THE NUMERICAL SOLUTION OF INTEGRAL EQUATIONS USING CHEBYSHEV POLYNOMIALS

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1. Introduction

An investigation has been made into the numerical solution of non-singular linear integral equations by the direct expansion of the unknown function $f(x)$ into a series of Chebyshev polynomials of the first kind. The use of polynomial expansions is not new, and was first described by Crout [1]. He writes $f(x)$ as a Lagrangian-type polynomial over the range in x, and determines the unknown coefficients in this expansion by evaluating the functions and integral arising in the equation at chosen points x_i. A similar method (known as collocation) is used here for cases where the kernel is not separable. From the properties of expansion of functions in Chebyshev series (see, for example, [2]), one expects greater accuracy in this case when compared with other polynomial expansions of the same order. This is well borne out in comparison with one of Crout's examples.

The most common method of solution of integral equations is by the use of finite differences. Fox and Goodwin [3] have made a thorough investigation of these methods, using the Gregory quadrature formula for the evaluation of the integral. Other methods for the algebraization of the integral equation using Gaussian quadrature have been described by Kopal [4].

The methods of this paper are not as versatile as the finite-difference techniques, since they depend to a much greater extent on the form of the given functions eg. kernel, arising in the equation. However, in cases where the method can be used without a prohibitive amount of labour, we obtain the value of the function throughout the range of x, instead of at a discrete number of points. Also, with the Chebyshev expansion of the function known, some estimate can generally be made, a posteriori, to its accuracy.

The crux of the problem is to find easily the Chebyshev expansion of the given functions in the equation. To find these, we confine ourselves to functions which can be represented as the solution of some linear differential equation with associated boundary conditions. The solution of the differential equation can then be found by a direct expansion of the function in Chebyshev polynomials. This method has been described by Clenshaw [5], and frequent use of it will be made throughout this paper. It is assumed that the reader is familiar with the methods and notation of [5]. For functions whose Chebyshev expansions cannot readily be found in this way, or which are given numerically, some curve fitting technique can be used [2]. It is felt that in such cases, the labour might better be spent using a finite-difference technique.

2. Method of Solution

Linear integral equations can be divided into two types depending upon the limits of the integral. An equation of the form

$$f(x) = F(x) + \lambda \int_a^b K(x, y) f(y)\, dy,$$

where F, K are given functions; λ, a, b are finite constants and $f(x)$ is the unknown function is known as a "Fredholm equation". When the upper limit of the integral is not a constant, but is the variable x, the equation takes the form

$$f(x) = F(x) + \lambda \int_a^x K(x, y) f(y)\, dy,$$

and is known as a "Volterra equation".

We shall be concerned with equations of the Fredholm type, and in order to use the Chebyshev polynomials we must change the range of the variable x from (a, b) to either $(-1, 1)$ or $(0, 1)$. In the former case we use the polynomials $T_n(x)$ where

$$T_n(x) = \cos n\theta, \qquad x = \cos\theta; \qquad -1 \leqq x \leqq 1.$$

When the range of x is $(0, 1)$, we use the $T_n^*(x)$ polynomials where

$$T_n^*(x) = \cos n\theta, \qquad 2x - 1 = \cos\theta; \qquad 0 \leqq x \leqq 1.$$

For tables and properties of these polynomials, see [2].

Before proceeding with the discussion of methods of solution, we shall need results for

(i) the product of two Chebyshev expansions

and (ii) the integral of a function whose Chebyshev expansion is given.

2.1 *Product of two Chebyshev expansions*

$$\text{(1)} \qquad \begin{cases} \text{Suppose } f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n T_n(x) \\ \text{and } g(x) = \tfrac{1}{2}b_0 + \sum_{n=1}^{\infty} b_n T_n(x), \end{cases}$$

and we want to find the Chebyshev expansion of the product of $f(x)$ and

$g(x)$. From the relation,

$$2T_m(x)T_n(x) = T_{m+n}(x) + T_{|m-n|}(x)$$

we find that,

$$(2)\qquad \begin{cases} f(x)\,g(x) = \tfrac{1}{2}d_0 + \sum_{n=1}^{\infty} d_n T_n(x) \\ \text{where } d_n = \tfrac{1}{2}[a_0 b_n + \sum_{m=1}^{\infty} a_m(b_{|m-n|} + b_{m+n})], \quad n \geqq 0. \end{cases}$$

