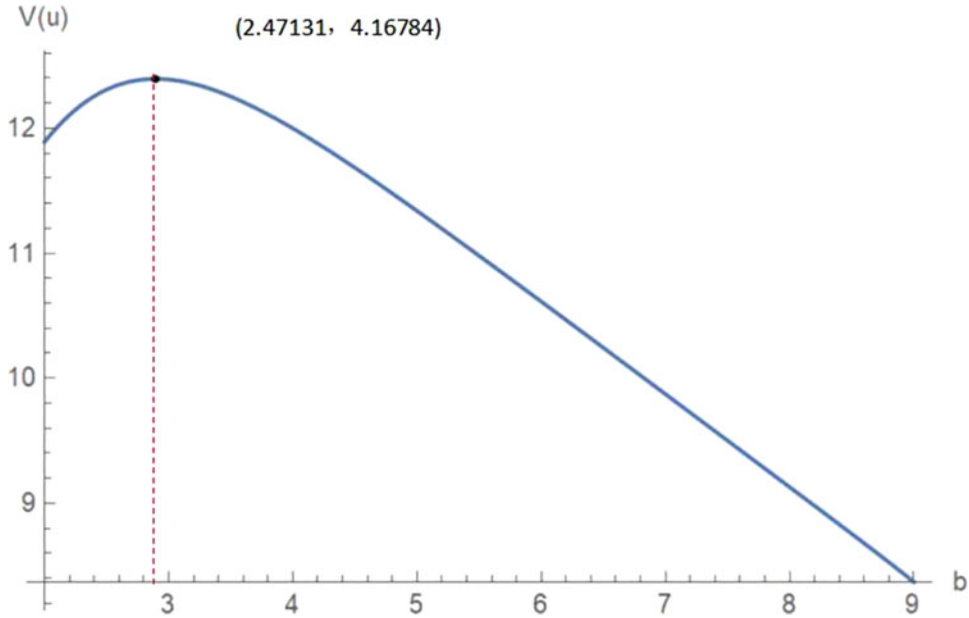

Figure 4. *$V(x)$ for $u = 2$ with parameters $c = 0.5$, $c_1 = 0.25$, $\beta = 1$, $a_1 = 1$, $\lambda = 1$, $d = 2$, $\delta = 0.1$.*

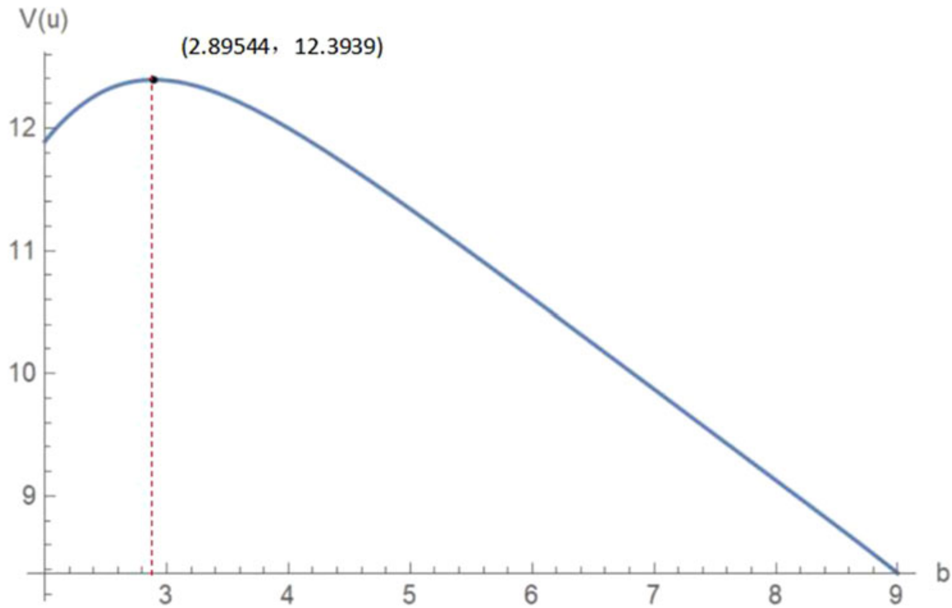

Figure 5. *$V(u)$ for $u = 12$ with parameters $c = 0.5$, $c_1 = 0.25$, $\beta = 1$, $a_1 = 1$, $\lambda = 1$, $d = 2$, $\delta = 0.1$.*

