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Mathematical Universalism

Pierre Cartier

Introduction

We begin with a typical quotation from the introduction to Bourbaki's *Treatise*:

> Since the Greeks mathematics has meant demonstration; some people even doubt whether, outside mathematics, demonstrations exist in the precise, rigorous sense that the word received from the Greeks and which we intend to give it here. We can say that meaning has not changed, since what was a demonstration for Euclid is still one for us; and at times when the idea threatened to disappear and hence mathematics was endangered, it was to the Greeks that people went to look for models. But in the last century important conquests have been added to that legacy.
>
> Indeed analysis of the mechanism of demonstrations in selected mathematical texts has allowed us to identify its structure from the twofold viewpoint of language and syntax. And so we come to the conclusion that a sufficiently explicit mathematical text could be expressed in a conventional language that consisted of only a small number of invariable 'words' arranged according to a syntax that would comprise a small number of inviolable rules. Such a text is known as *formalized*. Description of a chess game using the conventional notation, or a log table, is a formalized text; the formulae of ordinary algebraic calculus would be too – if the rules governing use of brackets were fully codified and there was strict conformity to them, whereas in fact some of those rules can only be learnt by use and usage allows certain derogations to be practised.

But demonstration is not everything in mathematics. For sure mathematics is part of mankind's universal heritage. Any civilization worthy of the name makes use of exchanges and counting, measurements and estimates. Mastery of numbers, counting and operations is part of the essential basic baggage, and the progress achieved can be measured by examining the improvements in methods of representation and calculation with numbers. The idea of number itself has been considerably extended over the centuries. The 'natural' whole integers 1, 2, 3, . . . have been successively joined by zero, fractions, negative numbers, then 'real' numbers, which include

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rational numbers (or fractions) and irrational ones. From the 16th to the 19th century 'complex' (or 'imaginary') numbers have gradually been recognized, first pragmatically then on a rigorous basis with Euler, Gauss and Cauchy. This evolution of the notion of number is not over, as we shall see later.

At the same time the needs of architecture or shipbuilding, for instance, as well as artists' 'formal' research,[1] led to cataloguing of forms and study of their measurements and transformations. Terms such as square, circle, sphere, cube, cylinder, straight line, angle, perpendicular designate basic notions in geometry but are also an integral part of ordinary language in the same way as the names of numbers. A civilization's memory is expressed in the conservation and description of many forms.

In a third area the need for location, anticipation and prediction were at the origin of other developments. The need to situate something in history or simply regulate economic and social life demanded that calendars be continually improved; the growing interconnection of the world forces us to live on two time levels: a local clock and a 'universal time' which we use as a reference. At any moment there are people who are at another point in the day with whom we nonetheless communicate instantaneously.

Since the start of travel and trade geography has continued to live off borrowing from geometry and astronomy. Since the rhythms of the heavens were the easiest to observe, calendars benefited from astronomy's advances. Still today it is space exploration – the modern form of astronomy – that has allowed us to create huge systems of immediate location (GPS, GALILEO) and communication. Navigation – by sea, land and air – has benefited from these advances but has also had a strong influence on them. Travelling and doing business forces us to evaluate risks and foresee their consequences in order to correct them. Management of large economic, social or financial entities requires forecasting tools. The basic ideas of statistics became indispensable in the late 17th century; the concepts of average, standard deviation (or margin of error), percentage, degree of confidence were part of the general baggage. Probability theory, a flourishing science nowadays, is merely the endpoint of that trend.

Though all the above-mentioned concepts correspond to chapters in orthodox mathematics (algebra, geometry, mechanics, probability), the important consideration of *scale*, which means we can differentiate a 100,000-euro project from a 10,000,000-euro one, does not belong to an accepted division of mathematics itself. Instead we hand it over to the know-how of the practitioner (financier, engineer), but this situation has no more than a temporary sociological value, and we shall have to invent a mathematical theory of small and large.