An exactly similar result holds for expansions in terms of the $T_n^*(x)$ polynomials. If,

$$(3)\qquad \begin{cases} f(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{\infty} A_n T_n^*(x) \\ \text{and } g(x) = \tfrac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n T_n^*(x) \end{cases}$$

$$\text{then } f(x)\,g(x) = \tfrac{1}{2}D_0 + \sum_{n=1}^{\infty} D_n T_n^*(x)$$

where,

$$(4)\qquad D_n = \tfrac{1}{2}[A_0 B_n + \sum_{m=1}^{\infty} A_m(B_{|m-n|} + B_{m+n})], \quad n \geqq 0.$$

2.2 *The Integral of $f(x)$*

We suppose that $f(x)$ is given in terms of its Chebyshev expansion in $T_n(x)$, and we want the expansion of $I(x)$, where

$$I(x) = \int_{-1}^{x} f(x)dx.$$

Following the methods of [5], we have that $I(x)$ is the solution of

$$\frac{dI}{dx} = f(x) \quad \text{with } I(-1) = 0.$$

Then, if

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{\infty} a_n T_n(x)$$

and

$$I(x) = \tfrac{1}{2}b_0 + \sum_{n=1}^{\infty} b_n T_n(x),$$

we find

$$(5)\qquad b_0 = a_0 - \tfrac{1}{2}a_1 - 2\sum_{n=2}^{\infty} \frac{(-1)^n}{n^2-1} a_n$$

and

$$b_n = \frac{a_{n-1} - a_{n+1}}{2n} \quad \text{for } n \geqq 1. \tag{6}$$

In many problems we want $I(1)$; this is given by

$$I(1) = a_0 - 2\sum_{n=1}^{\infty} \frac{a_{2n}}{4n^2 - 1}. \tag{7}$$

In solving Fredholm equations, we require the integral of the product of two functions between the limits -1 and 1. Defining $f(x)$ and $g(x)$ as in equation (1), and using equations (2) and (7), we find

$$\begin{aligned} \int_{-1}^{1} f(x)g(x)dx &= a_0\left(\tfrac{1}{2}b_0 - \sum_{r=1}^{\infty} \frac{b_{2r}}{4r^2 - 1}\right) \\ &\quad + \sum_{n=1}^{\infty} a_n\left[b_n - \sum_{r=1}^{\infty} \frac{b_{|n-2r|} + b_{n+2r}}{4r^2 - 1}\right] \end{aligned} \tag{8}$$

Similar results can be found for expansions in terms of the $T_n^*(x)$ polynomials. Defining $f(x)$ as in equation (3), then if

$$\begin{aligned} I(x) &= \int_0^x f(x)dx \\ &= \tfrac{1}{2}B_0 + \sum_{n=1}^{\infty} B_n T_n^*(x), \end{aligned}$$

we find

$$\begin{aligned} B_0 &= \tfrac{1}{2}A_0 - \tfrac{1}{4}A_1 - \sum_{n=2}^{\infty} \frac{(-1)^n}{n^2 - 1} A_{2n} \\ B_n &= \frac{A_{n-1} - A_{n+1}}{4n}, \quad \text{for } n \geqq 1. \end{aligned} \tag{9}$$

For $\int_0^1 f(x)dx$, we have,

$$I(1) = \tfrac{1}{2}A_0 - \sum_{n=1}^{\infty} \frac{1}{4n^2 - 1} A_{2n}. \tag{10}$$

Finally for the integral of the product of two functions, if $f(x)$ and $g(x)$ are defined as in equation (3), then

$$\begin{aligned} \int_0^1 f(x)g(x)dx &= \tfrac{1}{2}A_0\left[\tfrac{1}{2}B_0 - \sum_{r=1}^{\infty} \frac{1}{4r^2 - 1} B_{2r}\right] \\ &\quad + \tfrac{1}{2}\sum_{n=1}^{\infty} A_n\left[B_n - \sum_{r=1}^{\infty} \frac{B_{|n-2r|} + B_{n+2r}}{4r^2 - 1}\right] \end{aligned} \tag{11}$$