maximizing the expected discounted dividend function $V_b(u)$. The results are given in Figures 4–6 and are explained as follows. The three cases of initial earnings $u = 2$,$u = 12$ and $u = -2$ are considered here. The value range of dividend barrier b is 2 to 9, which is corresponding to three cases of initial surplus between 0 and b, greater than b and less than 0, respectively. It can be seen from Figures 4–6 that when the initial earnings u and the delay time d are determined, it might not be the case that the smaller the dividend barrier b is, the better. A smaller dividend barrier will make it easier to distribute the dividend, but it will also be more likely to cause a ruin of a company. Similarly, it is not the case that the bigger

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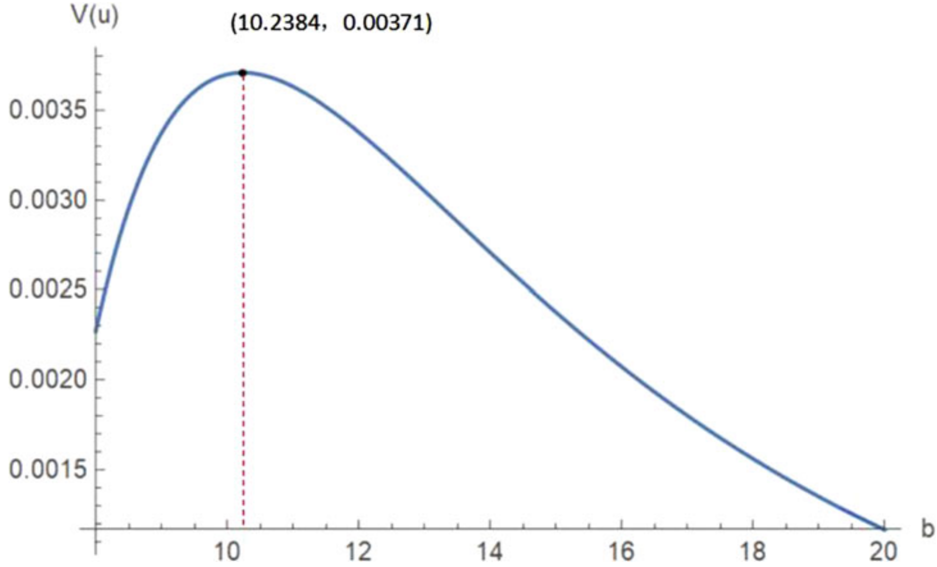

Figure 6. *$V(u)$ for $u = -2$ with parameters $c = 0.5$, $c_1 = 0.25$, $\beta = 1$, $a_1 = 1$, $\lambda = 1$, $d = 2$, $\delta = 0.1$.*

the dividend barrier b is, the better. A large dividend barrier will reduce the probability of ruin, but the dividend will occur later, and the total dividend amount will be reduced.The relationship between all dividend barriers b and the expected discounted total dividend $V_b(u)$ is shown in Figures 4–6.

Acknowledgments. The authors are very grateful to reviewers for their very thorough reading of the paper and valuable suggestions. This work is supported by the National Key R&D Program of China (No. 2020YFB2103503), the Natural Science Foundation of Hunan Province, China (No. 2021JJ30436), the Scientific Research Fund of Hunan Provincial Education Department, China (Nos.19B343, 20B381, 20K084), the Science and Technology Planning Project of Shenzhen Municipality (20220810155530001) and the Changsha Municipal Natural Science Foundation (No. kq2014072). The authors wish to express their gratitude for these financial supports.

Competing interests. The authors declare no conflict of interest.

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Cite this article: Hu K, Li J and Zhou J (2023). On the dual risk model with Parisian implementation delays under a mixed dividend strategy. *Probability in the Engineering and Informational Sciences* **37**, 442–461. https://doi.org/10.1017/S0269964822000481