What has gone before is a description of mathematical knowledge (and know-how) as it belongs to everyone. It may be that the creative juices, which flow so freely in our era, will dry up, as has happened several times throughout history. All that accumulated knowledge will remain available provided it is passed on through teaching, which could even be quite routine. In music not every country or century has produced Mozart, Bach or Beethoven, but provided there are a few music or singing teachers, and some enthusiasm for it, people will still be humming, dancing in rhythm, and listening to ballads and psalms. The disappearance of music or mathematics would mean civilization falling asleep, or coming to an end, and history does not allow us to be so optimistic that we can dismiss such an eventuality.

* * *

If, as I maintain, mathematics is an integral part of civilization, it must combine its particular characteristics with the general trends of the period, the *Zeitgeist*. The 19th century was marked by the legacy of the Enlightenment from the previous century, the *Aufklärung*, von Humboldt's effort to reshape the universities, Napoleon's reconstruction of France and Europe. Mathematics did not escape this movement.

Around 1800 it inherited a mass of results and concepts, some of which dated back to antiquity while others had been added in the early 18th century (Newton, Leibniz and their disciples), but it all rested on *uncertain foundations*. *Complex numbers* had been freely used since 1700 but with a bad conscience ('imaginary' numbers); they gave marvellous results in the hands of Euler (born in 1707), for example the 'magic' formula:

$$\cos\theta + i\sin\theta = e^{i\theta}$$

which contains the whole of trigonometry and opens the way to hyperbolic trigonometry, and from which stems the 'mystic' formula:

$$e^{i\pi} + 1 = 0$$

The *fundamental theorem of algebra*, which says that any algebraic equation has a solution, still awaited a convincing demonstration, which was eventually supplied by Gauss around 1800. The justification of complex numbers by their geometric representation, also due to Gauss, opened the way to Cauchy's marvellous work: line integrals, residues. This also allowed the introduction into geometry of 'imaginary' elements (points, straight lines, circles); as the geometer Darboux is supposed to have said in an apposite phrase, 'in geometry the shortest route between real elements often goes via a detour into the imaginary'. The most pragmatic technical mind enjoyed the benefit provided to the theory of electricity by complex factors, and quantum physics stated fundamental laws with another magic formula (Dirac, Feynman):

$$e^{iS/\hbar}$$

where $i = \sqrt{-1}$ is the imaginary unit.

Geometry itself was in difficulties. By trying to demonstrate Euclid's axiom of parallels or to square the circle, and failing every time, people started to think there were insurmountable obstacles. Could it be that there were geometric truths that were non-demonstrable? Could the presumed accord between logic and the physical reality of our familiar space be an illusion? Lobachevsky's and Gauss's discovery of a logically coherent geometry that nevertheless contradicted the evidence of parallels forced a rethink about the legitimacy of mathematical facts. Were they a 'natural' truth or conventions (as H. Poincaré maintained), or were the aim and use of mathematics to obtain models of reality, which were therefore falsifiable in Popper's sense?

We also notice that many of geometry's current concepts, like the orientation of a straight line or a plane, remained implicit in Euclid's axiomatics. It was physics that was to get those ideas accepted:

- Ampère's and Faraday's rule for currents and electrical forces;
- circular polarization of light;
- 'left' and 'right' crystals.

When critical analysis developed, people realized that it is not self-evident that a straight line and a circle bisect each other because the drawing seems to show it. Neither was it well understood what was the tangent to a curve, the area of a surface . . .

The situation was worse still in differential and integral calculus, despite Euler's and Lagrange's efforts at presentation, despite d'Alembert's 'encyclopedic' crusade to clarify the notion of derivative. And what are those infinitely small and large entities which the most rigorous folk do not completely rule out despite the logical absurdities surrounding their 'definition':

> Any quantity smaller than any assignable quantity is infinitely small

(therefore smaller than itself?). And what do those ratios $\frac{0}{0}$ different from one another mean?

The fact is that the nature of the continuum is unknown. What is quantity? The theory of Eudoxus and Archimedes, reproduced in Book V of Euclid, is rigorous but very ponderous and itself rests on a number of assumptions. If we want to prove basic theorems such as the intermediate value theorem[2] or the existence of maxima, we have to stop relying on geometric intuition and replace the vague idea of quantity with that of real (or decimal) number.