We will now examine in detail the numerical solution of Fredholm-type integral equations. The method depends entirely upon whether the kernel $K(x, y)$ is separable or not. In Section 3 we will discuss the case of a separable kernel; in Section 4 we will compare the method with one of Crout's examples, and in Sections 5 and 6 we will investigate the case of non-separable kernels.

3. Separable kernel

In general, when the kernel is separable we will have

$$K(x, y) = \sum_{m=1}^{M} g_m(x)h_m(y).$$

The Fredholm integral equation can then be written

$$f(x) = F(x) + \lambda \sum_{m=1}^{M} g_m(x) \int_{-1}^{1} h_m(y)f(y)dy \tag{12}$$

where the range in x has been normalized to $-1 \leqq x \leqq 1$. $F(x)$, $g_m(x)$, $h_m(y)$ are given functions and we assume that their expansions in $T_n(x)$ can be found by, for example, the method of [5] or some curve fitting technique. We assume that $f(x)$ is to be approximated by a polynomial of degree N,

$$f(x) = \tfrac{1}{2}a_0 + \sum_{n=1}^{N} a_n T_n(x)$$

If $F(x) \not\equiv 0$, we choose N to be the degree to which $F(x)$ is given to the required accuracy. If $F(x) \equiv 0$, then N can only be estimated a priori from, perhaps, some physical criterion. If N is originally chosen too small, this will be apparent from the series expansion for $f(x)$. The calculation will then have to be repeated with larger N. If N is chosen too large initially, then unnecessary extra work will have been done. Many integral equations, however, arise from physical problems where something is known of the form of $f(x)$ which will enable us to make a reasonable guess for N. Now

$$I_m = \int_{-1}^{1} h_m(y)f(y)dy$$

is a constant depending upon $a_0, a_1, \cdots, a_N$ and can be evaluated using equation (8). If $C_n(G)$ denotes the coefficient of $T_n(x)$ in the Chebyshev expansion of a function $G(x)$, then on equating coefficients of $T_n(x)$ on each side of equation (12) we find,

$$a_n = C_n(F) + \lambda \sum_{m=1}^{M} C_n(g_m) I_m(a_0, a_1, \cdots, a_N) \quad \text{for } n = 0, 1, \cdots, N \tag{13}$$

Equation (13) gives a system of $(N+1)$ linear equations for the $(N+1)$ unknowns $a_0, a_1, \cdots, a_N$. These equations can be solved numerically by standard methods to give the Chebyshev expansion of $f(x)$. From this series the value of the function can be found for any x in the range $-1 \leqq x \leqq 1$.

An exactly similar analysis holds for the range $0 \leqq x \leqq 1$, when the $T_n^*(x)$ polynomials are used.

EXAMPLE 1. Let us consider the integral equation

$$f(x) = -\frac{2}{\pi}\cos\left(\tfrac{1}{2}\pi x\right) + 2\int_0^1 \cos\tfrac{1}{2}\pi(x-y)f(y)dy,$$

whose solution is given by $f(x) = \sin\left(\frac{1}{2}\pi x\right)$. The kernel is separable with $M = 2$, where $I_1 = \int_0^1 f(y)\cos\left(\frac{1}{2}\pi y\right)dy$, and $I_2 = \int_0^1 f(y)\sin\left(\frac{1}{2}\pi y\right)dy$, say. Using the method of [5], we find that

$$\begin{bmatrix}\sin\\ \cos\end{bmatrix}\left(\tfrac{1}{2}\pi x\right)\tfrac{1}{2} = +\,0.602\,194 \pm 0.513\,625 T_1^*(x) - 0.103\,546\,T_2^*(x) \mp 0.013\,732\,T_3^*(x) + 0.001\,359\,T_4^*(x) \pm 0.000\,107\,T_5^*(x) - 0.000\,007\,T_6^*(x).$$

