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ON THE INVERSION OF RIGHT INVARIANT ELEMENTS

BY

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In this note we show that every (not necessarily commutative) integral domain R has a quotient ring which, although need not be a field, has the property that all of its right invariant elements are units. As an application this shows that every PRI (principal right ideal) domain can be embedded in a simple PRI domain which is, in general, not a field.

A nonzero element a in R is said to be *right invariant* if aR is a twosided ideal of R, i.e., if $Ra \subseteq aR$. Let I be the set of all right invariant elements of R and let $B=B(R)$ be the set of all factors of elements of I. We note that every factor b of a right invariant element a is actually a left factor; for if $a=xby$ then $a=byx'$ where x' is chosen to satisfy $xa=ax'$. Thus we may write

$$B = \{b \in R \mid aR \subseteq bR \text{ for some } a \in I\}.$$

The right quotient ring $K=RB^{-1}=\{rb^{-1} \mid r \in R, b \in B\}$ may be formed provided that B is a *right Ore system* in R, i.e., a submonoid of the monoid R^* of nonzero elements of R satisfying

$$bR \cap rB \neq \emptyset$$

for each $b \in B$, $r \in R$. We assume that all right Ore systems are *saturated.* For B, this means that

$$b_1 b_2 \in B \text{ iff } b_1 \text{ and } b_2 \in B.$$

Saturation insures that if $U(K)$ is the group of units of K then $U(K) \cap R=S$. The proof that B is a right Ore system in R is left as an exercise (cf. [2, p. 218]). Although we do not know that the right invariant elements of K are units in K, the center of K is a field (also left as an exercise).

In order to find a quotient ring of R all of whose right invariant elements are units we use a simple iteration based on a procedure which is borrowed from [1]. Let $S_1=U(R)$ and let $K_1=R$. Let $\alpha>1$ be an ordinal and assume that for each $\beta<\alpha$, S_β has been defined and is a right Ore system in R with $K_\beta=RS_\beta^{-1}$. We define S_α by

$$S_\alpha = \begin{cases} \bigcup_{\beta<\alpha} S_\beta \text{ if } \alpha \text{ is a limit ordinal} \\ B(K_{\alpha-1}) \cap R \text{ if } \alpha \text{ is not a limit ordinal.} \end{cases}$$

One may easily check that S_α is a right Ore system in R, and we put $K_\alpha=RS_\alpha^{-1}$ so that the induction is valid. Thus we obtain an increasing sequence of right Ore

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systems in R. Clearly we must have $S_\alpha = S_{\alpha+1}$ for some α, and we let γ be the least such ordinal. Then K_γ can have no nonunit right invariant element. We have established the following.

THEOREM 1. *Each integral domain can be embedded in a quotient ring whose right invariant elements are all units.*

As a corollary we find at if R is a PRI domain then it has a right quotient ring K_γ which has no nonunit right invariant elements; since K_γ is also a PRI domain [1] this means that K_γ is simple More specifically we can show that the sequence constructed above ends at $\gamma=2$. In other words, if $K=RB^{-1}$ then all of the right invariant elements in K are units in K. To prove this let $k=rb^{-1}$ be right invariant in K. Then r is also right invariant in K. Since $rK \cap R$ is a right ideal of R we may put $rK \cap R=aR$. Then $a=rd$ where d is a unit in K (since $aK=rK$). Thus a is also right invariant in K. From

$$Ra \subseteq Ka \cap R \subseteq aK \cap R = aR$$

we find that a is right invariant in R and hence a unit in K. Consequently $k=ad^{-1}b^{-1}$ is also a unit in K. We summarize in the following.

THEOREM 2. *Let R be a PRI domain and let B be the set of all right invariant elements of R together with all of their factors. Then the quotient ring $K=RB^{-1}$ is a simple PRI domain.*

We remark that for the case of a PRI domain R the set $B(R)$ is precisely the set of all elements $b \in R$ for which bR is a bounded right ideal as defined in [4, p. 38] (cf. also [2, p. 227]).

It is well known that in a PRI domain R, R^* is a right Ore system and so R has a quotient field. Thus the main interest in Theorem 2 is in the construction of simple PRI domains which are not fields (cf. also [3]). In order to achieve this we need only take R to be a PRI domain in which $B(R) \neq R^*$. For such an example see [5, p. 211].

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