I will let Henri Poincaré (*The Value of Science*, 1905) describe the solution provided by German mathematicians such as Dedekind or Weierstrass:

> Let us look at what happened, for instance to the idea of continuous function. At first it was just a perceptible image, for example a continuous line drawn in chalk on a blackboard. Then it was gradually purified, soon it was used to construct a complicated system of inequalities that reproduced, so to speak, all the lines of the initial image; when that construction was finished that gross representation was removed, so to speak, rejected, which had momentarily supported it and was now useless; there remained only the construction itself, irreproachable for the logician. And yet if the initial image had totally disappeared from our memory, how would we guess what whim caused all those inequalities to be erected thus one upon another?

I cannot recommend too warmly that chapter of Poincaré's on intuition and logic in mathematics.

Salvation came from an *arithmetization of the continuum*. We owe this to the obsessive meticulousness of someone like Frege for example, but also to a whole tradition of formal algebra in Britain stretching from Cayley, Boole and L. Carroll to Heaviside and even Dirac. There was a near-absolute confidence in the virtues of an operator calculus. The expression 'symbolic calculus' is Heaviside's.

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In the late 19th century, as Poincaré notes, the ultimate goal of rigour in mathematics seemed to have been reached. A staged construction took us from whole numbers, governed by the principle of recurrence (in the form of Peano's axioms), to real and complex numbers; notions of continuity, derivability and integral were finally clarified in exchange for sacrificing the infinitely small, and the theory of the functions of real or complex variables and differential and integral equations now rested on solid foundations. Thanks to the prodigious discovery of Cartesian coordinates geometry was aligned with algebra and analysis. What seemed to contemporaries to be the crowning moment of this *logicist* trend was the publication of Hilbert's *Foundations of Geometry*. In it Euclid's axiomatics was laid out unimpeded and the tour de force was achieved of a rigorous exposé of geometry without coordinates. In a way it was an outdated project but it was to be the model for the renovated axiomatic method in the following century.

* * *

Was science completed by around 1900? What happened in what has been called the 'short century' from 1914 to 1989? According to contemporary reports the start of the war in 1914 was perceived as the end of a European equilibrium reached at the Congress of Vienna in 1815. It was the beginning of a long period of disturbances, wars, tensions that would not come to an end until the collapse of the Soviet empire. In the meantime the Turkish, Austro-Hungarian and Russian empires have disappeared, and the French and British colonial empires have given way to shadowy commonwealths. It was also the century of revolutionary ideologies based on hope for a new world, and the century of hard questions about many certainties in psychology, moral sciences, economics, history and politics, not to mention space and time or the reality of atoms. It was also the century of great technological conquests: travelling all over the world in a few hours and communicating with anywhere on earth in a few seconds. People walked on the moon and will soon do the same on Mars.

In science quantum physics, arising from an *ad hoc* adjustment by Planck of experimental results concerning an apparently secondary phenomenon, introduced a new paradigm that we have still not mastered. In mathematics it was a century of great victories, paradigm shifts, crazy ideological ambitions, creation of convenient and efficient tools. As will be shown later, the universalism of mathematics emerged greatly strengthened.

In our description of 20th century mathematics we shall leave to one side purely internal progress such as the 23 problems mentioned by Hilbert in his Paris lecture in 1900. Instead we shall try to describe the external impact, the influence on society.

I mentioned earlier the new demand in the 19th century for rigour and logical precision. That demand is appositely described in the first lines of Bourbaki's *Treatise* quoted at the beginning of this article. But in the constellation of the Mathematical Sciences, as they were called then, there was still much that was far from formal logic: mechanics, probability, but also the most advanced parts of differential geometry, algebraic geometry and Lie's Group Theory. And we must add the *Analysis Situs* invented by Poincaré, which came to be called 'topology'.

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The first conquest was in *Functional Analysis*. Calculus of variations had been developing since the mid 18th century. Its spirit was to look for a configuration, optimized according to a certain criterion, in a mass of more or less arbitrary functions or figures. Direct methods were developed around 1900 under the name of *Dirichlet's principle*. Hilbert invented the space that bears his name, consisting of sequences of complex numbers $(z_1, z_2, z_3 \ldots)$ such that the series $\Sigma \mid z_n \mid^2$ converges; he applied it to solving integral equations. The contemporary invention of Lebesgue's integral, then towards 1920 the work of the Polish school around Banach familiarized mathematicians with this new kind of reasoning where *sets of functions* were being dealt with, just as in geometry figures that were *sets of points* were being handled. The last version, after a first attempt by Sobolev, was to be Laurent Schwartz's invention, around 1945, of distributions that finally justified Heaviside's and Dirac's symbolic methods. A whole school was created around Emile Borel, followed in Poland by Steinhaus and in the USA by Norbert Wiener, who developed the theory of *stochastic processes*. People became familiar with the idea of reasoning probabilistically about very irregular curves, which made it possible to model stock fluctuations or turbulence in fluid mechanics.