In this example, we see that to $6D$ we can represent the expansion of $F(x)$ by a polynomial of degree 6. Consequently we take $N = 6$, and assume that

$$f(x) = \tfrac{1}{2}A_0 + \sum_{n=1}^{6} A_n T_n^*(x)$$

Using equation (11), we find

$$\begin{bmatrix}I_1\\ I_2\end{bmatrix} = +\,0.318\,309\,A_0 \mp 0.173\,950\,A_1 - 0.249\,298\,A_2 \pm 0.109\,413\,A_3 - 0.020\,740\,A_4 \pm 0.022\,008\,A_5 - 0.013\,169\,A_6.$$

With these values we find on solving equation (13) the following Chebyshev expansion for $f(x)$

$$f(x) = 0.60220 + 0.51362\,T_1^*(x) - 0.10355\,T_2^*(x) - 0.01373\,T_3^*(x) + 0.00136\,T_4^*(x) + 0.00011\,T_5^*(x) - 0.00001\,T_6^*(x).$$

This expansion can be compared with that for $\sin\frac{1}{2}\pi x$ from which we see that there is an error of approximately 1×10^{-5}. Although starting with the expansion of all the given functions to $6D$, some accuracy has been lost in the sixth decimal place due to rounding errors.

With this Chebyshev expansion for $f(x)$ we might conclude from the rate of convergence of the last three coefficients, that the truncation error will be less than 1×10^{-5}. With a round-off error in each term less than $\frac{1}{2} \times 10^{-5}$, we might conclude just from the series expansion that its error is less than 4×10^{-5}. Consequently we can assume that the expansion will give values of $f(x)$ correct to $4D$ for all values of x in $0 \leqq x \leqq 1$. This we know to be correct from the analytic solution.

Finally, we note that whenever the kernel is separable, the integral equation is satisfied for all values of x when determining the relations between the coefficients A_n.

4. Comparison with Crout's method

We shall now compare by means of an example, the Chebyshev series expansion with the method of Crout. In this problem, the kernel is again separable, although it has a discontinuity in the first derivative.

EXAMPLE 2.

$$f(x) = \lambda \int_0^L K(x, y) f(y) dy$$

$$\text{where } K(x,y) = \begin{cases} \dfrac{x(L-y)}{EIL} & \text{for } y \geqq x \\ \dfrac{y(L-x)}{EIL} & \text{for } y \leqq x \end{cases}$$

This integral equation arises in the problem of the buckling of a beam of length L. It is an eigen-value problem in which we want to find those values of λ for which a non-trivial solution exists. In particular we wish to find the first mode of buckling where the mid-point of the beam is an anti-node. The analytic solution for this mode of buckling is

$$f(x) = \sin \frac{\pi x}{L} \quad \text{with } \lambda = \pi^2 \frac{EI}{L^2}.$$

$$\text{Defining } \zeta = \frac{x}{L}, \quad \eta = \frac{y}{L}, \quad \mu = \frac{\lambda L^2}{EI}$$

and writing,

$$f(L\zeta) \equiv u(\zeta) \quad \text{and } f(L\eta) \equiv u(\eta),$$

the equation can be written as,

$$u(\zeta) = \mu \left\{ (1-\zeta) \int_0^\zeta \eta u(\eta) d\eta + \zeta \int_\zeta^1 (1-\eta) u(\eta) d\eta \right\}.$$

Again the kernel is separable although each integral contains the variable as a limit. Write

$$u(\zeta) = \tfrac{1}{2} A_0 + \sum_{n=1}^{\infty} A_n T_n^*(\zeta),$$

$$I(\zeta) = \int_0^\zeta \eta u(\eta) d\eta, \quad \text{and } J(\zeta) = \int_\zeta^1 (1-\eta) u(\eta) d\eta.$$

The function $I(\zeta)$ satisfies the equation

$$\frac{dI}{d\zeta} = \zeta u(\zeta) \quad \text{with } I(0) = 0.$$

Applying the methods of [5], if $I(\zeta) = \frac{1}{2}\alpha_0 + \sum_{n=1}^{\infty} \alpha_n T_n^*(\zeta)$

we find at once that

$$\alpha_0 = 2 \sum_{n=1}^{\infty} (-1)^{n+1} \alpha_n \text{ where } \alpha_n = \frac{1}{16n} (A_{|n-2|} + 2A_{n-1} - 2A_{n+1} - A_{n+2}), \quad n \geqq 1.$$

A similar result can be found for the coefficients in the Chebyshev expansion for $J(\zeta)$. Returning to the integral equation, if $C_n^*(G)$ denotes the coefficient of $T_n^*(\zeta)$ in the Chebyshev expansion of $G(\zeta)$ then,

$$C_n^*(u) = \mu C_n^*[I - \zeta(I - J)] \quad \text{for all } n.$$

On simplifying this expression we find the following 3 term recurrence relation for A_n, valid for all $n \geqq 2$,

$$(14) \qquad (n+1)A_{n-2} + [16n(n^2-1)\varepsilon - 2n]A_n + (n-1)A_{n+2} = 0,$$

where $\varepsilon = 1/\mu$. Corresponding to $n = 0$, and using the values for α_0 and β_0 we find,

$$(15) \qquad (96\varepsilon - 6)A_0 + 7A_2 - 36 \sum_{n=2}^{\infty} \frac{1}{(n^2-1)(4n^2-1)} A_{2n} = 0.$$

A corresponding equation can be found for $n = 1$, but since we are interested only in the first mode of buckling which gives a solution symmetrical about $\zeta = \frac{1}{2}$, we have,

$$A_{2n+1} = 0, \qquad n \geqq 0.$$

Rewriting equation (14) with $2n$ in place of n we have,

$$(16) \quad (2n+1)A_{2n-2} + [32n(4n^2-1)\varepsilon - 4n]A_{2n} + (2n-1)A_{2n+2} = 0, \text{ for } n \geqq 1.$$

Equations (15) and (16) completely define the problem for symmetrical solutions.

Following Crout, we assume that $u(\zeta)$ can be approximated by a polynomial of degree four, so that

$$u(\zeta) = \tfrac{1}{2}A_0 + A_2 T_2^*(\zeta) + A_4 T_4^*(\zeta).$$

The three equations for A_0, A_2, A_4 obtained from equations (15) and (16) can be written in the matrix form

$$\mathbf{MA} = \varepsilon \mathbf{A}$$

where $\mathbf{A}$ is the column vector $\{A_0, A_2, A_4\}$ and $\mathbf{M}$ is the matrix

$$\begin{pmatrix} +\frac{1}{16} & -\frac{7}{96} & +\frac{1}{120} \\ -\frac{1}{32} & +\frac{1}{24} & -\frac{1}{96} \\ 0 & -\frac{1}{192} & +\frac{1}{120} \end{pmatrix}$$

The largest eigenvalue of this matrix corresponds to $\mu = 9.86958$ so that,

$$\lambda = 9.86958 \frac{EI}{L^2}$$

Crout finds $\lambda = 9.87605\ EI/L^2$ which must be compared with the analytic solution of $\lambda = 9.86960\ EI/L^2$, to $5D$.

Using the Chebyshev expansion to the same order as Crout's Lagrangian-type expansion we have found a much better approximation to the eigen-value. The errors are of magnitude 2×10^{-5} and 645×10^{-5} respectively. Such an accuracy in this case seems slightly fortuitous since on repeating the calculation with a sixth order polynomial, the eigen-value is $\lambda = 9.86966\ EI/L^2$, an error of 6×10^{-5} which is slightly larger than for the 4th order case.

For the eigenfunction $f(x)$, if we normalise the solution so that $f(L/2) = 1$, we find

$$f(x) = 0.47230 - 0.49971\, T_2^*\left(\frac{x}{L}\right) + 0.02799\, T_4^*\left(\frac{x}{L}\right)$$

The comparison with Crout's solution, and the analytic solution is shown in Table 1.