What was coming into being was a *new paradigm* of set theory. In order to study the convergence of Fourier series Cantor was the first to consider exceptional sets, increasingly complex sets of real numbers, which he undertook to classify. As has been said earlier, people dared to consider sets of very irregular points and functions for the requirements of Lebesgue-style integration theory, reincarnated in the Wiener-Levy theory of Brownian motion. With Banach people learnt to handle spaces of infinite dimensions, and the *General Topology* was one of the tools that transposed geometric intuition into the new world. In algebra Dedekind taught us to interpret Kummer's ideal numbers as sets of numbers, and algebraic geometry was re-founded by Krull and his successors by considering polynomial ideals.

And so we came, at first fragmentarily, with the algebraists' ideals, rings, and fields and with Hausdorff's topological spaces, to consider that all mathematical objects are to be interpreted as sets: sets of points, but also sets of straight lines or circles (which are themselves sets of points). In topology we became familiar with topological structures, or filters, which are sets of sets; we went up the scale with sets of sets of sets. Hilbert, who wanted to subject all mathematical sciences – even physics – to the axiomatic method, is credited with aspiring to base everything on sets.

- *Hilbert's prophet would be Bourbaki.* The group of mathematicians conceived the grandiose project of re-founding the whole of mathematics coherently on the idea of sets – and the structures they carry. That slogan about structures was an extraordinary motor to create the encyclopedic series, 40 volumes of which appeared between 1940 and 1980. Since then the great athlete has run out of steam: a small new 200-page volume came out in 1998, and the complete recasting of a volume published in 1958 is due in a few years' time. It is easy to mock the weaknesses of the enterprise:
- *The published encyclopedia covers only part of current mathematical knowledge*; in particular the subjects the founders of Bourbaki cherished, such as differential

geometry or arithmetic, have never come to fruition. Probability theory, partial differential equations or distributions are not dealt with.

- *The style is dogmatic and rigid*, with no concessions to the reader. But it is that style that ensures the project's unity and continues without weakening over 40 volumes.
- *The notion of structure* is a methodological tool, but it has never been promoted to the rank of epistemological concept. It is true that the explanation attempted by Bourbaki of the notion of structure is clumsy and unconvincing. But we should note that if categories have replaced structures to advantage, the reasoning specific to categories springs from Bourbaki's emphasis on sub-objects, quotient-objects, limits, etc.

With 30 years' hindsight we can say that, despite the excesses of some over-zealous disciples, Bourbaki's encyclopedia[3] left its mark on a whole era, 1945–75 particularly. It established a new standard for mathematical activity, fixed conventions, codified basic concepts (varieties, rings . . .) and renewed the use of the axiomatic method. It was a new episode in the creation of *universal norms*. Though it did not create a new world, the enterprise 'consolidated the achievements of the early 20th century revolutions'. The verdict is wholly positive overall.

* * *

The 20th century legacy in mathematics laid the foundations for a *new universalism*. The history of mankind has in part been connected with successive extensions of the notion of number and with skill in manipulating those extensions. It is a good thing to create reliable, robust tools that can be used easily and a know-how that is passed on from one generation to another. From that viewpoint the previous century laid up a rich harvest. The idea of formalization, a kind of sublimation of calculus, took off in logic but led to the creation of informatics and scientific computing. Within the field of extensions of number, matrices can be included: these tables of numbers that were almost unknown around 1925 have become indispensable tools in quantum physics as well as statistics, and are also applied in economics (Leontief model), optics, and electricity.. Combinatorics achieved a great flexibility and power of expression, as did methods of symbolic calculus: 'we are drawing calculus and calculating on drawings.' Methods from differential geometry invaded theoretical physics thanks in particular to S. S. Chern, who developed Elie Cartan's methods. A new geometry of forms arose and exotic objects such as fractal sets have become powerful hermeneutic tools.