TABLE 1

	Exact	Crout		Chebyshev expansions			
	$\lambda = 9.86960\ EI/L^2$	$\lambda = 9.87605 EI/L^2$		$\lambda = 9.86958 EI/L^2$		$\lambda = 9.86966 EI/L^2$	
x/L	$\sin \pi x/L$	4th degree	$\lvert\text{error}\rvert \times 10^5$	4th degree	$\lvert\text{error}\rvert \times 10^5$	6th degree	$\lvert\text{error}\rvert \times 10^5$
0.0, 1.0	0	0	0	0.00058	58	−0.00004	4
0.1, 0.9	0.30902	0.30716	186	0.30878	24	+0.30906	4
0.2, 0.8	0.58779	0.58716	63	0.58862	83	0.58785	6
0.3, 0.7	0.80902	0.80918	16	0.81000	98	0.80907	5
0.4, 0.6	0.95106	0.95119	13	0.95142	36	0.95107	1
0.5, 0.5	1.00000	1.00000	0	1.00000	0	+1.00000	0
		$10^{10}\ \Sigma(\text{error})^2 =$	38990	$10^{10}\Sigma(\text{error})^2 =$	21729		

For the given tabular points, the maximum error in the Chebyshev expansion (98×10^{-5}) is less than in Crout's case (186×10^{-5}). Also the sum of the squares of the errors at these points is less for the Chebyshev expansion.

Taking a sixth degree expansion for $f(x)$ we find,

$$f(x) = 0.47202 - 0.49943\, T_2^*\left(\frac{x}{L}\right) + 0.02795\, T_4^*\left(\frac{x}{L}\right) - 0.00060\, T_6^*\left(\frac{x}{L}\right).$$

The maximum error at the given points has now been reduced to 6×10^{-5}, a considerable improvement in accuracy obtained with little extra computation.

5. Non-separable kernel

In most problems where a numerical approach is required the kernel will not be separable. There are two possible methods of approach. We can try to approximate to the kernel by a function which is separable, and then use the method of Section 3. Alternatively, we can consider the equation as it stands and proceed by a method of collocation.

Suppose that the range of the independent variable x has been normalised to $-1 \leqq x \leqq 1$ and we have the following Fredholm equation,

$$f(x) = F(x) + \lambda \int_{-1}^{1} K(x, y) f(y) dy, \tag{17}$$

where λ, $F(x)$, $K(x, y)$ are given and we have to find $f(x)$. As before, write

$$f(x) = \tfrac{1}{2} a_0 + \sum_{n=1}^{N} a_n T_n(x)$$

where N in general is not known a priori but might be estimated from perhaps, some physical grounds. In order to determine the $(N+1)$ constants $a_0, a_1, \cdots, a_N$ we write down the integral equation at each of $(N+1)$ points x_i, say, where $i = 1, 2, \cdots, N+1$. Equation (17) is then replaced by the $(N+1)$ equations

$$f(x_i) = F(x_i) + \lambda \int_{-1}^{1} K(x_i, y) f(y) dy; \quad i = 1, 2, \cdots, N+1 \tag{18}$$

For each value of x_i, we now compute the Chebyshev expansion for $K(x_i, y)$ either from a differential equation or by some curve fitting process. Using equation (8), we obtain the value of

$$I(x_i, 1) = \int_{-1}^{1} K(x_i, y) f(y) dy$$

in terms of the coefficients $a_0, a_1, \cdots, a_N$. The quantity $F(x_i)$ is known immediately and using tables of Chebyshev Polynomials [2] we can write down $f(x_i)$ in terms of $a_0, a_1, \cdots, a_N$ for each value of x_i. Equation (18) becomes

$$f(x_i) = F(x_i) + \lambda I(x_i, 1) \quad \text{for } i = 1, 2, \cdots, N+1 \tag{19}$$

which is a system of $(N+1)$ linear equations for the $(N+1)$ unknown coefficients. These can be solved by standard methods.

We shall illustrate the method by means of an example taken from [3].

EXAMPLE 3.

$$f(x) \pm \frac{1}{\pi} \int_{-1}^{1} \frac{1}{[1 + (x-y)^2]} f(y) dy = 1$$

Let us consider first the equation with positive sign. We approximate to the function $f(x)$ by means of a polynomial of degree 6. Since $f(x)$ is an even function of x, we write

$$f(x) = \tfrac{1}{2} a_0 + a_2 T_2(x) + a_4 T_4(x) + a_6 T_6(x),$$

and only consider positive values of x_i which have been chosen as,

$$x_i = 0,\ 0.5,\ 0.8,\ 1.0.$$

The kernel $K(x_i, y)$ can be considered as satisfying the differential equation of zero order with polynomial coefficients, given by

$$(1 + x_i^2)K(x_i, y) - 2x_i yK(x_i, y) + y^2K(x_i, y) = 1. \tag{20}$$

If we write

$$K(x_i, y) = \tfrac{1}{2}b_0(x_i) + \sum_{n=1}^{\infty} b_n(x_i)T_n(y),$$

then substitution into equation (20) and using the formulae for $C_n(yK(x_i, y))$ and $C_n(y^2K(x_i, y))$ gives immediately the recurrence relation between the b_n for each value of x_i. The coefficients in the expansion of $K(x_i, y)$ for $x_i = 0$, 0.5, 0.8, 1.0 are given in Table 2.

TABLE 2

n	$b_n(0)$	$b_n(0.5)$	$b_n(0.8)$	$b_n(1.0)$
0	+1.414 214	+1.361 549	+1.252 701	+1.137 729
1	0	+0.31920	+0.42286	+0.43457
2	−0.24264	−0.12703	−0.00841	+0.04965
3	0	−0.08453	−0.06081	−0.03079
4	+0.04163	−0.00300	−0.02218	−0.01912
5	0	+0.01245	−0.00023	−0.00449
6	−0.00714	+0.00385	+0.00293	+0.00037
7	0	−0.00091	+0.00116	+0.00070
8	+0.00123	−0.00085	+0.00004	+0.00025
9	0	−0.00009	−0.00014	+0.00003
10	−0.00021	+0.00011	−0.00006	−0.00002
11	0	+0.00004	0	−0.00001
12	+0.00004	−0.00001	0	0
13	0	−0.00001	0	0
14	−0.00001	0	0	0

$$K(x_i, y) = \tfrac{1}{2}b_0(x_i) + \sum_{n=1}^{\infty} b_n(x_i)T_n(y)$$

With these coefficients known for $K(x_i, y)$, the evaluation of $I(x_i, 1)$ for each value of x_i can now be made by means of equation (8), to give

$$\begin{aligned}
I(0, 1) &= 0.78540a_0 - 0.71238a_2 + 0.03686a_4 - 0.04217a_6\\
I(0.5, 1) &= 0.72322a_0 - 0.57161a_2 - 0.04902a_4 - 0.02328a_6\\
I(0.8, 1) &= 0.63055a_0 - 0.41763a_2 - 0.10331a_4 - 0.02458a_6\\
I(1, 1) &= 0.55358a_0 - 0.32602a_2 - 0.11278a_4 - 0.02975a_6
\end{aligned}$$

Substituting these values into equation (19) gives the following system of equations,

$$\begin{aligned}
0.75000a_0 - 1.22676a_2 + 1.01173a_4 - 1.01342a_6 &= 1\\
0.73021a_0 - 0.68195a_2 - 0.51560a_4 + 0.99259a_6 &= 1\\
0.70071a_0 + 0.14706a_2 - 0.87608a_4 - 1.00494a_6 &= 1\\
0.67621a_0 + 0.89622a_2 + 0.96410a_4 + 0.99053a_6 &= 1
\end{aligned}$$

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the solution of which gives,

$$f(x) = 0.70758 + 0.04937\,T_2(x) - 0.00102\,T_4(x) - 0.00022\,T_6(x).$$

The comparison of this solution with that obtained by Fox and Goodwin is given in Table 3.