This universalism was expressed in the organization of the mathematician's profession. We can recall some of the stages: A network of Academies of Science was created and developed from 1650 to 1750. Then, until 1900, came the spread of mathematical journals. There was also a geographical expansion of mathematical activity. Eastern Europe, Turkey, Russia were part of the Germanic cultural area of expansion. It spread to Chile and the Japan of the Meiji era. Of course the linguistic vehicle was German and to a lesser extent French. Since Latin had lost its privileges during the 19th century, the problem of an international language of communication repeatedly raised its head. Since about 1950 English has come to the fore as the

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reference despite the reluctance of some French speakers. The spread through Africa and Asia of a European education by colonial enterprises created centres of mathematical activity in North Africa, Senegal, South Africa and of course India, Australia, Vietnam and more recently Korea and China.

At the same time national mathematical societies were set up. For historical or political reasons Great Britain or Russia do not have national societies (but instead a society in both London and Glasgow, one in Moscow and St Petersburg, and of course Kazan). The Spanish society has Catalan and Basque rivals. Several mathematical societies may co-exist within the same country, aiming at differentiated audiences: university mathematicians, industrial mathematicians and engineers, statisticians, teachers from the different levels of pre-university education, women mathematicians. As a general rule there are harmonious organic links between these various societies.

Unification of mathematical norms, the choice of one (or several) international languages, pooling of teaching experience favoured international exchanges. Since 1897 a world congress of mathematics (called the 'International Congress of Mathematicians' or ICM) brings together between 3000 and 5000 participants every four years. There was a break from 1936 to 1950 because of the Second World War, and the Warsaw congress was postponed from 1982 to 1983 (but kept the acronym ICM82) because of the situation in Poland. The majority of these congresses take place in Europe or North America, but the 2002 one was held in Beijing and the next is planned for Hyderabad (India) in 2010. An International Mathematical Union (IMU) now operates regularly, as well as a European Mathematical Society (EMS), which dreams of equalling the powerful American Mathematical Society (AMS).[4]

To describe this spirit of international cooperation the best course of action is perhaps to hand over to C. Procesi, an Italian colleague and IMU vice-president:

> . . . IMU is serving more and more the mathematical community. In fact for me this is the most important point, the fact that we 'the mathematicians' act and feel as being members of a community and want to make a serious effort towards the goal of making this community really universal.
>
> It was for me an element of pride to read that, after world war one, it was an Italian president of IMU, Salvatore Pincherle, who insisted on inviting to the 1928 ICM mathematicians from the countries who had lost the war: the 'enemies'.[5]
>
> This created a clash inside IMU and we all know how tragic it was the inability to bring back those countries into the international community at that time. We needed another terrible war to start to understand how to cooperate, at least in a part of the world. After the fall of the wall we have been able to knit the ties between the western and eastern European mathematicians and we face now the much harder task to integrate into our community mathematicians from countries which, for political, economic or even cultural reasons are isolated from us.[6] We are small players in the world, with very limited financial resources but with great human resources and a universal language to offer. Our society, IMU, can serve as a reference point for the many organizations[7] who work to build up mathematics in developing countries and, hopefully, also as a fund-raiser. (http://www.mathunion.org/imu-net/archive/2007/imu-net-23/)

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We cannot leave this section on international organization without mentioning the issue of education, 'from nursery school to university', in the words of one of our French professional bodies, the Association of Teachers of Mathematics. That implies an exchange of teaching experiences, international competitions for students (the International Mathematical Olympiad), the work of designing, publishing and distributing textbooks.[8] There is also what may be called acculturation (or indigenization) of the mathematical liturgy. As I said to my Vietnamese friends 30 years back, 'at doctorate level we need a universal language: for you it was French or Russian depending on historical chance, later it will probably be English. But we have to carry on trying to Vietnamize the mathematical vocabulary because primary and secondary education must be in the national language. In this respect using the Latin alphabet (Vietnamized by Alexandre de Rhodes around 1600) will make life easier for you than for the Japanese or Chinese.'