TABLE 3

	$f(x) + \frac{1}{\pi}\int_{-1}^{1} \frac{1}{[1+(x-y)^2]} f(y)dy = 1$			$f(x) - \frac{1}{\pi}\int_{-1}^{1} \frac{1}{[1+(x-y)^2]} f(y)dy = 1$			
x	Fox and Goodwin to $4D$	Chebyshev (6th degree)	Legendre (4th degree)	Fox and Goodwin to $4D$	Chebyshev (6th degree)	Legendre (4th degree)	x
0	0.6574	0.65741	0.65745	1.9191	1.91903	1.91925	0
± 0.25	0.6638	0.66385	0.66397	1.8997	1.89958	1.89966	± 0.25
± 0.5	0.6832	0.68318	0.68323	1.8424	1.84240	1.84261	± 0.5
± 0.75	0.7149	0.71482	0.71432	1.7520	1.75208	1.75318	± 0.75
± 1.0	0.7557	0.75571	0.75576	1.6397	1.63971	1.63987	± 1.0

Taking the integral equation with negative sign and proceeding as before, we find

$$f(x) = 1.77447 - 0.14003T_2(x) + 0.00490T_4(x) + 0.00037T_6(x).$$

The comparison of this solution with Fox and Goodwin's is also given in Table 3. Fox and Goodwin have presented their results only to $4D$ with an estimated maximum error of 1×10^{-4} due to round-off, and we see that the results found here agree exactly to within the prescribed error.

Of the computational labour in this solution of the problem, most was spent in the determination of the Chebyshev expansions of $K(x_i, y)$. With these expansions found, comparatively little labour was necessary for the evaluation of $I(x_i, 1)$ and the solution of the equation for the coefficients a_n. Had we found it necessary to use a higher degree polynomial for $f(x)$, all previous results for $K(x_i, y)$ and $I(x_i, 1)$ can be used again. When the degree of the polynomial approximation to $f(x)$ is not known a priori, we can start with a low N and increase the degree until the necessary accuracy in the solution is reached.

6. Use of Legendre Polynomials

In the above example, since the limits of integration are from -1 to $+1$, this suggests expanding all functions in terms of the Legrendre polynomials $P_n(x)$. The evaluation of $I(x_i, 1)$ is then almost trivial due to the orthogonality property of the Legendre polynomials, in the range $-1 \leqq x \leqq 1$. For suppose

$$f(x) = \sum_{n=0}^{N} a_n P_n(x),$$

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and for a given x_i we find that

$$K(x_i, y) = \sum_{n=0}^{M} b_n(x_i) P_n(y)$$

where $M \neq N$ in general, take $M > N$. Then since

$$\int_{-1}^{1} P_n(x) P_m(x) dx = \frac{2}{2n+1} \delta_{m,n}$$

we have that

$$I(x_i, 1) = \int_{-1}^{1} K(x_i, y) f(y) dy = \sum_{n=0}^{N} \frac{2a_n b_n(x_i)}{2n+1}. \tag{23}$$

This equation is considerably simpler than equation (8) for Chebyshev polynomials. The problem is now one of finding the expansion of $K(x_i, y)$ in terms of Legendre polynomials. This can be done in a similar way to the Chebyshev expansion from the direct solution of differential equations in Legendre polynomials. This method has been described by the author, [6]. However, we shall find in general that the recurrence relation between the coefficients b_n are more complicated for Legendre polynomials than for Chebyshev polynomials. The computing time saved in using equation (23) instead of equation (8) will generally be more than off-set in the computation of the expansions $K(x_i, y)$.

The integral equation of Example 3 has been solved by writing $f(x)$ as the fourth degree polynomial,

$$f(x) = a_0 P_0(x) + a_2 P_2(x) + a_4 P_4(x).$$

To determine the three unknown coefficients a_0, a_2, a_4 we have used collocation at the points $x_i = 0, 0.5, 1$. The following solutions were found

$$+\,ve \text{ sign; } f(x) = 0.69107 + 0.06615 P_2(x) - 0.00146 P_4(x)$$
$$-\,ve \text{ sign; } f(x) = 1.82129 - 0.18971 P_2(x) + 0.00829 P_4(x)$$

These results are also tabulated in Table 3, and agree excellently to 3D with the previous results.

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