* * *

And what now? What should we do with this universal tool in the 21st century? As a mathematical philosophy, structuralism lasted a long time. *Categories* have replaced structures as a principle of mathematical organization and introduced a principle of relativity. Mathematics developed within a topos may migrate towards another topos. The *search for the foundations*, that is complete rigour in construction of the architecture, appears to be resolved. But that seemed to be the case around 1900, and we still stumble over the same philosophical difficulty: what is an arbitrary series (for instance the decimals of an 'arbitrary' number), or an arbitrary function? Cantor taught us the fundamental distinction between two infinites: denumerable and continuum. The whole of modern analysis rests on that distinction, on the fact that the continuum is irreducibly greater than the denumerable. Logic – and its child informatics – have given an increasingly precise meaning to what is a 'constructible' function described using an algorithm or a program. Unfortunately there is only a denumerable infinity of series, or numbers, or constructible functions, whereas we would need an infinity of them with the power of the continuum. Logical research into the undecidability of the continuum hypothesis (Gödel, P. Cohen) has shown that the collection of sub-sets of a given set, even a very simple one, can be a bottomless pit. Indeed, despite modern claims to have eliminated paradoxes by axiomatizing set theory, have we not with Zermelo, in Poincaré's words, 'shut up the wolf in the sheep-pen, when we thought we'd built a strong fence'? It is likely that we will have to get used to the idea that every scientific theory conceals abysses whose edge we need to avoid approaching

The end of a certain structuralist hegemony also means the end of introspection, the return to a policy of exploration and adventure. There are so many new worlds out there where *string theory* beckons to us. This physicists' theory is engaged in the maybe illusory search for a unification of the two great 20th century advances in physics: Einstein's gravity (or general relativity) and quantum physics. Though there are for the moment no physical predictions, it has opened the way to a mass of very daring developments in mathematics. In a similar way the marriage between operator algebras, renamed 'noncommutative geometry' by Alain Connes and

Alexander Grothendieck's visionary forecasts, two attempts to deepen the structure of space and number, also promises a revolutionary transformation of the ideas of symmetry in mathematics and physics, a final version of Galois's theory on a cosmic scale.

This change in perspective has also been promoted by the end of the world divided by the cold war. Enthusiastic re-encounters between scientists from west and east have led to a meeting between two systems, one of them more formalist and logical in the west, the other more cosmopolitan, more open to rapprochement, less compartmentalized in the east.

Of course the incredible tools provided by the new, near-instantaneous communication media, immediate access via the internet to all the libraries and all the encyclopedias in the world, the opportunities for an experimental mathematics that is new in dimension if not in conception – all these things that mathematicians share with the rest of humanity, all the positive aspects of globalization, can be used to develop our science. After a long and fruitful marriage, over at least four centuries, between mathematics and physics there comes a time to understand mathematically the enormous body of information in biology, to understand the physiology of behaviour after the molecular biology of basic mechanisms. Both in so-called pure mathematics and in mathematical applications there is room for huge advances using tools created over the last 50 years.

This renewed universalism should not make us forget that, without stable enough social balances, without still imperfect mastery of educational methods, without being able to motivate a new generation to take over the reins, without being attentive to the diversity of situations, behaviours and cultures, we are likely to confirm Paul Valéry's pessimistic view at the moment of the historical turning point of 1914: 'We civilizations know we are mortal.'

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Translated from the French by Jean Burrell

Notes

1. By this I mean exploration of the world of form.
2. If a quantity varies continuously, from an initial value A to a final value $B>A$, any intermediate value C such that $A<C<B$ will be reached at least once.
3. There is a constant need for encyclopaedias, and there were other important projects during the 20th century. None will have the scope or the influence of Bourbaki's.
4. Which has also become an important publishing house.
5. And on correcting the affront some French mathematicians had given to the Germans at the 1920 Strasbourg congress. (PC)
6. The IMU has representatives from 65 countries of the 190 represented at the United Nations.
7. Among those organizations mention can be made of CIMPA in Nice, which undertakes to organize meetings and courses worldwide.
8. The Soviet MIR publications did a terrific job in this area. Unfortunately Russia's disastrous economic situation has all but halted this work.